Discrete Mathematics

Let \((\mathbb{N}, \leq)\) be the natural numbers under the usual ordering. Assuming that \((\mathbb{N}, \leq)\) is well-ordered, prove that the Cartesian product \((\mathbb{N} \times \mathbb{N})\) is well-ordered under the derived lexicographical ordering. [6 marks]

State the Principle of Well-Ordered Induction. [3 marks]

Define inductively \(f: (\mathbb{N} \times \mathbb{N}) \rightarrow \mathbb{N}\) as follows:

\[
\begin{align*}
  f(0, y) &= y + 1 \\
  f(x + 1, 0) &= f(x, 1) \\
  f(x + 1, y + 1) &= f(x, f(x + 1, y))
\end{align*}
\]

Show that \(f\) is defined for all pairs \((x, y)\). [2 marks]

Prove that for all \(y \in \mathbb{N}\):

\[
\begin{align*}
  f(2, y) &= 2y + 3 \\
  f(3, y) &= 2^{y+3} - 3
\end{align*}
\]

[9 marks]