1994 Paper 8 Question 12

Semantics of Programming Languages

Let D be a complete partial order with bottom. What does it mean for a subset of D (regarded as a predicate) to be *inclusive*? State Scott's principle of fixed-point induction. [6 marks]

Let $\mathbf{B}$ be the usual flat complete partial order of truth values consisting of elements $\bot$, $true$ and $false$. For a complete partial order D with bottom, let the conditional function

$$\cdot \rightarrow \cdot \,|\, \cdot \quad : \quad \mathbf{B} \times D \times D \longrightarrow D$$

be given by

$$b \rightarrow d \,|\, e \quad = \quad \begin{cases} d & \text{if } b = true, \\ e & \text{if } b = false, \\ \bot & \text{if } b = \bot. \end{cases}$$

Let $p : D \longrightarrow \mathbf{B}$ and $h : D \longrightarrow D$ be continuous functions with h strict (i.e. $h(\bot) = \bot$). Let $f : D \times D \longrightarrow D$ be the least continuous function which satisfies

$$f(x, y) \quad = \quad p(x) \rightarrow y \,|\, h(f(h(x), y)) \qquad \text{for all } (x, y) \in D \times D.$$

Show that the following predicate

$$P(g) \quad \equiv \quad \forall (x, y) \in D \times D . h(g(x, y)) = g(x, h(y))$$

is inclusive. [4 marks]

Prove that $h(f(x, y)) = f(x, h(y))$, for all $(x, y) \in D \times D$. [10 marks]