MPhil Advanced Computer Science
Topics in Logic and Complexity

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Exercise Sheet 2

1. In the lecture we saw an illustration of a construction to show that *acyclicity* of graphs is not definable in first-order logic. Write out a proof of this result.

   Prove that *acyclicity* is not definable in $\mathsf{Mon}.\Sigma^1_1$. Is it definable in $\mathsf{Mon}.\Pi^1_1$?

2. Prove (using Hanf's theorem or otherwise) that 3-colourbility of graphs is not definable in first-order logic.

   Graph 3-colourability (and, indeed, 2-colourability) are definable in $\mathsf{Mon}.\Sigma^1_1$. Are they definable in $\mathsf{Mon}.\Pi^1_1$? Are they definable in *universal second-order logic*?

3. Prove the lemma stated in the lecture that any formula that is positive in the relation symbol R defines a monotone operator.

4. Prove that the formula of LFP $\neg[\mathbf{lfp}_{R,\mathbf{x}}\neg\phi(R/\neg R)](\mathbf{x})$, where $\phi(R/\neg R)$ denotes the result of replacing all occurrences of R in ϕ by $\neg R$, defines the greatest fixed point of the operator defined by ϕ.

5. In the lectures, we saw how definitions by simultaneous induction can be replaced by a single application of the **lfp** operator. In this exercise, you are asked to show the same for *nested* applications of the **lfp** operator.

   Suppose $\phi(\mathbf{x}, \mathbf{y}, S, T)$ is a formula in which the relational variables S (of arity s) and T (of arity t) only appear positively, and $\mathbf{x}$ and $\mathbf{y}$ are tuples of variables of length s and t respectively. Show that (for any t-tuple of terms $\mathbf{t}$) the predicate expression

   $$[\mathbf{lfp}_{S,\mathbf{x}}([\mathbf{lfp}_{T,\mathbf{y}}\phi](\mathbf{t}))]$$

   is equivalent to an expression with just one application of **lfp**.

6. Consider a vocabulary consisting of two unary relations P and O, one binary relation E and two constants s and t. We say that a structure $\mathbb{A} = (A, P, O, E, s, t)$ in this vocabulary is an *arena* if $P \cup O = A$ and $P \cap O = \emptyset$. That is, P and O partition the universe into two disjoint sets.

   An arena defines the following game played between a *player* and an *opponent*. The game involves a *token* that is initially placed on the element s. At each move, if the token is currently on an element of P it is *player* who plays and if it is on an element of O, it is *opponent* who plays. At each move, if the token is on an element a, the one who plays choses an element b such that $(a, b) \in E$ and moves the token from a to b. If the token reaches t at any point then *player* has won the game.

We define Game to be the class of arenas for which *player* has a strategy for winning the game. Note that in an arena $\mathbb{A} = (A, P, O, E, s, t)$, *player* has a strategy to win from an element a if *either* $a \in P$ and there is some move from a so that *player* still has a strategy to win after that move *or* $a \in O$ and for every move from a, *player* can win after that move.

(a) Give a sentence of LFP that defines the class of structures Game.

We say that a collection $\mathcal{C}$ of decision problems is *closed under logarithmic space reductions* if whenever $A \in \mathcal{C}$ and $B \leq_L A$ (i.e. B is reducible to A by a logarithmic-space reduction) then $B \in \mathcal{C}$.

The class of structures Game defined above is known to be P-complete under logarithmic-space reductions.

(b) Explain why this, together with (a) implies that the class of problems definable in LFP is *not* closed under logarithmic-space reductions.

7. Give a sentence of LFP that defines the class of linear orders with an even number of elements.

8. The *directed graph reachability problem* is the problem of deciding, given a structure (V, E, s, t) where E is an arbitrary binary relation on V, and $s, t \in V$, whether (s, t) is in the reflexive-transitive closure of E. This problem is known to be decidable in NL.

   Transitive closure logic is the extension of first-order logic with an operator **tc** which allows us to form formulae

   $$\phi \equiv [\mathbf{tc}_{\mathbf{x},\mathbf{y}}\psi](\mathbf{t}_1, \mathbf{t}_2)$$

   where $\mathbf{x}$ and $\mathbf{y}$ are k-tuples of variables and $\mathbf{t}_1$ and $\mathbf{t}_2$ are k-tuples of terms, for some k; and all occurrences of variables $\mathbf{x}$ and $\mathbf{y}$ in ψ are bound in ϕ. The semantics is given by saying, if $\mathbf{a}$ is an interpretation for the free variables of ϕ, then $\mathcal{A} \models \phi[\mathbf{a}]$ just in case $(\mathbf{t}_1^{\mathbf{a}}, \mathbf{t}_2^{\mathbf{a}})$ is in the reflexive-transitive closure of the binary relation defined by $\psi(\mathbf{x}, \mathbf{y})$ on A^k.

