Advanced Topics in Category Theory

Lectured by Jamie Vicary

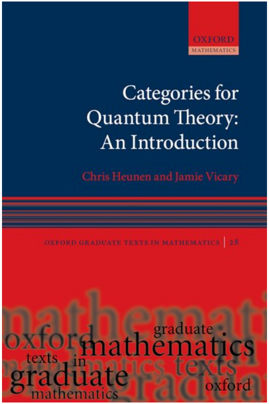
Department of Computer Science
University of Cambridge

Lent Term 2024

Welcome to ATCT!

Topics. We will cover these topics:

- Monoidal and higher categories
- The graphical calculus
- Type theory for higher categories
- Linear structure and duality
- Monoids and comonoids
- Frobenius and Hopf structures



Assessment. Three modes:

- Exercise sheets (50%)
- Practical portfolio (30%)
- Class presentation (20%)

Book. The course is based on the book *Categories for Quantum Theory: An Introduction* (OUP).

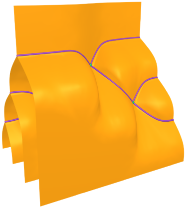
Notes. All notes and slides are on the website.
Exercise sheets will be released during the term.

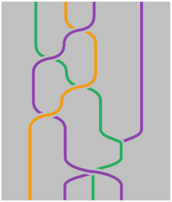
Practical (30% credit)

For the practical, we will learn to use a proof assistant for higher categories, called *homotopy.io*.

It is web-based and hosted here:

http://beta.homotopy.io





We will take a first look at it today. You can follow along if you have a suitable device.

Over the term you will build up a portfolio of formalised proofs, working in your own time, and supported by 4 practical classes.

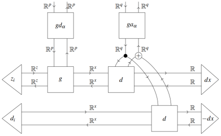
Over time you will build up to tackling more complex proofs, supported by a list of suggestions, available on the webpage.

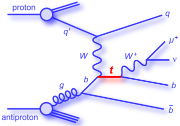
At the end of the course, you will submit a portfolio of 5 of the most interesting and challenging proofs that you have constructed.

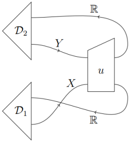
Research seminars (20% credit)

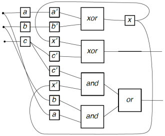
Monoidal and higher categories are part of the standard toolkit of modern theoretical computer science.

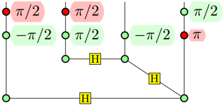
They are used across broad range of areas, including foundations, type theory, game theory, machine learning, natural language processing, programming languages, proof assistants, and process algebras.

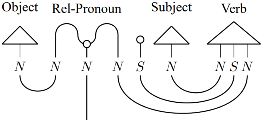


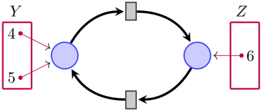












Research seminars (20% credit)

The second half of this course will be a research seminar, exploring the fascinating area of applied category theory.

These techniques are well-represented at research events such as the *Applied Category Theory* (ACT) conferences, and the workshop series *Symposium on Compositional Structures* (SYCO):

- ACT 2018, ACT 2019, ACT 2021, ACT 2022, ACT 2023
- SYCO 1, SYCO 2, SYCO 3, SYCO 4, SYCO 5, SYCO 6, SYCO 7, SYCO 8, SYCO 9, SYCO 10, SYCO 11

Each student will give a **20-minute talk** on their choice of paper. Together we will discuss and learn about the research frontier.

This is a core part of the course. The highest marks will go to students who deliver a clear and interesting presentation, including some research-level technical content, and who interact well with the seminar series as a whole.

Research seminars (20% credit)

Here are some notes and advice about the research seminars.

- A list of suggested papers is on the course webpage.
- You can also find your own paper (try the ACT/SYCO pages.)
- Use the planning spreadsheet to indicate your chosen paper.
- We can have multiple talks on the same paper or topic, but they should cover different aspects.
- Talks must be written and delivered independently, but you can coordinate so the talks work well together.
- Email other students as necessary to deconflict.
- If you prefer an earlier talk, indicate this on the spreadsheet.
- Final decisions about timetabling will be made by me.
- Deliver your talk however you like. Slides are recommended.
- In your talk, try to communicate two main things:

  Why is this interesting? *What's the key technical idea?*
- Practise your talk in advance.
- **The research seminar environment will be supportive. Have fun and don't be nervous!**

Chapter 0

Basic ideas

Basic ideas

Chapter 0 of the notes covers some simple topics that are a good background for the course:

- Section 0.1: Category theory
- Section 0.2: Hilbert spaces
- Section 0.3: Quantum information

We will cover in the lectures everything that we need directly, but you may find these sections useful if you have not studied these topics before.

Chapter 1

Monoidal categories

1.1 Monoidal structure

Category theory describes systems and processes:

- physical systems, and physical processes governing them;
- data types, and algorithms manipulating them;
- algebraic structures, and structure-preserving functions;
- logical propositions, and implications between them.

Monoidal category theory adds the idea of *parallelism*:

- independent physical systems evolve simultaneously;
- running computer algorithms in parallel;
- products or sums of algebraic or geometric structures;
- using separate proofs of P and Q to construct a proof of the conjunction (P and Q).

1.1 Monoidal structure

Why should this theory be interesting?

- Let A, B and C be processes, and let $\otimes$ be parallel composition
- What *relationship* should there be between these processes?

$$(A \otimes B) \otimes C \qquad\qquad A \otimes (B \otimes C)$$

- It's not right to say they're *equal*, since even just for sets,

$$(S \times T) \times U \neq S \times (T \times U).$$

- Maybe they should be *isomorphic* — but then what *equations* should these isomorphisms satisfy?
- How do we treat *trivial* systems?
- What should the relationship be between $A \otimes B$ and $B \otimes A$?

1.1 Monoidal structure

Definition 1.1. A *monoidal category* is a category $\mathbf{C}$ equipped with the following data:

- a *tensor product* functor
$$\otimes \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C};$$
- a *unit object*
$$I \in \mathrm{Ob}(\mathbf{C});$$
- an *associator* natural isomorphism
$$(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C);$$
- a *left unitor* natural isomorphism
$$I \otimes A \xrightarrow{\lambda_A} A;$$
- and a *right unitor* natural isomorphism
$$A \otimes I \xrightarrow{\rho_A} A.$$

1.1 Monoidal structure

This data must satisfy the *triangle* and *pentagon* equations, for all objects A, B, C and D:

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes \mathrm{id}_B"'] & & A \otimes (I \otimes B) \arrow[dl, "\mathrm{id}_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
& \big(A \otimes (B \otimes C)\big) \otimes D \arrow[r, "\alpha_{A,B\otimes C,D}"] & A \otimes \big((B \otimes C) \otimes D\big) \arrow[dr, "\mathrm{id}_A \otimes \alpha_{B,C,D}"] & \\
\big((A \otimes B) \otimes C\big) \otimes D \arrow[ur, "\alpha_{A,B,C} \otimes \mathrm{id}_D"] \arrow[dr, "\alpha_{A\otimes B,C,D}"'] & & & A \otimes \big(B \otimes (C \otimes D)\big) \\
& (A \otimes B) \otimes (C \otimes D) \arrow[urr, "\alpha_{A,B,C\otimes D}"'] & &
\end{tikzcd}$$

Theorem 1.2 (Coherence for monoidal categories). *If the pentagon and triangle equations hold, then so does any well-typed equation built from α, λ, ρ and their inverses.*

To appreciate this, try to prove $\lambda_I = \rho_I$ (see exercises.)

1.1 Monoidal structure

The monoidal structure on **Set** is given by Cartesian product.

Definition 1.4. The monoidal structure on the category **Set**, and also by restriction on **FSet**, is defined as follows:

- **the tensor product** is Cartesian product of sets, written $\times$, acting on functions $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$ as $(f \times g)(a, c) = \big(f(a); g(c)\big)$
- **the unit object** is a chosen singleton set $\{\bullet\}$;
- **associators** $(A \times B) \times C \xrightarrow{\alpha_{A,B,C}} A \times (B \times C)$ are the functions given by $\big((a, b), c\big) \mapsto \big(a, (b, c)\big)$;
- **left unitors** $I \times A \xrightarrow{\lambda_A} A$ are the functions $(\bullet, a) \mapsto a$;
- **right unitors** $A \times I \xrightarrow{\rho_A} A$ are the functions $(a, \bullet) \mapsto a$.

Other tensor products exist, but this one plays a canonical role in our interpretation of classical reality.

1.1 Monoidal structure

Monoidal categories satisfy the *interchange law*, which governs the interaction between composition and tensor product.

Theorem 1.7 (Interchange). *Any morphisms $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, $D \xrightarrow{h} E$ and $E \xrightarrow{j} F$ in a monoidal category satisfy the interchange law:*

$$(g \circ f) \otimes (j \circ h) = (g \otimes j) \circ (f \otimes h)$$

Proof. This holds because of properties of the category $\mathbf{C} \times \mathbf{C}$, and from the fact that $\otimes \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is a functor:

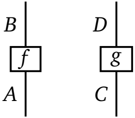
$$\begin{aligned}
(g \circ f) \otimes (j \circ h) &\equiv \otimes(g \circ f, j \circ h) \\
&= \otimes\big((g,j) \circ (f,h)\big) && \text{(composition in } \mathbf{C} \times \mathbf{C}\text{)} \\
&= \big(\otimes(g,j)\big) \circ \big(\otimes(f,h)\big) && \text{(functoriality of } \otimes\text{)} \\
&= (g \otimes j) \circ (f \otimes h)
\end{aligned}$$

Remember the functoriality property: $F(g \circ f) = F(g) \circ F(f)$.

1.1 Monoidal structure

Monoidal categories have an elegant graphical calculus.

For morphisms $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$, we draw their tensor product $A \otimes C \xrightarrow{f \otimes g} B \otimes D$ like this:



The idea is that f and g represent distinct processes taking place at the same time.

Inputs are drawn at the bottom, and outputs are drawn at the top; in this sense, "time" runs upwards.

1.1 Monoidal structure

The monoidal unit object I is drawn as the empty diagram:

The left unitor $I \otimes A \xrightarrow{\lambda_A} A$, the right unitor $A \otimes I \xrightarrow{\rho_A} A$ and the associator $(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C)$ are also not depicted:

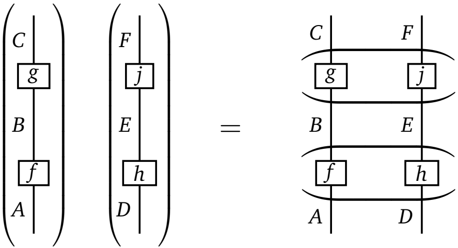
λ_A ρ_A $\alpha_{A,B,C}$

The coherence of α, λ and ρ is essential for the graphical calculus to function. Since there can only be a single morphism built from their components of any given type, it *doesn't matter* that their graphical calculus encodes no information.

1.1 Monoidal structure

Now let's look at the interchange law (1.4):

$$(g \circ f) \otimes (j \circ h) \quad = \quad (g \otimes j) \circ (f \otimes h)$$



Graphically it's trivial.

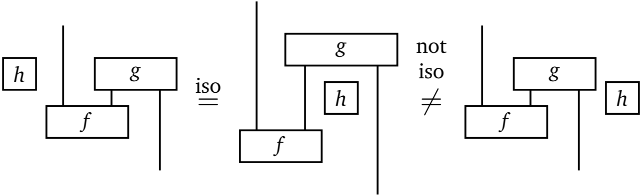
The apparent complexity of the theory of monoidal categories—α, λ, ρ, coherence, interchange—was in fact complexity of the *geometry of the plane*. So when we use a geometrical notation, the complexity vanishes.

1.1 Monoidal structure

Two diagrams are *planar isotopic* when one can be deformed continuously into the other, such that:

- diagrams remain confined to a rectangular region of the plane;
- input and output wires terminate at the lower and upper boundaries of the rectangle;
- components of the diagram never intersect.

Here are examples of isotopic and non-isotopic diagrams:



We will allow heights of the diagrams to change, and allow input and output wires to slide horizontally along the boundary, although they must never change order.

1.1 Monoidal structure

We can now state the correctness theorem.

Theorem 1.8 (Correctness of the graphical calculus for monoidal categories). *A well-formed equation between morphisms in a monoidal category follows from the axioms if and only if it holds in the graphical language up to planar isotopy.*

Let f and g be morphisms such that the equation $f = g$ is well-formed, and consider the following statements:

- $P(f,g)$ = 'under the axioms of a monoidal category, $f = g$'
- $Q(f,g)$ = 'graphically, f and g are planar isotopic'

Soundness is the assertion that for all such f and g, $P(f,g) \Rightarrow Q(f,g)$. It is easy to prove: just check each axiom.

Completeness is the reverse assertion, that for all such f and g, $Q(f,g) \Rightarrow P(f,g)$. It is hard to prove; one must show that planar isotopy is generated by a finite set of moves, each being implied by the monoidal axioms.

1.1 Monoidal structure

The category **Hilb** has a canonical monoidal structure, given by quantum theory.

Definition 1.3. The monoidal structure on the category **Hilb**, and also by restriction on **FHilb**, is defined in the following way:

- **the tensor product** $\otimes$: **Hilb** $\times$ **Hilb** $\to$ **Hilb** is the tensor product of Hilbert spaces, as defined in Section 0.2.5;
- **the unit object** I is the one-dimensional Hilbert space $\mathbb{C}$;
- **associators** $(H \otimes J) \otimes K \xrightarrow{\alpha_{H,J,K}} H \otimes (J \otimes K)$ are the unique linear maps satisfying $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$ for all $u \in H$, $v \in J$ and $w \in K$;
- **left unitors** $\mathbb{C} \otimes H \xrightarrow{\lambda_H} H$ are the unique linear maps satisfying $1 \otimes u \mapsto u$ for all $u \in H$;
- **right unitors** $H \otimes \mathbb{C} \xrightarrow{\rho_H} H$ are the unique linear maps satisfying $u \otimes 1 \mapsto u$ for all $u \in H$.

