Numerical Analysis II

A cubic spline \( \phi(x) \) is defined over \([a, b]\) with knots \( x_1, x_2, \ldots x_n \) such that \( a < x_1, x_n < b \). The spline takes the values \( y_1, y_2, \ldots y_n \) at the knots. What continuity conditions are usually imposed on the cubic spline at each knot? \([2\text{ marks}]\)

If \( d_j = x_{j+1} - x_j \) and \( \mu_j = \phi''(x_j) \), the spline has the following formula for \( x \in [x_j, x_{j+1}] \)

\[
\phi(x) = \frac{(x-x_j)y_{j+1} + (x_{j+1} - x)y_j}{d_j} - \frac{(x-x_j)(x_{j+1} - x)(d_j + x_{j+1} - x)\mu_j + (d_j + x - x_j)\mu_{j+1}}{6d_j}.
\]

By differentiating this formula:

(a) find formulae for \( \phi'(x_j) \) and \( \phi'(x_{j+1}) \) for \( x \in [x_j, x_{j+1}] \) \([4\text{ marks}]\)

(b) verify that \( \phi''(x_j) = \mu_j, \phi''(x_{j+1}) = \mu_{j+1} \) \([4\text{ marks}]\)

(c) deduce the equation which expresses the continuity condition on \( \phi'(x) \) at \( x_j \) \([6\text{ marks}]\)

If the equations derived in part (c) are solved as a simultaneous system, what are the unknowns? If the end conditions specify the spline to be linear in \([a, x_1]\) and \([x_n, b]\) how does this simplify the calculation? State the most important properties of the resulting equations. \([4\text{ marks}]\)