   (a) Show that any class of structures definable by a sentence ϕ, as above, where ψ is first-order, is decidable in NL.

   (b) Show that, if K is an isomorphism-closed class of structures in a relational signature including $<$, such that each structure in K interprets $<$ as a linear order and

   $$\{[\mathcal{A}]_< \mid \mathcal{A} \in K\}$$

   is decidable in NL, then there is a sentence of transitive-closure logic that defines K.

9. For a binary relation E on a set A, define its *deterministic transitive closure* to be the set of pairs (a, b) for which there are $c_1, \ldots, c_n \in A$ such

that $a = c_1$, $b = c_n$ and for each $i < n$, c_{i+1} is the *unique* element of A with $(c_i, c_{i+1}) \in E$.

Let DTC denote the logic formed by extending first-order logic with an operator **dtc** with syntax analogous to **tc** above, where $[\mathbf{dtc}_{\mathbf{x},\mathbf{y}}\psi]$ defines the deterministic transitive closure of $\psi(\mathbf{x}, \mathbf{y})$.

(a) Show that every sentence of DTC defines a class of structures decidable in L.

(b) Show that, if K is an isomorphism-closed class of structures in a relational signature including $<$, such that each structure in K interprets $<$ as a linear order and

$$\{[\mathcal{A}]_< \mid \mathcal{A} \in K\}$$

is decidable in L, then there is a sentence of DTC that defines K.

10. On page 47 of Handout 4, the correspondence between k-pebble games and the equivalence $\equiv^k$ is stated. This exercise asks you to prove the easy direction of that equivalence. That is, show that if Duplicator has a winning strategy in the k-pebble game for q moves starting from position $(\mathbb{A}, \mathbf{a})$ and $(\mathbb{B}, \mathbf{b})$ then $(\mathbb{A}, \mathbf{a}) \equiv^k_q (\mathbb{B}, \mathbf{b})$.

11. We say that a graph $G = (V, E)$ has a *perfect matching* if there is a set $M \subseteq E$ of edges such that for every vertex $v \in V$ there is *exactly one* edge in M that includes v. Prove that the property of having a perfect matching is not definable in LFP.

12. Let $\mathcal{E}$ be the class of *equivalence relations*. That is, it consists of all structures $\mathbb{A} = (A, R)$ where R is a binary relation on A that is reflexive, symmetric and transitive. Prove that, for any k there are no more than k^k equivalence classes of the relation $\equiv^k$ on $\mathcal{E}$.

    Prove that LFP is no more expressive that first-order logic on $\mathcal{E}$. That is, for any formula ϕ of LFP, there is a first-order formula ψ such that, for any $\mathbb{A} \in \mathcal{E}$, $\mathbb{A} \models \phi$ if, and only if, $\mathbb{A} \models \psi$.

13. In Handout 5, pages 10–12, we saw a sketch of the proof that $\mathsf{CNF\text{-}SAT}$ is NP-complete under first-order reductions.

    Define the problem Clique to be the class of structures (V, E, U) where E is a binary relation on V and U is a unary relation and the graph (V, E) contains a clique of size $|U|$. Show that there is a first-order reduction from $\mathsf{CNF\text{-}SAT}$ to Clique.

14. Give a definition of what it would mean for the complexity class L to be *recursively indexable*. Show that it is recursively indexable if, and only if, it has a complete problem under first-order reductions.

    The complexity class $\mathsf{PolyLogSpace}$ is defined to be class of those problems decidable on a deterministic machine in time $O((\log n)^k)$ for some k. Show that this class has no complete problems under first-order reductions. (*Hint:* recall the space hierarchy theorem).

15. The aim of this exercise is to show that Hanf's theorem can be extended to the logic with counting quantifiers. That is, write $\mathbb{A} \equiv_p^C \mathbb{B}$ to denote that $\mathbb{A}$ and $\mathbb{B}$ cannot be distinguished by any sentence of *first-order logic with counting quantifiers* that has quantifier rank at most p. Also recall that $\mathbb{A} \simeq_r \mathbb{B}$ denotes that $\mathbb{A}$ and $\mathbb{B}$ are Hanf equivalent with radius r (see page 21 of Handout 3). Show that for every vocabulary σ and every p there is an r such that if $\mathbb{A}$ and $\mathbb{B}$ are σ-structures, then $\mathbb{A} \simeq_r \mathbb{B}$ implies $\mathbb{A} \equiv_p^C \mathbb{B}$.