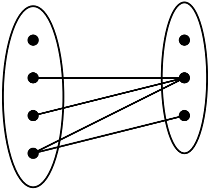
1.1 Monoidal structure

Relations give another notion of process between sets.

Definition 0.4. Given sets A and B, a *relation* $A \xrightarrow{R} B$ is a subset $R \subseteq A \times B$.

We can think of a relation $A \xrightarrow{R} B$ in a dynamical way, as specifying how states of A can evolve into states of B:

$$A \xrightarrow{R} B$$



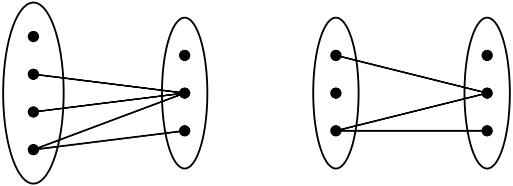
(1)

This is nondeterministic, because an element of A can be related to more than one element of B, or to none.

1.1 Monoidal structure

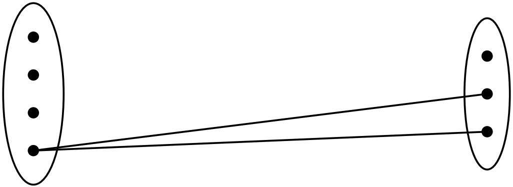
Suppose we have a pair of head-to-tail relations:

$$A \xrightarrow{R} B \qquad B \xrightarrow{S} C$$



Then our interpretation gives a natural notion of composition:

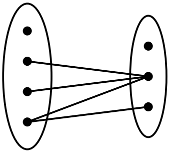
$$A \xrightarrow{S \circ R} C$$



1.1 Monoidal structure

We can write relations as (0,1)-valued matrices:

$$A \xrightarrow{R} B$$



$$\leftrightsquigarrow \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Composition of relations is then ordinary matrix multiplication, with logical disjunction (OR) and conjunction (AND) for + and $\times$.

1.1 Monoidal structure

The intuition we have developed leads to the following definition of the category **Rel**.

Definition 0.5 (Rel, FRel). The category **Rel** of sets and relations is defined as follows:

- **objects** are sets $A, B, C, \ldots$;
- **morphisms** are relations $R \subseteq A \times B$, with $(a, b) \in R$ written aRb;
- **composition** of $A \xrightarrow{R} B$ and $B \xrightarrow{S} C$ is the relation $$\{(a, c) \in A \times C \mid \exists b \in B \colon aRb, bSc\};$$
- **the identity morphism** on A is the relation $$\{(a, a) \in A \times A \mid a \in A\}.$$

Define the category **FRel** to be the restriction of **Rel** to finite sets.

While **Set** is a setting for classical physics, and **Hilb** is a setting for quantum physics, **Rel** is somewhere in the middle.

It seems like **Rel** should be a lot like **Set**, but we will discover it behaves a lot more like **Hilb**.

1.1 Monoidal structure

There is a canonical monoidal structure on the category **Rel**.

Definition 1.5. The monoidal structure on the category **Rel** is defined in the following way:

- **the tensor product** is Cartesian product of sets, written $\times$, acting on relations $A \xrightarrow{R} B$ and $C \xrightarrow{S} D$ by setting $(a, c)(R \times S)(b, d)$ if and only if aRb and cSd;
- **the unit object** is a chosen singleton set $= \{\bullet\}$;
- **associators** $(A \times B) \times C \xrightarrow{\alpha_{A,B,C}} A \times (B \times C)$ are the relations defined by $\big((a, b), c\big) \sim \big(a, (b, c)\big)$;
- **left unitors** $I \times A \xrightarrow{\lambda_A} A$ are the relations defined by $(\bullet, a) \sim a$;
- **right unitors** $A \times I \xrightarrow{\rho_A} A$ are the relations defined by $(a, \bullet) \sim a$.

The Cartesian product is *not* a categorical product in **Rel**, so although this monoidal structure looks like that of **Set**, it is more similar to the structure on **Hilb**.

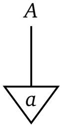
1.1 Monoidal structure

In a category, we cannot 'look inside' an object to inspect its elements. We have do everything using the morphisms.

Definition 1.10. In a monoidal category, a *state* of an object A is a morphism $I \to A$.

The monoidal unit object represents the trivial system, so a state is a way for the system A to be 'brought into existence'.

We draw a state $I \xrightarrow{a} A$ like this:



1.1 Monoidal structure

Example 1.11. Let's examine the states in our example categories.

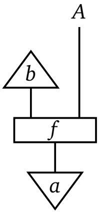
- In **Hilb**, states of a Hilbert space H are linear functions $\mathbb{C} \to H$, which correspond to *elements* of H by considering the image of $1 \in \mathbb{C}$.
- In **Set**, states of a set A are functions $\{\bullet\} \to A$, which correspond to *elements* of A by considering the image of $\bullet$.
- In **Rel**, states of a set A are relations $\{\bullet\} \xrightarrow{R} A$, which correspond to *subsets* by considering all elements related to $\bullet$.

1.1 Monoidal structure

The dual notion of state is effect.

Definition 1.15. In a monoidal category, an *effect* on an object A is a morphism $A \to I$.

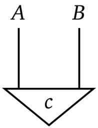
We can use states, effects and other morphisms to build up interesting diagrams, which give 'histories' for a family of systems:



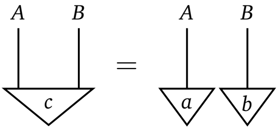
We can interpret an effect as a *property observation* of a system. Overall this composite gives a state of A.

1.1 Monoidal structure

A morphism $I \xrightarrow{c} A \otimes B$ is a *joint state* of A and B. We depict it graphically in the following way.



Definition 1.13. A joint state $I \xrightarrow{c} A \otimes B$ is a *product state* when it is of the form $I \xrightarrow{\lambda_I^{-1}} I \otimes I \xrightarrow{a \otimes b} A \otimes B$:



Definition 1.13. A joint state is *entangled* when it is not a product state.

1.1 Monoidal structure

Example 1.14. Let's investigate joint states, product states, and entangled states in our example categories.

- In **Hilb**:
  - **joint states** of H and K are elements of $H \otimes K$;
  - **product states** are factorizable states;
  - **entangled states** are elements of $H \otimes K$ which cannot be factorized, i.e. entangled states in the quantum sense.
- In **Set**:
  - **joint states** of A and B are elements of $A \times B$;
  - **product states** are elements $(a, b) \in A \times B$;
  - **entangled states** don't exist.
- In **Rel**:
  - **joint states** of A and B are subsets of $A \times B$;
  - **product states** are subsets $U \subseteq A \times B$ such that, for some $V \subseteq A$ and $W \subseteq B$, $(v, w) \in U$ if and only if $v \in V$, $w \in W$;
  - **entangled states** are subsets that aren't of this form.

1.2 Braiding and symmetry

In many theories, the systems $A \otimes B$ and $B \otimes A$ can be considered essentially equivalent. Developing this idea gives rise to *braided* and *symmetric* monoidal categories.

Definition 1.17. A *braided monoidal category* is a monoidal category equipped with a natural isomorphism

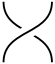
$$A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A$$

satisfying the following *hexagon equations*:

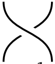
$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[rr, "\sigma_{A,B\otimes C}"] \arrow[d, "\alpha^{-1}_{A,B,C}"] & & (B \otimes C) \otimes A \\
(A \otimes B) \otimes C \arrow[d, "\sigma_{A,B} \otimes \mathrm{id}_C"] & & B \otimes (C \otimes A) \arrow[u, "\alpha^{-1}_{B,C,A}"'] \\
(B \otimes A) \otimes C \arrow[rr, "\alpha_{B,A,C}"'] & & B \otimes (A \otimes C) \arrow[u, "\mathrm{id}_B \otimes \sigma_{A,C}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \otimes B) \otimes C \arrow[rr, "\sigma_{A\otimes B,C}"] \arrow[d, "\alpha_{A,B,C}"] & & C \otimes (A \otimes B) \\
A \otimes (B \otimes C) \arrow[d, "\mathrm{id}_A \otimes \sigma_{B,C}"] & & (C \otimes A) \otimes B \arrow[u, "\alpha_{C,A,B}"'] \\
A \otimes (C \otimes B) \arrow[rr, "\alpha^{-1}_{A,C,B}"'] & & (A \otimes C) \otimes B \arrow[u, "\sigma_{A,C} \otimes \mathrm{id}_B"']
\end{tikzcd}$$

1.2 Braiding and symmetry

We include the braiding in our graphical notation like this:



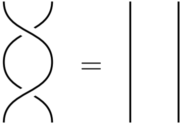
$$A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A$$

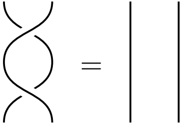


$$B \otimes A \xrightarrow{\sigma_{A,B}^{-1}} A \otimes B$$

The strands of a braiding cross over each other, so the diagrams are not planar; they are inherently 3-dimensional.

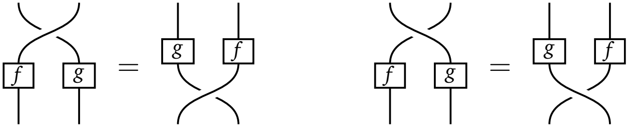
Invertibility takes the following graphical form:



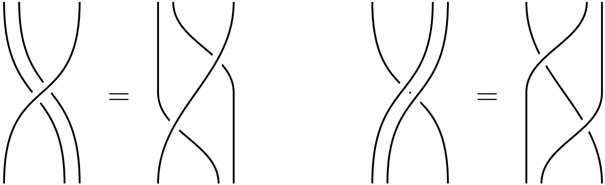


1.2 Braiding and symmetry

Naturality has the following graphical representation:



The hexagon equations look like this:



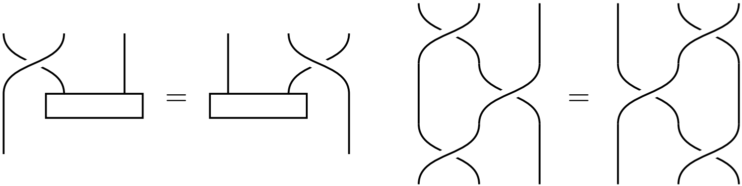
So braiding with a tensor product of two objects is the same as braiding with one then the other separately.

1.2 Braiding and symmetry

Braided monoidal categories have a sound and complete graphical calculus, as established by the following theorem.

Theorem 1.18 (Correctness of graphical calculus for braided monoidal categories). *A well-formed equation between morphisms in a braided monoidal category follows from the axioms if and only if it holds in the graphical language up to 3-dimensional isotopy.*

The coherence theorem is very powerful. Try to show that the following equations hold (Exercise 1.4.4):



The second equation is called the *Yang–Baxter equation*, which plays an important role in the mathematical theory of knots.

1.2 Braiding and symmetry

Let's consider this structure for our example categories.

Definition 1.19. The monoidal categories **Hilb**, **Set** and **Rel** can all be equipped with a canonical braiding.

- In **Hilb**, $H \otimes K \xrightarrow{\sigma_{H,K}} K \otimes H$ is the unique linear map extending $a \otimes b \mapsto b \otimes a$ for all $a \in H$ and $b \in K$.
- In **Set**, $A \times B \xrightarrow{\sigma_{A,B}} B \times A$ is defined by $(a, b) \mapsto (b, a)$ for all $a \in A$ and $b \in B$.
- In **Rel**, $A \times B \xrightarrow{\sigma_{A,B}} B \times A$ is defined by $(a, b) \sim (b, a)$ for all $a \in A$ and $b \in B$.

1.2 Braiding and symmetry

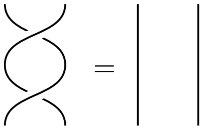
In **Hilb**, **Rel** and **Set**, the braidings satisfy an extra property.

Definition 1.20. A braided monoidal category is *symmetric* when

$$\sigma_{B,A} \circ \sigma_{A,B} = \mathrm{id}_{A \otimes B}$$

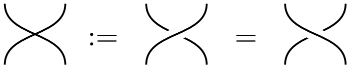
for all objects A and B, in which case we call σ the *symmetry*.

The symmetry condition has the following representation:



The strings can pass through each other, and knots can't be formed.

Lemma 1.21. In a symmetric monoidal category $\sigma_{A,B} = \sigma_{B,A}^{-1}$, with the following graphical representation:



1.3 Coherence

Some monoidal categories have a particularly simple structure.

Definition 1.25. A monoidal category is *strict* if the morphisms $\alpha_{A,B,C}$, λ_A and ρ_A are all identities.

Later we will sketch the proof of the following theorem.

Theorem 1.38. Every monoidal category is monoidally equivalent to a strict monoidal category.

This seems like a very useful thing. *But beware!* This is not enough:

$$(A \otimes B) \otimes C = A \otimes (B \otimes C) \qquad\qquad I \otimes A = A = A \otimes I$$

In particular, it does not ensure that $(f \otimes g) \otimes h = f \otimes (g \otimes h)$.
The identity $(A \otimes B) \otimes C \xrightarrow{\mathrm{id}} A \otimes (B \otimes C)$ might not be natural!

Definition 0.10. A category is *skeletal* when any two isomorphic objects are equal.

Theorem. Not every monoidal category is monoidally equivalent to a strict monoidal skeletal category.

1.3 Coherence

For the case of **FHilb**, everything works nicely.

Definition 0.36. The skeletal category $\mathbf{Mat}_{\mathbb{C}}$ is defined as follows:

- **objects** are natural numbers $0, 1, 2, \ldots$;
- **morphisms** $n \to m$ are matrices of complex numbers with m rows and n columns;
- **composition** is matrix multiplication;
- **identities** $n \xrightarrow{\mathrm{id}_n} n$ are identity matrices.

Definition 1.26. The following structure makes $\mathbf{Mat}_{\mathbb{C}}$ strict monoidal:

- **tensor product** is given on objects by $n \otimes m = nm$, and on morphisms by Kronecker product of matrices (0.32);
- **the monoidal unit** is the natural number 1;
- **associators, left unitors and right unitors** are identity matrices.

