Numerical Analysis I

The mid-point rule can be expressed in the form

\[ I_n = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(x) \, dx = f(n) + e_n \]

where

\[ e_n = \frac{f''(\theta_n)}{24} \]

for some \( \theta_n \) in the interval \((n - \frac{1}{2}, n + \frac{1}{2})\). Assuming that a formula for \( \int f(x) \, dx \) is known, and using the notation

\[ S_{p,q} = \sum_{n=p}^{q} f(n), \]

describe a method for estimating the sum of a slowly convergent series \( S_{1,\infty} \), by summing only the first \( N \) terms and estimating the remainder by integration. [7 marks]

Assuming that \( f''(x) \) is a positive decreasing function, derive an estimate of the error \( |E_N| \) in the method. [5 marks]

Given

\[ \int \frac{dx}{1 + x^2} = \tan^{-1} x, \]

apply the method to

\[ \sum_{n=1}^{\infty} \frac{1}{1 + n^2}. \]

What is the integral remainder to be added to \( S_{1,N} \)? [4 marks]

To the nearest power of 10, how large should \( N \) be to achieve an absolute error of approximately \( 10^{-16} \)? [4 marks]