Semantics of Programming Languages

State the Tarski-Knaster fixed-point theorem. Give a brief justification for the importance of the fixed-point theorem in denotational semantics. [8 marks]

Prove that the least fixed-point operator \( \text{fix} \) is a continuous function. You may assume the following result:

Let \( \langle D, \sqsubseteq \rangle \) be a complete partial order. Every doubly-increasing chain \( \langle d_{ij} \rangle_{i,j \in \omega} \) in \( D \) (i.e. for any \( i, j, i', j' \) in \( \omega \), if \( i \leq i' \) and \( j \leq j' \) then \( d_{ij} \sqsubseteq d_{i'j'} \)) has a least upper bound \( l \). Further,

\[
l = \bigsqcup_{i \in \omega} \bigsqcup_{j \in \omega} d_{ij} = \bigsqcup_{j \in \omega} \bigsqcup_{i \in \omega} d_{ij} = \bigsqcup_{k \in \omega} d_{kk}
\]

[12 marks]