1994 Paper 11 Question 10

Discrete Mathematics

Let $(\mathbb{N}, \leqslant)$ be the natural numbers under the usual ordering. Assuming that $(\mathbb{N}, \leqslant)$ is well-ordered, prove that the Cartesian product $(\mathbb{N} \times \mathbb{N})$ is well-ordered under the derived lexicographical ordering. [6 marks]

State the *Principle of Well-Ordered Induction.* [3 marks]

Define inductively f: $(\mathbb{N} \times \mathbb{N}) \to \mathbb{N}$ as follows:

$$\begin{cases} f(0,y) & = \; y+1 \\ f(x+1,0) & = \; f(x,1) \\ f(x+1,y+1) & = \; f(x, f(x+1,y)) \end{cases}$$

Show that f is defined for all pairs (x,y). [2 marks]

Prove that for all $y \in \mathbb{N}$:

$$\begin{cases} f(2,y) & = \; 2y+3 \\ f(3,y) & = \; 2^{y+3}-3 \end{cases}$$

[9 marks]