1.3 Coherence

Definition 1.27. A *monoidal functor* $F\colon \mathbf{C} \to \mathbf{D}$ between monoidal categories is a functor equipped with natural isomorphisms

$$(F_2)_{A,B}\colon F(A) \otimes F(B) \to F(A \otimes B)$$
$$F_0\colon I \to F(I)$$

making the following diagrams commute:

$$\begin{tikzcd}
\big(F(A) \otimes F(B)\big) \otimes F(C) \arrow[r, "\alpha_{F(A),F(B),F(C)}"] \arrow[d, "(F_2)_{A,B} \otimes \mathrm{id}_{F(C)}"'] & F(A) \otimes \big(F(B) \otimes F(C)\big) \arrow[d, "\mathrm{id}_{F(A)} \otimes (F_2)_{B,C}"] \\
F(A \otimes B) \otimes F(C) \arrow[d, "(F_2)_{A \otimes B,C}"'] & F(A) \otimes F(B \otimes C) \arrow[d, "(F_2)_{A,B \otimes C}"] \\
F\big((A \otimes B) \otimes C\big) \arrow[r, "F(\alpha_{A,B,C})"'] & F\big(A \otimes (B \otimes C)\big)
\end{tikzcd}$$

$$\begin{tikzcd}
F(A) \otimes I \arrow[r, "\rho_{F(A)}"] \arrow[d, "\mathrm{id}_{F(A)} \otimes F_0"'] & F(A) \arrow[d, "F(\rho_A^{-1})"'] \\
F(A) \otimes F(I) \arrow[r, "(F_2)_{A,I}"'] & F(A \otimes I)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes F(A) \arrow[r, "\lambda_{F(A)}"] \arrow[d, "F_0 \otimes \mathrm{id}_{F(A)}"] & F(A) \arrow[d, "F(\lambda_A^{-1})"] \\
F(I) \otimes F(A) \arrow[r, "(F_2)_{I,A}"'] & F(I \otimes A)
\end{tikzcd}$$

1.3 Coherence

Definition 1.33. A *monoidal equivalence* is a monoidal functor that is an equivalence as a functor.

Theorem. There is a monoidal equivalence $R\colon \mathbf{Mat}_{\mathbb{C}} \to \mathbf{FHilb}$.

Proof. We define R like this:

$$\begin{aligned} R(n) &:= \mathbb{C}^n \\ R(n \xrightarrow{f} m) &:= f \text{ as a linear map} \end{aligned}$$

$$\begin{aligned} (R_2)_{m,n} &: |i\rangle \otimes |j\rangle \mapsto |ni + j\rangle \\ R_0 &: 1 \mapsto 1 \end{aligned}$$

This is full, faithful and essentially surjective, and satisfies the monoidal functor conditions. $\square$

1.3 Coherence

We now prove the strictification theorem.

Theorem 1.38. Every monoidal category is monoidally equivalent to a strict monoidal category.

Proof sketch. Let **C** be a monoidal category, and define **D** like this:

- an object is $F\colon \mathbf{C} \to \mathbf{C}$ equipped with a natural isomorphism

$$F(A) \otimes B \xrightarrow{\gamma_{A,B}} F(A \otimes B);$$

- a morphism $(F, \gamma) \to (F', \gamma')$ is $\theta\colon F \Longrightarrow F'$ such that:

$$\begin{tikzcd}
F(A) \otimes B \arrow[r, "\gamma_{A,B}"] \arrow[d, "\theta_A \otimes \mathrm{id}_B"'] & F(A \otimes B) \arrow[d, "\theta_{A \otimes B}"] \\
F'(A) \otimes B \arrow[r, "\gamma'_{A,B}"'] & F'(A \otimes B)
\end{tikzcd}$$

1.3 Coherence

Proof sketch (continued).

- the tensor product is $(F, \gamma) \otimes (F', \gamma') := (F \circ F', \delta)$, where δ is

$$F(F'(A)) \otimes B \xrightarrow{\gamma_{F'(A),B}} F(F'(A) \otimes B) \xrightarrow{F(\gamma'_{A,B})} F(F'(A \otimes B)).$$

We can then calculate these products:

$$\big((F, \gamma) \otimes (F', \gamma')\big) \otimes (F'', \gamma'') = (F, \gamma) \otimes \big((F', \gamma') \otimes (F'', \gamma'')\big)$$

They are equal, and indeed the category is strict monoidal.

Now build a monoidal functor $L\colon \mathbf{C} \to \mathbf{D}$ in the following way:

$$L(A) := (A \otimes -, \alpha_{A,-,-})$$

You can show that L is full and faithful.

Finally, restrict $\mathbf{D}$ to the strict monoidal subcategory containing objects isomorphic to those in the image of L. Then L is a monoidal equivalence of $\mathbf{C}$ with a strict monoidal category. $\square$

1.3 Coherence

The final topic in this chapter is *coherence*: any well-formed equation built from α, α^{-1}, λ, λ^{-1}, ρ, ρ^{-1}, id, $\otimes$ and $\circ$ holds.

An equation is *well-formed* when it does not make use of any 'accidental equalities' of objects. For example, suppose that $(A \otimes A) \otimes A = A \otimes (A \otimes A) = A$. Then

$$\alpha_{A,A,A} = \mathrm{id}_A$$

is not well-formed.

To make this precise, let a *bracketing* be a fixed way to bracket a list of objects of a given length, including empty brackets. For example, we could define the following bracketings v, w:

$$v(A, B, C, D) = ((A \otimes B) \otimes ()) \otimes (C \otimes D)$$
$$w(A, B, C, D) = \big(() \otimes (A \otimes (B \otimes C))\big) \otimes (() \otimes (() \otimes D)))$$

Then we can consider transformations of bracketings $\theta, \theta' : \nu \Rightarrow \mu$.

1.3 Coherence

We now give a proof of the coherence theorem.

Theorem 1.39 (Coherence for monoidal categories). Let v, w be bracketings; then any two transformations $\theta, \theta' : v \Rightarrow w$ built from α, α^{-1}, λ, λ^{-1}, ρ, ρ^{-1}, id, $\otimes$, and $\circ$ are equal.

Proof. We can define a canonical morphism

$$v(L(A), \ldots, L(Z)) \xrightarrow{L_v} L(v(A, \ldots, Z))$$

using the fact that L is a monoidal functor, and similarly for w. Then the following diagram commutes, for both θ and θ':

$$\begin{tikzcd}
v(L(A), \ldots, L(Z)) \arrow[r, "\theta_{(L(A),\ldots,L(Z))}"] & w(L(A), \ldots, L(Z)) \arrow[d, "L_w"] \\
L(v(A, \ldots, Z)) \arrow[u, "{L_v}^{-1}"] \arrow[r, "L(\theta_{(A,\ldots,Z)})"'] & L(w(A, \ldots, Z))
\end{tikzcd}$$

But $\theta_{(L(A),\ldots,L(Z))} = \theta'_{(L(A),\ldots,L(Z))} = \mathrm{id}$! So $L(\theta_{(A,\ldots,Z)}) = L(\theta'_{(A,\ldots,Z)})$, and hence $\theta_{(A,\ldots,Z)} = \theta'_{(A,\ldots,Z)}$, since L is faithful. $\square$

Chapter 2

Linear structure

2.1 Scalars

From the monoidal structure of **Hilb**, we can extract some of the structure of the complex numbers.

- As a set, we can find them as $\mathbf{Hilb}(\mathbb{C}, \mathbb{C})$ — the endomorphisms of the unit object.
- Multiplication of complex numbers is given by composition.
- We can verify commutativity, by checking that $ab = ba$ for all elements of $\mathbf{Hilb}(\mathbb{C}, \mathbb{C})$.

Using this as inspiration, we make the following definition.

Definition 2.1. In a monoidal category, the *scalars* are the morphisms $I \to I$.

We can use this to replicate linear algebra in any monoidal category.

2.1 Scalars

We start with the following proof.

Lemma 2.3. In a monoidal category, the scalars are commutative.

Proof. Consider the following diagram, for any two scalars $I \xrightarrow{a,b} I$:

$$\begin{tikzcd}
I \arrow[rr, "a"] \arrow[dr, "b"] \arrow[dd, "\lambda_I^{-1}"', "\rho_I^{-1}"] & & I \arrow[dr, "b"] \arrow[dd, "\lambda_I^{-1}"', "\rho_I^{-1}"] & \\
& I \arrow[rr, "a"] & & I \\
I \otimes I \arrow[rr, "a \otimes \mathrm{id}_I"'] \arrow[dr, "\mathrm{id}_I \otimes b"'] & & I \otimes I \arrow[dr, "\mathrm{id}_I \otimes b"'] & \\
& I \otimes I \arrow[rr, "a \otimes \mathrm{id}_I"'] \arrow[uu, "\lambda_I", "\rho_I"'] & & I \otimes I \arrow[uu, "\lambda_I", "\rho_I"']
\end{tikzcd}$$

The four side cells use naturality of λ_I and ρ_I, the bottom cell commutes by the interchange law, and the vertical arrows use coherence. Hence we have $ab = ba$. □

2.1 Scalars

We draw a scalar $I \xrightarrow{a} I$ as a circle:

$$\textcircled{a} \tag{2}$$

Commutativity of scalars then has the following graphical representation:

$$\begin{matrix} \textcircled{b} \\ \\ \textcircled{a} \end{matrix} \quad = \quad \begin{matrix} \textcircled{a} \\ \\ \textcircled{b} \end{matrix} \tag{3}$$

The diagrams are isotopic, so it follows from correctness of the graphical calculus that scalars are commutative.

Again, a nontrivial property of monoidal categories follows straightforwardly from the graphical calculus.

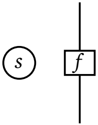
2.1 Scalars

For a linear map $H \xrightarrow{f} J$ and a number $c \in \mathbb{C}$, we can multiply to form $H \xrightarrow{c \cdot f} J$. We can mimic this in any monoidal category.

Definition 2.5. For a scalar $I \xrightarrow{a} I$ and a morphism $A \xrightarrow{f} B$, the *left scalar multiplication* $A \xrightarrow{a \bullet f} B$ is the following composite:

$$\begin{tikzcd} A \arrow[r, "a \bullet f"] \arrow[d, "\lambda_A^{-1}"'] & B \\ I \otimes A \arrow[r, "a \otimes f"] & I \otimes B \arrow[u, "\lambda_B"'] \end{tikzcd}$$

Graphically, it looks like this:



2.1 Scalars

This satisfies many familiar properties.

Lemma 2.6 (Scalar multiplication). In a monoidal category, the following properties hold for scalars $I \xrightarrow{a,b} I$ and morphisms $A \xrightarrow{f} B$, $B \xrightarrow{g} C$:

(a) $\mathrm{id}_I \bullet f = f$;

(b) $a \bullet b = a \circ b$;

(c) $a \bullet (b \bullet f) = (a \bullet b) \bullet f$;

(d) $(b \bullet g) \circ (a \bullet f) = (b \circ a) \bullet (g \circ f)$.

Proof. Easy to see using graphical calculus. □

Example 2.7. Scalar multiplication looks like this for our examples.

- In **Hilb**: if $a \in \mathbb{C}$ is a scalar and $H \xrightarrow{f} K$ a morphism, then $H \xrightarrow{a \bullet f} K$ is the morphism $v \mapsto af(v)$.
- In **Set**, scalar multiplication is trivial: if $A \xrightarrow{f} B$ is a function, then $\mathrm{id}_1 \bullet f = f$ is again the same function.
- In **Rel**: for any relation $A \xrightarrow{R} B$, true $\bullet R = R$, and false $\bullet R = \emptyset$.

2.2 Superposition

Given two linear maps $H \xrightarrow{f,g} J$, we can construct their sum $H \xrightarrow{f+g} J$. This is how we form superpositions of quantum states.

There is also a zero linear map $H \xrightarrow{0_{H,J}} J$ which is the unit for $+$.

We will now think about how to model this categorically.

Definition 0.23 (Terminal object, initial object). An object 1 is *terminal* if for any A there is a unique morphism $A \to 1$. An object 0 is *initial* if for any A there is a unique morphism $0 \to A$.

Definition 2.8 (Zero object, zero morphism). An object 0 is a *zero object* when it is both initial and terminal, a *zero morphism* $A \xrightarrow{0_{A,B}} B$ is the unique morphism $A \to 0 \to B$ factoring through a zero object.

Lemma 2.9. Initial, terminal and zero objects are unique up to unique isomorphism.

2.2 Superposition

Lemma 2.10. Composition with a zero morphism always gives a zero morphism; that is, for any objects A, B and C, and any morphism $A \xrightarrow{f} B$, we have the following:

$$f \circ 0_{C,A} = 0_{C,B} \qquad\qquad 0_{B,C} \circ f = 0_{A,C}$$

Example 2.11. Of our example categories, **Hilb** and **Rel** have zero objects, whereas **Set** does not.

- In **Hilb**, the 0-dimensional vector space is a zero object, and the zero morphisms are the linear maps sending all vectors to the zero vector.
- In **Rel**, the empty set is a zero object, and the zero morphisms are the empty relations.
- In **Set**, the empty set is an initial object, and the one-element set is a terminal object. As they are not isomorphic, **Set** cannot have a zero object.

2.2 Superposition

Definition 2.12. An operation $(f,g) \mapsto f+g$, that is defined for morphisms $A \xrightarrow{f,g} B$ between any objects A and B, is a *superposition rule* if it has the following properties:

- **Commutativity:**
$$f+g=g+f$$
- **Associativity:**
$$(f+g)+h=f+(g+h)$$
- **Units:** for all A, B there is a unit morphism $A \xrightarrow{u_{A,B}} B$ such that:
$$f+u_{A,B}=f$$
- **Addition is compatible with composition:**
$$(g+g')\circ f=(g\circ f)+(g'\circ f)$$
$$g\circ (f+f')=(g\circ f)+(g\circ f')$$
- **Units are compatible with composition:**
$$u_{B,C}\circ u_{A,B}=u_{A,C}$$

2.2 Superposition

In category theory, a superposition rule is sometimes called an *enrichment in commutative monoids*.

Example 2.13. **Hilb** and **Rel** have a superposition rule; **Set** doesn't.

