Semantics

The language \( \text{IMP} \) comprises integer and boolean expressions and commands, defined by

\[\begin{align*}
\text{ie} & \in Iexp ::= n \mid x \mid \text{ie}_1 \text{oop} \text{ie}_2 \\
\text{be} & \in Bexp ::= b \mid \text{ie}_1 \text{bop} \text{ie}_2 \\
C & \in \text{Com} ::= \text{skip} \mid x := \text{ie} \mid (C_1; C_2) \mid \\
& \quad \text{if be then } C_1 \text{ else } C_2 \mid \text{repeat } C \text{ until } be
\end{align*}\]

where \( n \in \mathbb{Z} \), \( b \in \{\text{true, false}\} \), \( \text{iop} \in \{+, \times, -\} \), \( \text{bop} \in \{<, =\} \) and \( x \in Pvar \), a set of program variables.

(a) Give an annotated evaluation semantics for \( \text{IMP} \), expressing the usual behaviour of each command and expression form, which derives statements of the forms

\[\begin{align*}
\text{ie}, S \Rightarrow_I n; R & \quad \text{be}, S \Rightarrow_B b; R \\
C, S \Rightarrow_C S'; R, W
\end{align*}\]

for \( S, S' \in \text{States} = (Pvar \rightarrow \mathbb{Z}) \) and \( R, W \subseteq Pvar \). \( C, S \Rightarrow_C S'; R, W \) means ‘in state \( S \), command \( C \) executes to state \( S' \) whilst reading the set of variables \( R \) and writing the set of variables \( W \)’. Similarly, if \( e \in Iexp \cup Bexp \) then \( e, S \Rightarrow v; R \) means ‘in state \( S \), \( e \) reads variables \( R \) in evaluating to \( v \).’

[5 marks]

(b) For \( be \in Bexp \), \( ie \in Iexp \), \( C \in \text{Com} \) use induction on the structure of phrases to give simple definitions of sets

\[\mathcal{R}(ie), \mathcal{R}(be), \mathcal{PR}(C), \mathcal{PW}(C) \subseteq Pvar\]

where \( \mathcal{R}(e) \) is the set of variables accessed by \( e \), \( \mathcal{PR}(C) \) is a set of variables possibly read during the execution of \( C \) and \( \mathcal{PW}(C) \) is a set of variables possibly written to during the execution of \( C \). Give an example to show that it is not in general true that

\[C, S \Rightarrow_C S'; R, W \implies W = \mathcal{PW}(C).\]  

[5 marks]

(c) Prove that for any \( C, S, S', R, W \)

\[C, S \Rightarrow_C S'; R, W \implies (\forall x \in Pvar. \ x \notin W \Rightarrow S(x) = S'(x))\]

and that for any \( be, S, S', b, R \)

\[(\forall x \in \mathcal{R}(be), S(x) = S'(x)) \implies (be, S \Rightarrow_B b; R) \iff (be, S' \Rightarrow_B b; R)\]

[5 marks]

(d) Prove that for any \( C_1, C_2, C_3 \) and \( be \) that if \( \mathcal{R}(be) \cap \mathcal{PW}(C_1) = \emptyset \) then

\[(C_1; \text{if be then } C_2 \text{ else } C_3) \approx \text{if be then } (C_1; C_2) \text{ else } (C_1; C_3)\]

You may assume without proof that if \( C, S \Rightarrow_C S'; R, W \) then \( W \subseteq \mathcal{PW}(C) \).

[5 marks]