Numerical Analysis I

The Newton–Raphson formula for solution of \( f(x) = 0 \) is

\[
x = x - \frac{f(x)}{f'(x)}.
\]

By means of a sketch graph, describe how the method works in a simple case. [4 marks]

When the method converges, what rate of convergence is expected? Describe one circumstance in which the method may fail to converge. [4 marks]

Consider the simultaneous equations

\[
\begin{align*}
f_1(x_1, x_2) &= x_2 - x_1^2 - 2 = 0 \\ f_2(x_1, x_2) &= x_1(x_2 - 3x_1) = 0
\end{align*}
\]

Suppose the iterative scheme

\[
\begin{pmatrix}
-2x_1 & 1 \\
 x_2 - 6x_1 & x_1
\end{pmatrix}
\begin{pmatrix}
h_1 \\ h_2
\end{pmatrix}
= 
\begin{pmatrix}
-f_1(x_1, x_2) \\ -f_2(x_1, x_2)
\end{pmatrix}
\]

is applied to the equations (1). If \( \{x_1, x_2\} \) is the starting approximation, the improved approximation is given by

\[
\begin{pmatrix}
\hat{x}_1 \\ \hat{x}_2
\end{pmatrix}
= 
\begin{pmatrix}
x_1 \\ x_2
\end{pmatrix}
+ 
\begin{pmatrix}
h_1 \\ h_2
\end{pmatrix}.
\]

Suppose \( x_1 = 0 \). Show, by solving the equations (2) that the first iteration always produces the same improved approximation for any non-zero \( x_2 \). [10 marks]

Verify that the method converges if \( x_1 \) is set to 0, and \( x_2 \neq 0 \). [2 marks]