- In **Hilb** the superposition rule is addition of linear maps, given by $(f+g)(v) = f(v) + g(v)$.
- In **Rel**, the superposition rule is given by union of subsets: $R + S = R \cup S$. In the matrix representation of relations (2), this corresponds to entrywise disjunction.
- **Set** cannot be given a superposition rule. If it had one there would be a unit morphism $A \xrightarrow{u_{A,\emptyset}} \emptyset$, but there are no such functions for nonempty sets A.

Lemma 2.14. In a category with a zero object and a superposition rule, $u_{A,B} = 0_{A,B}$ for any objects A and B.

Proof. Since units are compatible with composition, $u_{A,B} = u_{0,B} \circ u_{A,0}$. But by definition of zero morphisms, this equals $0_{A,B}$. □

We can see this is true for our example categories.

2.2 Superposition

Lemma 2.15. If a monoidal category has a zero object and a superposition rule, its scalars form a *commutative semiring with an absorbing zero*:

$$
\begin{aligned}
(a+b)c &= ac + bc \\
a(b+c) &= ab + ac \\
a + b &= b + a \\
a + 0 &= a \\
a0 = 0 &= 0a
\end{aligned}
$$

Example 2.16. In **Hilb** and **Rel** we have the following semirings.

- In **Hilb**, the scalar semiring is the field $\mathbb{C}$ with its usual multiplication and addition.
- In **Rel**, it is the Boolean semiring {true, false}, with multiplication given by logical conjunction (AND) and addition given by logical disjunction (OR).

2.2 Superposition

The $\otimes$ and $+$ don't necessarily interact well. But consider this lemma.

Lemma 2.30. In a monoidal category with a zero object, $0 \otimes 0 \simeq 0$.

Proof. First note that $I \otimes 0$ is a zero object. Consider these maps:

$$\begin{tikzcd}
0 \arrow[r, "\lambda_0^{-1}"] & I \otimes 0 \arrow[r, "0_{I,0} \otimes \mathrm{id}_0"] & 0 \otimes 0 \\
0 \otimes 0 \arrow[r, "0_{0,I} \otimes \mathrm{id}_0"] & I \otimes 0 \arrow[r, "\lambda_0"] & 0
\end{tikzcd}$$

Composing in one direction we obtain a morphism of type $0 \to 0$, necessarily the identity. The other composite is also the identity:

$$\begin{tikzcd}
0 \otimes 0 \arrow[r, "0_{0,I} \otimes \mathrm{id}_0"] \arrow[dd, swap, "\begin{array}{r} 0_{0,0} \otimes \mathrm{id}_0 \\ = \mathrm{id}_0 \otimes \mathrm{id}_0 \\ = \mathrm{id}_{0\otimes 0} \end{array}"] & I \otimes 0 \arrow[dd, swap, "\mathrm{id}_{I\otimes 0}"] \arrow[dr, "\lambda_0"] & \\
& & 0 \arrow[dl, "\lambda_0^{-1}"] \\
0 \otimes 0 & I \otimes 0 \arrow[l, "0_{I,0} \otimes \mathrm{id}_0"] &
\end{tikzcd}$$

This completes the proof. □

2.2 Superposition

Given Hilbert spaces H and J, we can form their direct sum $H \oplus J$. This comes equipped with canonical maps into and out of H and J. It forms an instance of a *biproduct*.

Definition 2.18. In a category with a zero object and a superposition rule, the *biproduct* of A and B is an object $A \oplus B$ equipped with morphisms

$$A \xrightarrow{i_A} A \oplus B \qquad\qquad A \oplus B \xrightarrow{p_A} A$$
$$B \xrightarrow{i_B} A \oplus B \qquad\qquad A \oplus B \xrightarrow{p_B} B$$

satisfying the following equations:

$$\mathrm{id}_A = p_A \circ i_A \qquad\qquad 0_{A,B} = p_B \circ i_A$$
$$\mathrm{id}_B = p_B \circ i_B \qquad\qquad 0_{B,A} = p_A \circ i_B$$
$$\mathrm{id}_{A \oplus B} = i_A \circ p_A + i_B \circ p_B$$

This generalizes to an arbitrary finite number of objects.

2.2 Superposition

Lemma 2.19. If $A \oplus B$ is a biproduct with structure maps

$$A \xrightarrow{i_A} A \oplus B \xleftarrow{i_B} B \qquad\qquad A \xleftarrow{p_A} A \oplus B \xrightarrow{p_B} B$$

then it is also a product p_1, p_2, and a coproduct with i_1, i_2.

Proof. We will verify the universal property for products. Let $X \xrightarrow{f} A$ and $X \xrightarrow{g} B$ be arbitrary morphisms. Make the following definition:

$$\begin{pmatrix} f \\ g \end{pmatrix} := X \xrightarrow{i_A \circ f + i_B \circ g} A \oplus B$$

Then we compute as follows (and similarly for p_B):

$$\begin{aligned} p_A \circ \begin{pmatrix} f \\ g \end{pmatrix} &= p_A \circ (i_A \circ f + i_B \circ g) \\ &= p_A \circ i_A \circ f + p_A \circ i_B \circ g = f + 0 = f \end{aligned}$$

Now suppose $X \xrightarrow{x} A \oplus B$ satisfies $p_A \circ x = f$ and $p_B \circ x = g$:

$$x = (i_A \circ p_A + i_B \circ p_B) \circ x = i_A \circ p_A \circ x + i_B \circ p_B \circ x = i_A \circ f + i_B \circ g$$

So x is unique satisfying these constraints. The coproduct proof is the same, just with all the arrows reversed. □

2.2 Superposition

Since they are a categorical product, biproducts *aren't* a good choice of monoidal product if we want to generalize quantum theory: all joint states would be product states.

However, biproducts are perfect for modelling *classical* information. Later in the course we will see this a lot.

Let's see what biproducts look like in our example categories.

Example 2.20. Both **Hilb** and **Rel** have all finite biproducts; **Set** has no superposition rule so can't have biproducts.

- In **Hilb**, the direct sum of Hilbert spaces provides biproducts. Projections $p_H\colon H \oplus K \to H$ and $p_K\colon H \oplus K \to K$ are given by $(v, w) \mapsto v$ and $(v, w) \mapsto w$. Injections $i_H\colon H \to H \oplus K$ and $i_K\colon K \to H \oplus K$ are given by $v \mapsto (v, 0)$ and $w \mapsto (0, w)$.
- In **Rel**, the disjoint union $A \sqcup B$ of sets provides biproducts. Projections $A \sqcup B \to A$ and $A \sqcup B \to B$ are given by $a \sim a$ and $b \sim b$. Injections $A \to A \sqcup B$ and $B \to A \sqcup B$ are given by $a \sim a$ and $b \sim b$.

2.2 Superposition

The definition of biproducts seemed to rely on a chosen rule $+$. But in fact, biproducts make superpositions unique.

Lemma 2.21 (Unique superposition)**.** If a category has biproducts and a zero object, then it has a unique superposition rule.

Proof. Write $+$ and $\boxplus$ for the two superposition rules, and use a biproduct structure $A \xrightarrow{i_1,i_2} A \oplus A \xrightarrow{p_1,p_2} A$. Then for $A \xrightarrow{f,g} B$:

$$
\begin{aligned}
f + g &= (f \boxplus 0_{A,B}) + (0_{A,B} \boxplus g) \\
&= \big((f \circ p_1 \circ i_1) \boxplus (f \circ p_1 \circ i_2)\big) + \big((g \circ p_2 \circ i_1) \boxplus (g \circ p_2 \circ i_2)\big) \\
&= \big((f \circ p_1) \circ (i_1 \boxplus i_2)\big) + \big((g \circ p_2) \circ (i_1 \boxplus i_2)\big) \\
&= \big((f \circ p_1) + (g \circ p_2)\big) \circ (i_1 \boxplus i_2) \\
&= \big(((f \circ p_1) + (g \circ p_2)) \circ i_1\big) \boxplus \big(((f \circ p_1) + (g \circ p_2)) \circ i_2\big) \\
&= \big((f \circ p_1 \circ i_1) + (g \circ p_2 \circ i_1)\big) \boxplus \big((f \circ p_1 \circ i_2) + (g \circ p_2 \circ i_2)\big) \\
&= (f + 0_{A,B}) \boxplus (0_{A,B} + g) \\
&= f \boxplus g
\end{aligned}
$$

Note we don't actually use the full biproduct structure. □

2.2 Superposition

In a category with biproducts, we can use a matrix notation. For example, given $A \xrightarrow{f} C$, $A \xrightarrow{g} D$, $B \xrightarrow{h} C$ and $B \xrightarrow{j} D$, we can write

$$A \oplus B \xrightarrow{\begin{pmatrix} f & h \\ g & j \end{pmatrix}} C \oplus D$$

as shorthand for the following map:

$$A \oplus B \xrightarrow{(i_C \circ f \circ p_A) + (i_D \circ g \circ p_A) + (i_C \circ h \circ p_B) + (i_D \circ j \circ p_B)} C \oplus D$$

Matrices with any finite number of rows and columns are defined in a similar way.

2.2 Superposition

Lemma 2.26 (Matrix representation)**.** Every morphism $\bigoplus_{m=1}^{M} A_m \xrightarrow{f} \bigoplus_{n=1}^{N} B_n$ has a matrix representation.

Proof. We construct a matrix representation explicitly, for clarity just in the case when the source and target are biproducts of two objects only:

$$\begin{aligned}
f &= \mathrm{id}_{C\oplus D} \circ f \circ \mathrm{id}_{A\oplus B} \\
&= \big((i_C \circ p_C) + (i_D \circ p_D)\big) \circ f \circ \big((i_A \circ p_A) + (i_B \circ p_B)\big) \\
&= i_C \circ (p_C \circ f \circ i_A) \circ p_A + i_C \circ (p_C \circ f \circ i_B) \circ p_B \\
&\qquad + i_D \circ (p_D \circ f \circ i_A) \circ p_A + i_D \circ (p_D \circ f \circ i_B) \circ p_B \\
&= \begin{pmatrix} p_C \circ f \circ i_A & p_C \circ f \circ i_B \\ p_D \circ f \circ i_A & p_D \circ f \circ i_B \end{pmatrix}
\end{aligned}$$

This gives an explicit matrix representation for f. The general case is similar.

2.2 Superposition

Composition of matrices is just like ordinary matrix composition, except with morphism composition instead of multiplication:

$$\begin{pmatrix} s & p \\ q & r \end{pmatrix} \circ \begin{pmatrix} f & g \\ h & j \end{pmatrix} = \begin{pmatrix} (s \circ f) + (p \circ h) & (s \circ g) + (p \circ j) \\ (q \circ f) + (r \circ h) & (q \circ g) + (r \circ j) \end{pmatrix}$$

Identities have a predictable matrix representation:

$$\mathrm{id}_{A \oplus B} = \begin{pmatrix} \mathrm{id}_A & 0_{B,A} \\ 0_{A,B} & \mathrm{id}_B \end{pmatrix}$$

Example 2.29. Consider matrices in our example categories.

- In **Hilb**, the matrix notation gives block matrices between direct sums of Hilbert spaces, and ordinary matrix multiplication.
- In **Rel**, we can think of relations as $\{\text{false}, \text{true}\}$–valued matrices, as explored in Section 0.1.3.

2.3 Dagger structure

In the definition of **FHilb**, something was a bit strange: we didn't use the inner products of the Hilbert space at all.

Inner products allow us to construct adjoint linear maps, with nice properties:

$$(g \circ f)^\dagger = f^\dagger \circ g^\dagger \qquad\qquad \mathrm{id}_H{}^\dagger = \mathrm{id}_H \qquad\qquad (f^\dagger)^\dagger = f$$

So taking the adjoint has the following properties:

- it's contravariant and functorial;
- it's the identity on objects;
- it's involutive.

Also, we can *recover* the inner products from this functor:

$$(\mathbb{C} \xrightarrow{w} H \xrightarrow{v^\dagger} \mathbb{C}) \equiv v^\dagger(w(1)) = \langle 1 | v^\dagger(w(1)) \rangle = \langle v | w \rangle$$

So $\dagger$ and $\langle - | - \rangle$ encode *equivalent* information.

2.3 Dagger structure

This inspires the following abstract definition.

Definition 2.32. A *dagger functor* on a category $\mathbf{C}$ is an involutive contravariant functor $\dagger\colon \mathbf{C} \to \mathbf{C}$ that is the identity on objects. A *dagger category* is a category equipped with a dagger functor.

Let's consider our examples.

- **Hilb** is a dagger category using adjoint linear maps.
- $\mathbf{Mat}_{\mathbb{C}}$ is a dagger category using the conjugate transpose.
- **Rel** can be given a dagger functor by relational converse: for $S \xrightarrow{R} T$, define $T \xrightarrow{R^\dagger} S$ by setting $t\,R^\dagger\,s$ if and only if $s\,R\,t$.
- **Set** cannot be made into a dagger category: $\mathbf{Set}(A,B)$ has size $|B|^{|A|}$, while $\mathbf{Set}(B,A)$ has size $|A|^{|B|}$.
- **Vect** cannot be given a dagger functor: $\mathbf{Vect}(\mathbb{C},V)$ has a smaller cardinality than $\mathbf{Vect}(V,\mathbb{C})$ when V is infinite-dimensional.
- **FVect** *can* be equipped with a dagger functor (e.g. by assigning an inner product to objects and constructing adjoints.) But there is no *canonical* dagger functor.

2.3 Dagger structure

A different use of daggers is in classical probability theory, to construct the *Bayesian converse* of conditional distributions.

Definition. The dagger category **Bayes** is defined as follows:

- **objects** (A, p) are finite sets A equipped with *prior probability distributions*, functions $p : A \to \mathbb{R}^+$ such that $\sum_{a \in A} p(a) = 1$;
- **morphisms** $(A, p) \xrightarrow{f} (B, q)$ are *conditional probability distributions*, functions $f : A \times B \to \mathbb{R}^{\geq 0}$ such that $\forall a \sum_{b \in B} f(a, b) = 1$ and $\forall b \sum_{a \in A} p(a) f(a, b) = q(b)$;
- **composition** is composition of probability distributions as matrices of real numbers;
- the **dagger functor** is the *Bayesian converse*, acting on $f : A \times B \to \mathbb{R}^{\geq 0}$ to give $f^\dagger : B \times A \to \mathbb{R}^{\geq 0}$, defined as $f^\dagger(b, a) := f(a, b) p(a) / q(b)$.

The Bayesian converse is always well-defined since we require our prior probability distributions to be nonzero at every point.

2.3 Dagger structure

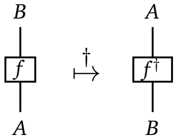
In a dagger category we give special names to some basic properties of morphisms. These generalize terms usually reserved for bounded linear maps between Hilbert spaces.

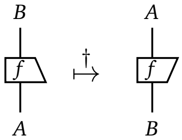
Definition 2.34. A morphism $A \xrightarrow{f} B$ in a dagger category is:

- the *adjoint* of $B \xrightarrow{g} A$ when $g = f^\dagger$;
- *self-adjoint* when $f = f^\dagger$;
- a *projection* when $f = f^\dagger$ and $f \circ f = f$;
- *unitary* when both $f^\dagger \circ f = \mathrm{id}_A$ and $f \circ f^\dagger = \mathrm{id}_B$;
- an *isometry* when $f^\dagger \circ f = \mathrm{id}_A$;
- a *partial isometry* when $f^\dagger \circ f$ is a projector;
- *positive* when $f = g^\dagger \circ g$ for some morphism $H \xrightarrow{g} K$.

2.3 Dagger structure

We depict taking daggers in the graphical calculus by flipping the graphical representation about a horizontal axis.



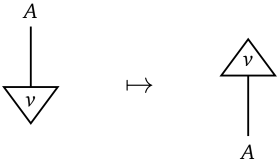


To help differentiate between these morphisms, we draw morphisms in a way that breaks their symmetry.

We also drop the label † from the morphism box.

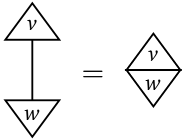
2.3 Dagger structure

We use this notation for states:



A dagger functor gives a correspondence between states and effects.

We can apply this notation to compute the inner product between two states:

$\langle v|w\rangle =$ 

The right-hand side is a rotated form of Dirac's bra-ket notation. So the graphical calculus for dagger categories can be seen as a *generalized* Dirac notation.

2.3 Dagger structure

The adjoint of a matrix is the conjugate transpose. This follows abstractly given the existence of dagger biproducts.

Definition 2.39. In a dagger category with biproducts, a *dagger biproduct* is a biproduct $A \oplus B$ satisfying $i_A^\dagger = p_A$ and $i_B^\dagger = p_B$.

While ordinary biproducts are unique up to isomorphism, dagger biproducts are unique up to *unitary* isomorphism.

Example 2.40. Let's investigate dagger biproducts in our examples.

- In **Rel**, every biproduct is a dagger biproduct.
- In **Hilb**, dagger biproducts are *orthogonal* direct sums. The notion of orthogonality relies on the inner product.

2.3 Dagger structure

Lemma 2.41. In a dagger category with dagger biproducts, the adjoint of a matrix is its conjugate transpose:

$$\begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \cdots & f_{mn} \end{pmatrix}^\dagger = \begin{pmatrix} f_{11}^\dagger & f_{21}^\dagger & \cdots & f_{m1}^\dagger \\ f_{12}^\dagger & f_{22}^\dagger & \cdots & f_{m2}^\dagger \\ \vdots & \vdots & \ddots & \vdots \\ f_{1n}^\dagger & f_{2n}^\dagger & \cdots & f_{mn}^\dagger \end{pmatrix}$$

Lemma 2.42. In a dagger category with dagger biproducts, daggers distribute over addition:

$$(f+g)^\dagger = f^\dagger + g^\dagger$$

Proof. We perform the following calculation:

$$(f+g)^\dagger = \left(\begin{pmatrix} f & g \end{pmatrix} \circ \begin{pmatrix} \mathrm{id}_B \\ \mathrm{id}_B \end{pmatrix} \right)^\dagger = \begin{pmatrix} \mathrm{id}_B \\ \mathrm{id}_B \end{pmatrix}^\dagger \circ \begin{pmatrix} f & g \end{pmatrix}^\dagger$$

$$= \begin{pmatrix} \mathrm{id}_B & \mathrm{id}_B \end{pmatrix} \circ \begin{pmatrix} f^\dagger \\ g^\dagger \end{pmatrix} = f^\dagger + g^\dagger$$

$\square$

2.3 Dagger structure

We can require a dagger functor to be compatible with the monoidal structure.

Definition 2.37. A *monoidal dagger category* is a dagger category that is also monoidal, such that:

- $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ for all morphisms f and g;
- the natural isomorphisms α, λ and ρ are unitary at every stage.

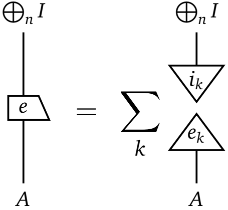
A *braided monoidal dagger category* is a monoidal dagger category equipped with a unitary braiding.

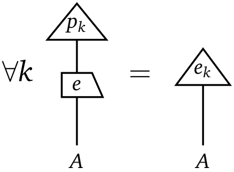
A *symmetric monoidal dagger category* is a braided monoidal dagger category for which the braiding is a symmetry.

2.4 Measurements

Suppose we have a family of n effects $A \xrightarrow{e_k} I$.

We can equivalently encode them as a biproduct effect $A \xrightarrow{e} \bigoplus_n I$:





This is a process that ‘observes’ a system, and converts it into classical information.

To ensure that some effect always takes place, we can require e to have zero kernel.

Chapter 3

Dual objects

3.1 Dual objects

Dual objects have two basic interpretations:

- Topologically, they allow wires to bend
- Quantum mechanically, they model full-rank entangled states

Definition 3.1 (Dual object). An object L is *left-dual* to an object R, and R is *right-dual* to L, written $L \dashv R$, when there is a unit morphism $I \xrightarrow{\eta} R \otimes L$ and a counit morphism $L \otimes R \xrightarrow{\varepsilon} I$ such that:

$$\begin{tikzcd}
L \arrow[r, "\rho_L^{-1}"] \arrow[d, "\mathrm{id}_L"'] & L \otimes I \arrow[r, "\mathrm{id}_L \otimes \eta"] & L \otimes (R \otimes L) \arrow[d, "\alpha_{L,R,L}^{-1}"] \\
L & I \otimes L \arrow[l, "\lambda_L"] & (L \otimes R) \otimes L \arrow[l, "\varepsilon \otimes \mathrm{id}_L"]
\end{tikzcd}$$

$$\begin{tikzcd}
R \arrow[r, "\lambda_R^{-1}"] \arrow[d, "\mathrm{id}_R"'] & I \otimes R \arrow[r, "\eta \otimes \mathrm{id}_R"] & (R \otimes L) \otimes R \arrow[d, "\alpha_{R,L,R}"] \\
R & R \otimes I \arrow[l, "\rho_R"] & R \otimes (L \otimes R) \arrow[l, "\mathrm{id}_R \otimes \varepsilon"]
\end{tikzcd}$$

3.1 Dual objects

We draw an object L as a wire with an upward-pointing arrow, and a right dual R as a wire with a downward-pointing arrow.

The unit $I \xrightarrow{\eta} R \otimes L$ and counit $L \otimes R \xrightarrow{\varepsilon} I$ are drawn as bent wires:

This notation is chosen because of the attractive form it gives to the duality equations:

They are also called the *snake equations*.

3.1 Dual objects

The monoidal category **FHilb** has all duals. Every finite-dimensional Hilbert space H is both right dual and left dual to its dual Hilbert space H^*, in a canonical way.

Of course, this is the origin of the terminology.

The counit $H \otimes H^* \xrightarrow{\varepsilon} \mathbb{C}$ is defined like this:

$$\varepsilon\colon |\phi\rangle \otimes \langle\psi| \mapsto \langle\psi|\phi\rangle$$

The unit $\mathbb{C} \xrightarrow{\eta} H^* \otimes H$ is defined like so, for any orthonormal basis $|i\rangle$:

$$\eta\colon 1 \mapsto \sum_i \langle i| \otimes |i\rangle$$

These definitions sit together rather oddly: η seems basis-dependent, while ε is clearly not.

In fact the same value of η is obtained whatever orthonormal basis is used, as we will see in Lemma 3.5 below.

Infinite-dimensional spaces do not have duals. We will prove this later.

3.1 Dual objects

In **Rel**, every object is its own dual, even sets of infinite cardinality. The unit $1 \xrightarrow{\eta} S \times S$ and counit $S \times S \xrightarrow{\varepsilon} 1$ can be defined like this:

$$\bullet \sim_\eta (s,s) \;\text{ for all } s \in S$$
$$(s,s) \sim_\varepsilon \bullet \;\text{ for all } s \in S$$

In $\mathbf{Mat}_{\mathbb{C}}$, every object n is its own dual, with a canonical choice of η and ε given as follows:

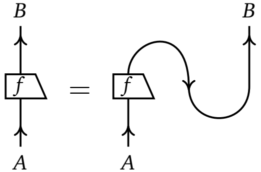
$$\eta : 1 \mapsto \sum_i |i\rangle \otimes |i\rangle \qquad\qquad \varepsilon : |i\rangle \otimes |j\rangle \mapsto \delta_{ij} 1$$

3.1 Dual objects

The category **Set** only has duals for sets of size 1. Let's see why.

Definition 3.3. In a monoidal category with dualities $A \dashv A^*$ and $B \dashv B^*$, given a morphism $A \xrightarrow{f} B$, we define its *name* $I \xrightarrow{\ulcorner f \urcorner} A^* \otimes B$ and *coname* $A \otimes B^* \xrightarrow{\llcorner f \lrcorner} I$ as the following morphisms:

Morphisms can be recovered from their names or conames:



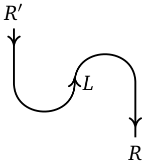
In **Set** 1 is terminal, and so all conames $A \otimes B^* \xrightarrow{\llcorner f \lrcorner} 1$ must be equal. If **Set** had duals this would imply all functions $A \longrightarrow B$ were equal.

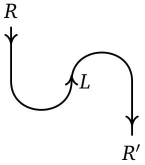
3.1 Dual objects

We first show duals are well-defined up to canonical isomorphism.

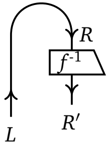
Lemma 3.4. In a monoidal category with $L \dashv R$, then $L \dashv R'$ if and only if $R \simeq R'$. Similarly, if $L \dashv R$, then $L' \dashv R$ if and only if $L \simeq L'$.

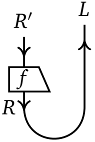
Proof. If $L \dashv R$ and $L \dashv R'$, define maps $R \to R'$ and $R' \to R$ as follows:





The snake equations imply that these are inverse. Conversely, if $L \dashv R$ and $R \xrightarrow{f} R'$ is invertible, we can construct a duality $L \dashv R'$:





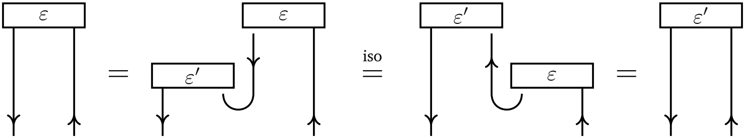
□

3.1 Dual objects

Given a duality, the unit determines the counit, and vice-versa.

Lemma 3.5. In a monoidal category, if $(L, R, \eta, \varepsilon)$ and $(L, R, \eta, \varepsilon')$ both exhibit a duality, then $\varepsilon = \varepsilon'$. Similarly, if $(L, R, \eta, \varepsilon)$ and $(L, R, \eta', \varepsilon)$ both exhibit a duality, then $\eta = \eta'$.

Proof. For the first case, we use the following graphical argument.



The second case is similar.

3.1 Dual objects

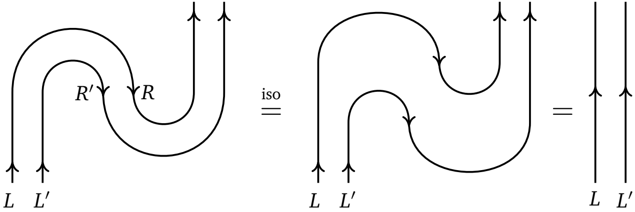
The following lemma shows that dual objects interact well with the monoidal structure.

Lemma 3.6. In a monoidal category, $I \dashv I$.

Proof. Taking $\eta = \lambda_I^{-1} : I \to I \otimes I$ and $\varepsilon = \lambda_I : I \otimes I \to I$ shows that $I \dashv I$. The snake equations follow from the coherence theorem. □

Lemma 3.7. In a monoidal category,
$L \dashv R, L' \dashv R' \;\Rightarrow\; L \otimes L' \dashv R' \otimes R$.

Proof. Suppose that $L \dashv R$ and $L' \dashv R'$. We make the new unit and counit maps from the old ones, and compute as follows:



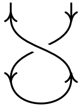
The other snake equation follows similarly. □

3.1 Dual objects

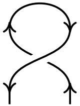
If the monoidal category has a braiding then a duality $L \dashv R$ gives rise to a duality $R \dashv L$, as the next lemma investigates.

Lemma 3.8. In a braided monoidal category, $L \dashv R \Rightarrow R \dashv L$.

Proof. Construct a new duality as follows:

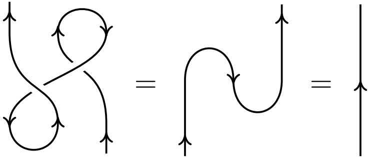


$I \xrightarrow{\eta'} L \otimes R$



$R \otimes L \xrightarrow{\varepsilon'} I$

We can then test the snake equations:



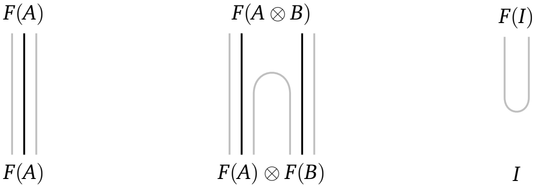
The other snake equation can be proved in a similar way. □

3.1 Dual objects

Next we will prove some nice theorems showing the relationship between duals and monoidal functors.

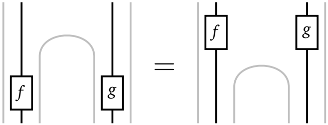
To understand them, we will need to develop a graphical calculus for monoidal functors.

We depict a monoidal functor $F\colon \mathbf{C} \to \mathbf{D}$ and the isomorphisms $(F_2)_{A,B}\colon F(A) \otimes F(B) \to F(A \otimes B)$ and $F_0\colon I \to F(I)$ like this:

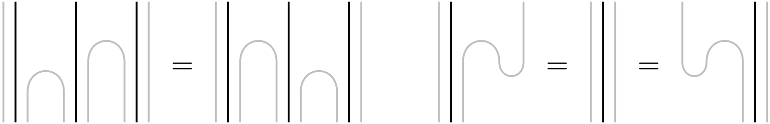


3.1 Dual objects

Naturality means that morphisms can pass through the gaps:



The coherence equations look like this:



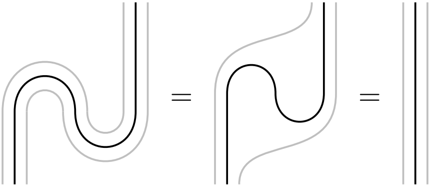
They have a nice topological flavour.

3.1 Dual objects

Let's prove our first theorem using these techniques.

Theorem 3.14. Monoidal functors preserve duals.

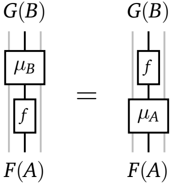
Proof. If we apply our monoidal functor to the unit and counit, we can show that the duality equations are still satisfied:



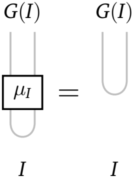
The other duality equation can be proved in a similar way. □

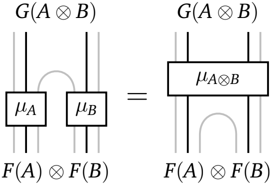
3.1 Dual objects

Given two functors $F, G : \mathbf{C} \to \mathbf{D}$ and a natural transformation $\mu\colon F \Longrightarrow G$, we can denote it like this:



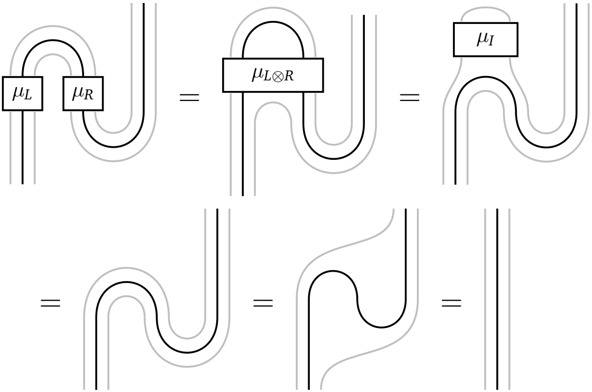
If **C**, **D**, F, G and μ are monoidal, then we have following extra properties:





Theorem 3.15. Let $\mu\colon F \Longrightarrow G$ be a monoidal natural transformation. If $A \in \mathrm{Ob}(\mathbf{C})$ has a left or a right dual, $F(A) \xrightarrow{\mu_A} G(A)$ is invertible.

Proof. Choose $A = L$ with $L \dashv R$ in $\mathbf{C}$. Then we perform the following computation:

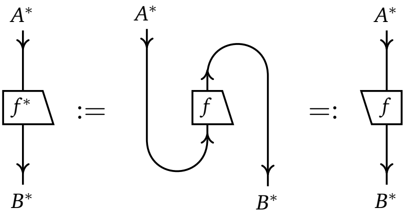


The rest of the proof uses similar techniques. $\square$

3.1 Dual objects

Choosing duals for objects extends functorially to morphisms.

Definition 3.9. For a morphism $A \xrightarrow{f} B$ and chosen dualities $A \dashv A^*$, $B \dashv B^*$, the *right dual* $B^* \xrightarrow{f^*} A^*$ is defined in the following way:



We represent this graphically by rotating the box representing f, as shown in the third image above.

3.1 Dual objects

The dual can 'slide' along the unit and counit.

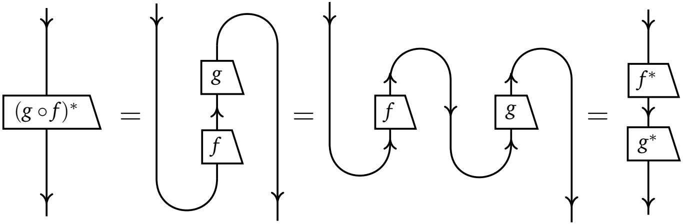
Lemma 3.12. In a monoidal category with chosen dualities $A \dashv A^*$ and $B \dashv B^*$, the following equations hold for all morphisms $A \xrightarrow{f} B$:

Proof. Let's write it out on the board. $\square$

3.1 Dual objects

Lemma 3.11. If a monoidal category has assigned right duals, the right-duals construction $(-)^*$ defines a functor.

Proof. Let $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$. Then we perform the following calculation:



Similarly, $(\mathrm{id}_A)^* = \mathrm{id}_{A^*}$ follows from the snake equations. □

3.1 Dual objects

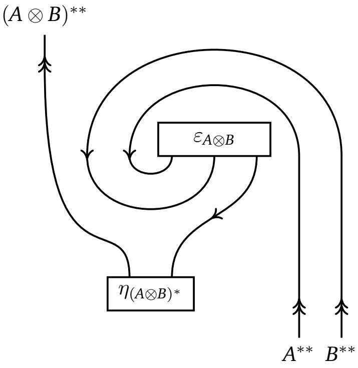
Example 3.13. Let's see how the right duals functor acts for our example categories, with chosen right duals as given by Example 3.2.

- In **FVect** and **FHilb**, the right dual of a morphism $V \xrightarrow{f} W$ is $W^* \xrightarrow{f^*} V^*$, acting as $f^*(e) := e \circ f$, where $W \xrightarrow{e} \mathbb{C}$ is an arbitrary element of W^*.
- In $\mathbf{Mat}_{\mathbb{C}}$, the dual of a matrix is its transpose.
- In **Rel**, the dual of a relation is its converse. So the right duals functor and the dagger functor have the same action: $R^* = R^\dagger$ for all relations R.

3.1 Dual objects

Lemma 3.16. For a monoidal category with chosen right duals for objects, the double duals functor $(-)^{**} : \mathbf{C} \to \mathbf{C}$ is monoidal.

Proof. The isomorphism $A^{**} \otimes B^{**} \simeq (A \otimes B)^{**}$ looks like this:



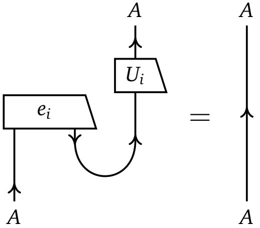
Showing this satisfies the monoidal functor axioms is a monster! □

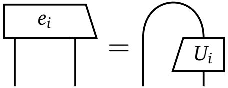
3.1 Dual objects

Dual objects give a nice way to model quantum teleportation.

Definition. In a monoidal category with biproducts and right duals, a *teleportation procedure* is a finite family of effects $e_i : A \otimes A^* \to I$ and unitaries $U_i : A \to A$ such that:

- the biproduct effect $\sum_{k=1}^{N} i_k \circ e_k : A \otimes A^* \to I^{\oplus N}$ has zero kernel;
- the following equation holds for each i:

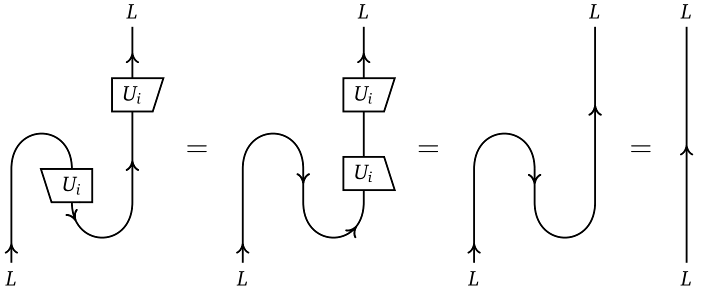


This can be solved to give 

.

3.1 Dual objects

We can use the graphical calculus to simplify the history:



So if the original history occurs, the result is for the state of the original system to be transmitted faithfully.

If the biproduct effect has zero kernel, then it will always succeed: there is no prior history which yields the null process.

3.1 Dual objects

Let's examine this in **Hilb**. Choose $L = R = \mathbb{C}^2$ and $\eta^\dagger = \varepsilon = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$, and the following unitaries U_i:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

This gives rise to the following family of effects:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}$$

This is a complete set of effects, since it forms a basis for the vector space $\mathbf{Hilb}(\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C})$. So it is guaranteed to be successful.

This is traditional qubit teleportation.

3.1 Dual objects

We can also implement teleportation in **Rel**. Choose $L = R = \{0, 1\}$ and $\eta^\dagger = \varepsilon = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$, and the following unitaries:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad\qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This gives rise to the following family of effects:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \qquad\qquad \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}$$

These form a complete set of effects.

This is classical encrypted communication with a one-time pad.

3.1 Dual objects

We now investigate interaction between duals and linear structure.

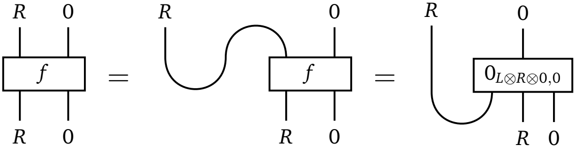
Lemma 3.19. In a monoidal category with a zero object 0:

(a) $0 \dashv 0$;

(b) if $L \dashv R$, then $L \otimes 0 \simeq R \otimes 0 \simeq 0 \simeq 0 \otimes L \simeq 0 \otimes R$.

Proof. For (a), because $0 \otimes 0 \simeq 0$, there are unique morphisms $I \xrightarrow{\eta} 0 \otimes 0$ and $0 \otimes 0 \xrightarrow{\varepsilon} I$. It also follows that $0 \otimes (0 \otimes 0) \simeq 0$, so that both sides of the snake equation must equal $0 \to 0$.

For (b), let $R \otimes 0 \xrightarrow{f} R \otimes 0$ be an arbitrary morphism. Then:



So there is only one morphism $R \otimes 0 \to R \otimes 0$, hence $R \otimes 0 \simeq 0$. The other claims follow similarly. □

3.1 Dual objects

This lets us prove the following lemma.

Lemma 3.20. In a monoidal category with $A \xrightarrow{f} B$ a morphism, if one of A or B has either a left or a right dual, then:

$$f \otimes 0_{C,D} = 0_{A \otimes C, B \otimes D}$$
$$0_{C,D} \otimes f = 0_{C \otimes A, D \otimes B}$$

Proof. Suppose A has a left or a right dual; then $A \otimes 0 \simeq 0$, and so $f \otimes 0_{C,D}$ is a zero morphism. A similar argument holds for B. □

The next result is harder to prove.

Theorem 3.22. In a monoidal category with biproducts and a zero object, let $A \xrightarrow{f} B$ and $C \xrightarrow{g,h} D$ be morphisms. If A has a left or a right dual, then:

$$(f \otimes g) + (f \otimes h) = f \otimes (g + h)$$
$$(g \otimes f) + (h \otimes f) = (g + h) \otimes f$$

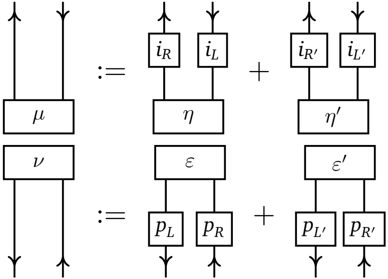
Proof. See the notes! □

3.1 Dual objects

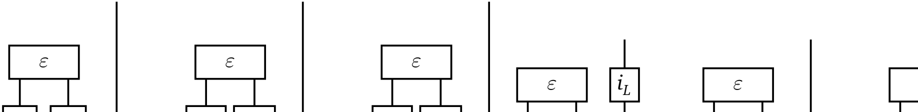
Finally, we show that taking biproducts preserves dual objects.

Lemma 3.23. In a monoidal category with duals and biproducts, $L \dashv R$ and $L' \dashv R'$ imply $L \oplus L' \dashv R \oplus R'$.

Proof. Define the following candidates for the duality $L \oplus L' \dashv R \oplus R'$:



The first snake equation can then be established like this:

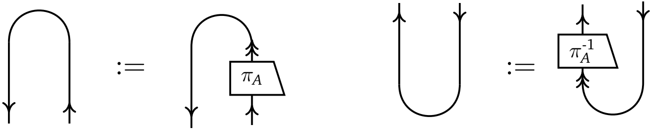


3.2 Pivotality

Definition 3.24. A monoidal category with right duals is *pivotal* when it is equipped with a monoidal natural transformation $A \xrightarrow{p_A} A^{**}$.

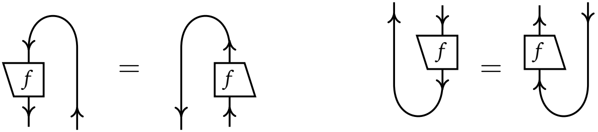
By Theorem 3.15, this will necessarily be invertible.

In a pivotal category, we extend the graphical calculus:



We can use this to rotate boxes arbitrarily.

Lemma. In a pivotal category, the following equations hold for all morphisms $A \xrightarrow{f} B$:



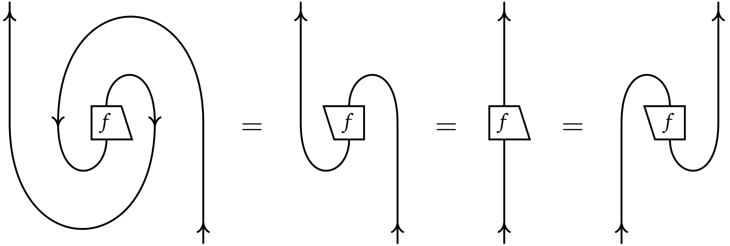
Proof. Let's write it out on the board. □

3.2 Pivotality

We can formalize this as follows.

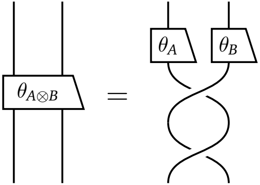
Theorem 3.28. A well-formed equation between morphisms in a pivotal category follows from the axioms if and only if it holds in the graphical language up to planar oriented isotopy.

The new feature is the word *oriented*. The wires of our diagram have arrows, and an isotopy must preserve them:



3.2 Pivotality

Definition 3.29. A braided monoidal category is *balanced* when it is equipped with a natural isomorphism $\theta_A \colon A \to A$ called a *twist*, satisfying the following equations:



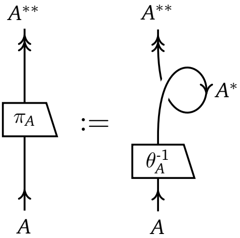
The second equation here says $\theta_I = \mathrm{id}_I$.

These equations look strange—we will see later what they mean!

3.2 Pivotality

Theorem 3.33. For a braided monoidal category with duals, a pivotal structure uniquely induces a twist structure, and vice versa.

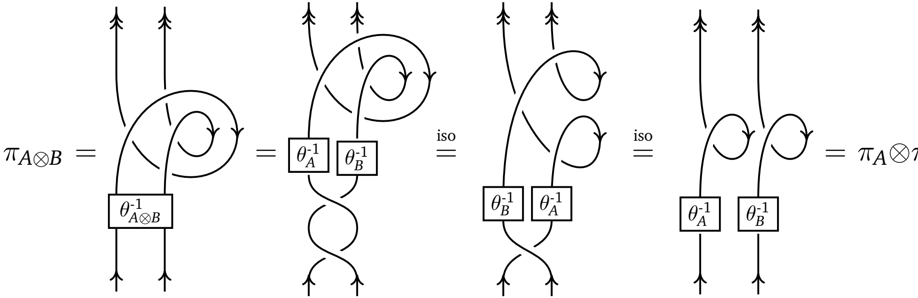
Proof. Suppose we have a twist structure $\theta_A : A \to A$. Then define a pivotal structure as follows:



We must verify that it is a monoidal natural transformation, and that it is natural.

3.2 Pivotality

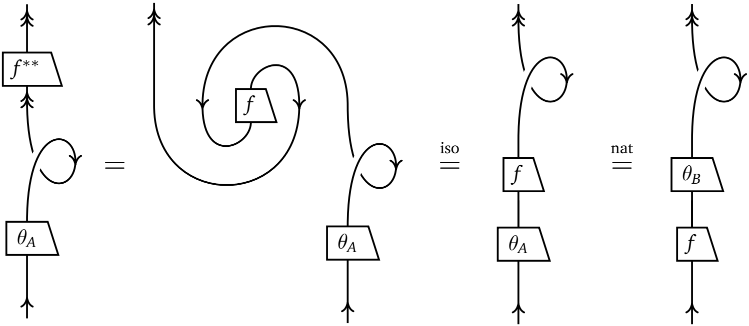
For the monoidal property, perform the following calculation:



For simplicity we have ignored the isomorphism $(A \otimes B)^{**} \simeq A^{**} \otimes B^{**}$.

3.2 Pivotality

To check naturality, we perform the following calculation:



Conversely, we can use a pivotal structure to define a twist. □

3.2 Pivotality

A symmetric monoidal category with duals has a canonical twist.

Definition 3.34. A *compact category* is a pivotal symmetric monoidal category with duals where the canonical twist is the identity $\theta_A = \mathrm{id}_A$.

Our example categories **FHilb**, **FVect** and **Rel** are all compact categories.

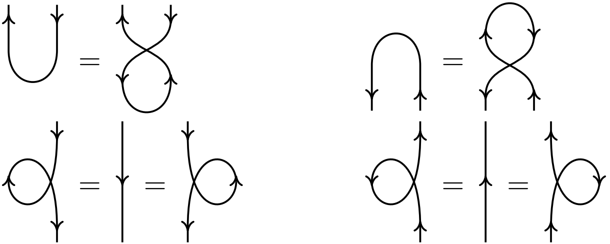
Note that *in general*, other balancings may exist: that is, it is possible for a symmetric monoidal category with duals and a twist *not* to be a compact category.

An example is **SuperHilb**, where $\theta_F = -\mathrm{id}_F$.

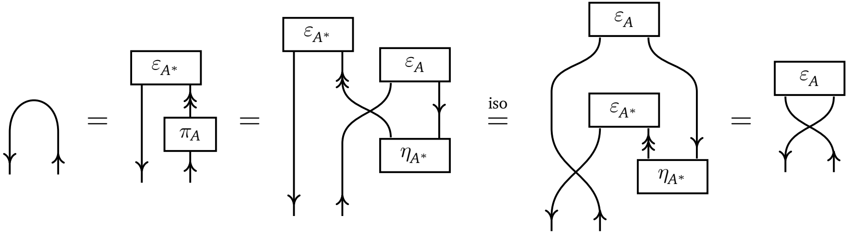
In general the twist is nontrivial extra data: for **Fib**, $\theta_\tau = e^{4\pi i/5} \cdot \mathrm{id}_\tau$.

3.2 Pivotality

Lemma 3.37. In a compact category, the following equations hold:



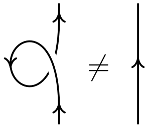
Proof. Let's prove the second equation in the top row:



The others can be proved in a similar way. □

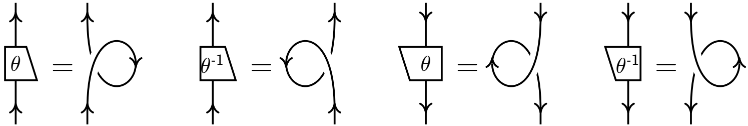
3.2 Pivotality

In a braided pivotal category, we must be careful with loops:



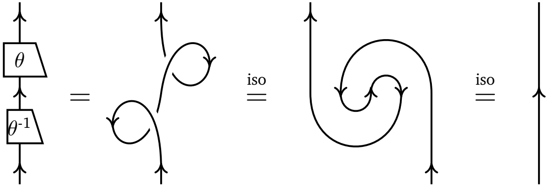
In fact, a loop on a single strand is directly related to the twist.

Lemma 3.38. In a braided pivotal category, the following hold:



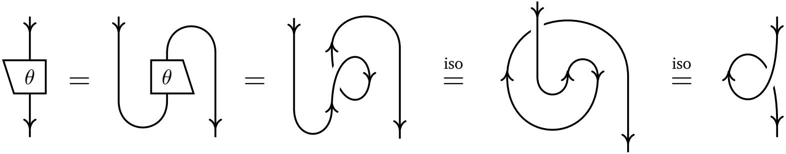
3.2 Pivotality

Proof. Let's verify the expression for θ^{-1}:



The equation $\theta \circ \theta^{-1} = \mathrm{id}$ can be checked in a similar way. Since inverses in a category are unique, this proves θ^{-1} is correct.

We demonstrate the graphical form of θ^{*} as follows:



The rest of the theorem can be proved similarly. ☐

3.2 Pivotality

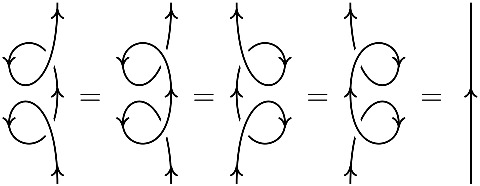
Thinking about ribbons inspires the following definition.

Definition 3.39. A *ribbon* or *tortile* category is a balanced monoidal category with duals, such that $(\theta_A)^* = \theta_{A^*}$.

This is equivalent to either of these graphical equations:

Lemma 3.41. A compact category is a ribbon category.

Lemma ??. In a ribbon category, the following equations hold:



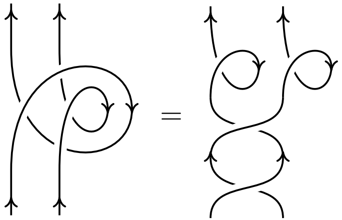
3.2 Pivotality

These are the equations we would expect to be satisfied by *ribbons*.

Theorem 3.28. A well-formed equation between morphisms in a ribbon category follows from the axioms if and only if it holds in the graphical language up to framed isotopy in three dimensions.

'Framed isotopy' is the name for the version of isotopy where the strands are thought of as ribbons, rather than just wires.

To get a feeling for framed isotopy, find some ribbons, or make some by cutting long, thin strips from a piece of paper. Verify (112) and (3.31), and also (3.24) specialized to ribbon categories:

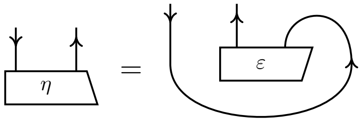


3.2 Pivotality

Lemma 3.45. In a monoidal dagger category, $L \dashv R \;\Leftrightarrow\; R \dashv L$.

Proof. Follows directly from the axiom $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ of a monoidal dagger category.

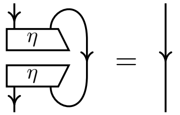
Definition 3.46. In a dagger category with a pivotal structure, a *dagger dual* is a duality $A \dashv A^*$ witnessed by morphisms $I \xrightarrow{\eta} A^* \otimes A$ and $A \otimes A^* \xrightarrow{\varepsilon} I$, satisfying the following condition:



3.2 Pivotality

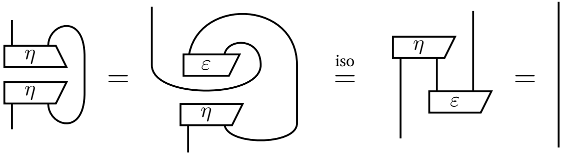
We can describe maximally entangled states like this.

Definition 3.47. In a dagger category with a pivotal structure, a *maximally entangled state* is a bipartite state with this property:



Lemma 3.48. In a dagger category with a pivotal structure, a state is maximally entangled if and only if it is part of a dagger duality.

Proof. We give the following graphical argument:

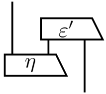


The rest of the proof is similar. □

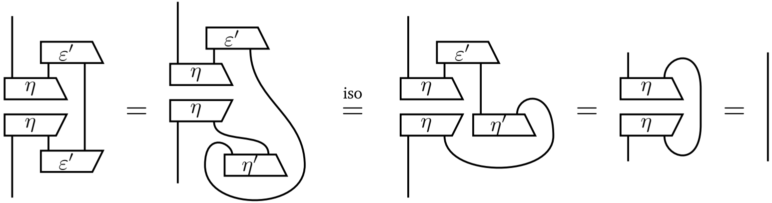
3.2 Pivotality

Lemma 3.49. In a dagger category with a pivotal structure, dagger duals are unique up to unique unitary isomorphism.

Proof. Given dagger duals $(L \vdash R, \eta, \varepsilon)$ and $(L \vdash R', \eta', \varepsilon')$, we construct an isomorphism $R \simeq R'$ as for Lemma 3.4 as follows:



To establish the first part of the unitarity condition, we perform the following calculation:

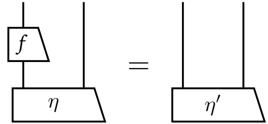


The rest is similar. $\square$

3.2 Pivotality

We can use this to prove an important result about maximally-entangled states.

Theorem 3.50. In a dagger category with a pivotal structure, for any two maximally entangled states $I \xrightarrow{\eta,\eta'} A \otimes B$ there is a unique unitary $A \xrightarrow{f} A$ satisfying the following equation:



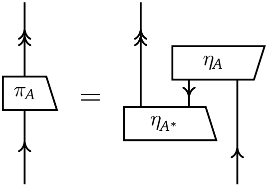
The proof follows from what we have just seen.

So maximally-entangled states are unique up to a unique unitary.

3.2 Pivotality

Definition 3.51. A *dagger pivotal category* is a dagger monoidal category with a pivotal structure, such that the chosen duals are all dagger duals.

Lemma 3.52. In a pivotal dagger category, the pivotal structure is this:



Proof. See notes. □

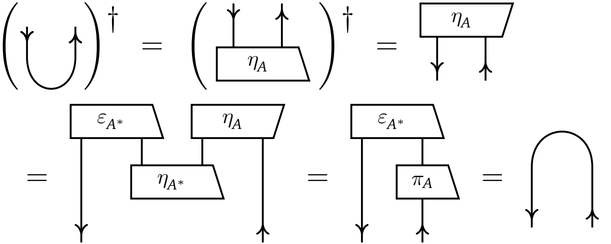
Theorem. In a dagger pivotal category, π_A is unitary.

3.2 Pivotality

Dagger pivotal categories have a good graphical calculus.

Lemma 3.54. In a dagger pivotal category, the following equations hold:

Proof. We prove the first of these in the following way:



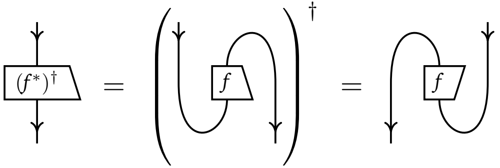
The second then follows by uniqueness of counits. □

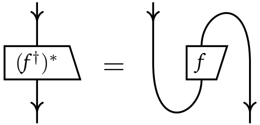
3.2 Pivotality

Lemma 3.55. In a dagger pivotal category, every morphism satisfies the following equation:

$$(f^*)^\dagger = (f^\dagger)^*$$

Proof. We compute both sides as follows:





These are isotopic, and hence equal by correctness of the graphical calculus for pivotal categories. □

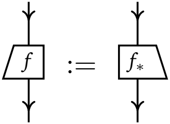
3.2 Pivotality

Definition 3.56. On a dagger pivotal category, *conjugation* $(-)_*$ is defined as the composite of the dagger functor and the right-duals functor:

$$(-)_* := (-)^{*\dagger} = (-)^{\dagger *}$$

Since taking daggers is the identity on objects we have $A_* := A^*$.

We denote conjugation by flipping the morphism about a vertical axis:



Since $(-)^*$ and $\dagger$ are contravariant, $(-)_*$ is covariant.

3.2 Pivotality

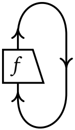
Definition 3.57. A *dagger compact category* is a symmetric dagger pivotal category with unitary symmetry, and $\theta = \mathrm{id}$.

Example 3.58. Our example categories **FHilb**, $\mathbf{Mat}_{\mathbb{C}}$ and **Rel** are all dagger compact categories.

- On **FHilb**, the conjugation functor gives the conjugate of a linear map.
- On $\mathbf{Mat}_{\mathbb{C}}$, the conjugation functor gives the conjugate of a matrix, with each matrix entry replaced by its conjugate as a complex number.
- On **Rel**, the conjugation functor is the identity.

3.2 Pivotality

Definition 3.59. In a pivotal category, the *trace* of a morphism $A \xrightarrow{f} A$, denoted $\mathrm{Tr}_A(f)$, is the following scalar:



A trace can also be defined for a braided monoidal category with duals, but we focus on the pivotal notion here.

Definition 3.60. In a pivotal category, the dimension of an object A is the scalar $\dim(A) := \mathrm{Tr}_A(\mathrm{id}_A)$.

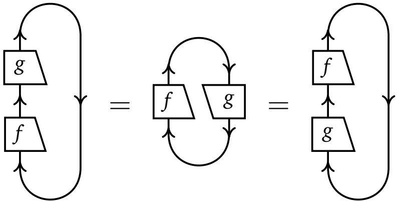
The trace in **FHilb** is the ordinary matrix trace.

3.2 Pivotality

We can prove the cyclic property abstractly.

Lemma 3.61. In a pivotal category, $\mathrm{Tr}_A(g \circ f) = \mathrm{Tr}_B(f \circ g)$.

Proof. We can show this graphically in the following way:



The morphism g slides around the circle, and ends up underneath the morphism f. □

3.2 Pivotality

Many more properties also follow.

Lemma 3.63. In a pivotal category, the trace has the following properties:

(a) $\mathrm{Tr}_A(f+g) = \mathrm{Tr}_A(f) + \mathrm{Tr}_A(g)$;

(b) $\mathrm{Tr}_{A\oplus B}\begin{pmatrix} f & g \\ h & j \end{pmatrix} = \mathrm{Tr}_A(f) + \mathrm{Tr}_B(j)$;

(c) $\mathrm{Tr}_I(s) = s$;

(d) $\mathrm{Tr}_A(0_{A,A}) = 0_{I,I}$;

(e) $\mathrm{Tr}_{A\otimes B}(f \otimes g) = \mathrm{Tr}_A(f) \circ \mathrm{Tr}_B(g)$ in a braided pivotal category;

(f) $\big(\mathrm{Tr}_A(f)\big)^\dagger = \mathrm{Tr}_A(f^\dagger)$ in a dagger pivotal category.

Proof. See notes. □

3.2 Pivotality

This immediately yields some properties of dimensions of objects.

Lemma 3.64. In a braided pivotal category, the following properties hold:

(a) $\dim(A \oplus B) = \dim(A) + \dim(B)$ if there are biproducts;

(b) $\dim(I) = \mathrm{id}_I$;

(c) $\dim(0) = 0_{I,I}$ if there is a zero object;

(d) $A \simeq B \Rightarrow \dim(A) = \dim(B)$;

(e) $\dim(A \otimes B) = \dim(A) \circ \dim(B)$ in a braided pivotal category.

Proof. See notes. □

3.2 Pivotality

Using these results, we can give a simple argument that infinite-dimensional Hilbert spaces cannot have duals.

Lemma 3.65. Infinite-dimensional Hilbert spaces do not have duals.

Proof. Suppose H is an infinite-dimensional Hilbert space. Then there is an isomorphism $H \oplus \mathbb{C} \simeq H$.

If H had a dual, then since $\dim(A \oplus B) = \dim(A) + \dim(B)$ and $A \simeq B \Rightarrow \dim(A) = \dim(B)$, we conclude $\dim(H) + 1 = \dim(H)$.

But this is a contradiction, since there is no complex number with that property. □

This argument would not apply in **Rel**, since we have $\mathrm{id}_1 + \mathrm{id}_1 = \mathrm{id}_1$ in that category. And indeed, every set has a dual in **Rel**, even those of infinite cardinality.

Chapter 4

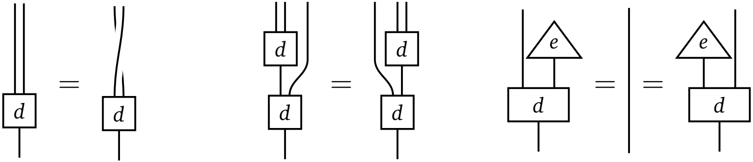
Monoids and comonoids

4.1 Monoids and comonoids

Consider how to formalize a 'copying' operation on an object A.

Type should be $A \xrightarrow{d} A \otimes A$. What does it mean for d to copy?

- Shouldn't matter if we switch both output copies.
- If copying twice, shouldn't matter if take first or second copy.
- Output should equal input: uses *deletion* $A \xrightarrow{e} I$.



cocommutativity coassociativity counitality

Definition 4.1. In a monoidal category, a *comonoid* is a triple $(A, d : A \to A \otimes A, e : A \to I)$ satisfying coassociativity and counitality. It is *cocommutative* when it satisfies the extra axiom.

4.1 Monoids and comonoids

Example 4.2. Here are some comonoids in our example categories.

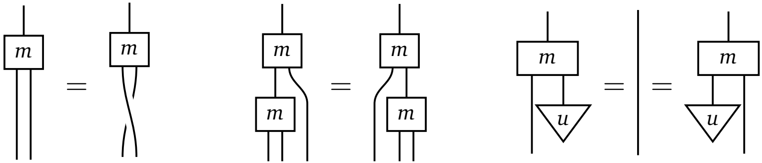
- In **Set**, the tensor product is a Cartesian product.
  Every object carries a unique comonoid with comultiplication $a \mapsto (a, a)$ and counit $a \mapsto \bullet$, which is cocommutative.

- In **Rel**, any group G forms a comonoid with comultiplication $g \sim (h, h^{-1}g)$ and counit $1 \sim \bullet$.
  Counitality: LHS is $g \sim h$ where $h^{-1}g = 1$, RHS is $g \sim 1^{-1}g$.
  The comonoid is cocommutative iff the group is abelian.
  Cocommutativity: LHS is $g \sim (h^{-1}g, h)$, RHS is $g \sim (k, k^{-1}g)$.

- In **FHilb**, a basis choice $\{e_i\}$ for a Hilbert space gives a cocommutative comonoid, with comultiplication $e_i \mapsto e_i \otimes e_i$ and counit $e_i \mapsto 1$.

4.1 Monoids and comonoids

We can dualize these concepts:



commutativity *associativity* *unitality*

Definition 4.3. In a monoidal category, a *monoid* is a triple (A, m, u) satisfying associativity and unitality. It is commutative when it satisfies the corresponding extra axiom.

Example 4.4. There are many examples of monoids:

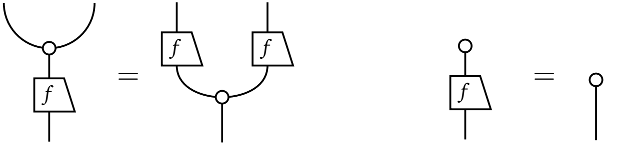
- The tensor unit I, with multiplication $\rho_I = \lambda_I$ and unit id_I.
- A monoid in **Set** is just an ordinary monoid; e.g. any group.
- A monoid in **Vect** is an *algebra*: a set where we can add vectors and multiply with scalars, and also multiply vectors bilinearly. E.g. $\mathbb{C}^n$ under pointwise multiplication and unit $(1, 1, \ldots, 1)$. E.g. vector space of n-by-n matrices with matrix multiplication.

4.1 Monoids and comonoids

Will abbreviate comultiplication to , counit to ,
and multiplication to , unit to . Use colour to differentiate.

Choice of bases $\{d_i\}$ and $\{e_j\}$ for H and K in **FHilb** makes them into comonoids. The functions $f\colon \{d_i\} \to \{e_j\}$ play a special role: they respect the comultiplication and counit.

Definition 4.5. A *comonoid homomorphism* from a comonoid $(A, ,)$ to a comonoid $(B, ,)$ is a morphism $A \xrightarrow{f} B$ such that:



Dual notion: *monoid homomorphism*.

Given a monoidal category, we can build new category with objects (co)monoids, and morphisms (co)monoid homomorphisms.

4.1 Monoids and comonoids

Example 4.6. Consider again our examples of comonoids.

- In **Set**, any function $A \xrightarrow{f} B$ is a comonoid homomorphism: $(f \times f)(a, a) = \big(f(a), f(a)\big)$, and $f(a) = \bullet$.
- In **Rel**, any surjective homomorphism $G \xrightarrow{f} H$ of groups is a comonoid homomorphism. Preservation of comultiplication: LHS is $g \sim (h, h^{-1}f(g))$, RHS is $g \sim (f(g'), f(g')^{-1}f(g))$.
- In **FHilb**, any function $\{a_i\} \xrightarrow{f} \{b_j\}$ between bases extends linearly to a comonoid homomorphism: $d'(f(a_i)) = f(a_i) \otimes f(a_i)$ and $e'(f(a_i)) = 1 = e(a_i)$.

4.1 Monoids and comonoids

Can combine two (co)monoids to single one on tensor product.

Lemma 4.8. In a braided monoidal category, given a pair of comonoids, we can produce a new comonoid:

When braiding is symmetry, this gives a categorical product in the category of comonoids.

Example 4.9. Products of our example comonoids:

- In **Set**, the product comonoid on sets A and B is the unique comonoid on $A \times B$.
- In **Rel**, the product comonoid of groups G and H is comonoid of $G \times H$ with multiplication $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$.
- In **FHilb**, the product of comonoids on H and K that copy bases $\{d_i\}$ and $\{e_j\}$ is the comonoid copying basis $\{d_i \otimes e_j\}$ of $H \otimes K$.

4.1 Monoids and comonoids

In a monoidal dagger category there is duality between monoids and comonoids.

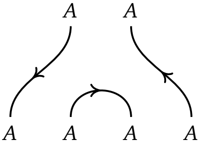
Lemma 4.10. In a monoidal dagger category, (A, d, e) is a comonoid if and only if $(A, d^\dagger, e^\dagger)$ is a monoid.

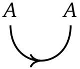
This relates our previous examples in **Rel**:

- Dagger in **Rel** constructs converse relation. Comultiplication $g \sim (h, h^{-1}g)$ for group G turns into multiplication $(g, h) \sim gh$.

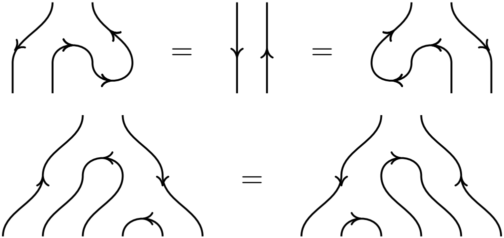
4.1 Monoids and comonoids

Lemma 4.11. If $A \dashv A^*$ are dual objects in a monoidal category, then $A^* \otimes A$ is a monoid as follows:





Proof.



□

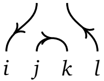
4.1 Monoids and comonoids

Example 4.12. The pair of pants algebra on $\mathbb{C}^n$ in **FHilb** is the algebra $\mathbb{M}_n$ of n-by-n matrices under matrix multiplication.

Proof. Fix basis $\{|i\rangle\}$ for $A = \mathbb{C}^n$, so $A^* \otimes A$ has basis $\{\langle j| \otimes |i\rangle\}$.

Define map $A^* \otimes A \to \mathbb{M}_n$ by mapping $\langle j| \otimes |i\rangle$ to the matrix e_{ij}, with a single entry 1 on row i and column j and zeroes elsewhere.

This bijection respects multiplication:

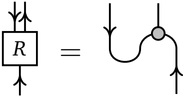


$$= \begin{bmatrix} \langle i| \otimes |l\rangle & \text{if } j = k \\ 0 & \text{if } j \neq k \end{bmatrix} \longmapsto \begin{bmatrix} e_{il} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{bmatrix} = e_{ij}e_{kl}$$

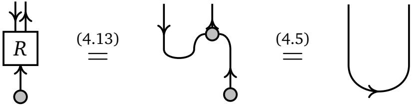
This completes the proof. $\square$

4.1 Monoids and comonoids

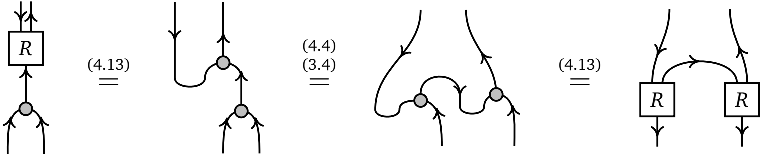
Proposition 4.13. Any monoid $(A, \ , \ )$ in a monoidal category with $A \dashv A^*$ has retractable monoid homomorphism to $(A^* \otimes A, \ , \ )$.



Proof. R preserves units:



R preserves multiplication:



Finally, R has a retraction given by . □

4.2 Uniform deleting and copying

Counit $A \xrightarrow{e} I$ tells us we can 'delete' A if we want to.
What does it mean to have deletion *systematically* on every object?

Definition 4.14. A monoidal category has *uniform deleting* if there is a natural transformation $A \xrightarrow{e_A} I$ with $e_I = \mathrm{id}_I$, such that:

$$\begin{tikzcd}
 & A \otimes B \arrow[dl, "e_A \otimes e_B"'] \arrow[dr, "e_{A\otimes B}"] & \\
I \otimes I \arrow[rr, "\lambda_I"'] & & I
\end{tikzcd}$$

Proposition 4.15. A monoidal category has uniform deleting just when I is a terminal object.

Proof. Uniform deleting gives a morphism $A \xrightarrow{e_A} I$ for each object A. Naturality and $e_I = \mathrm{id}_I$ then show any morphism $A \xrightarrow{f} I$ equals e_A. Conversely, if I is terminal, choose $e_A : A \to I$ uniquely. □

4.2 Uniform deleting and copying

Uniform deleting makes compact categories collapse.

Definition 4.19. A *preorder* is a category that has at most one morphism $A \to B$ for any pair of objects A, B.

Preorders are degenerate, with only one process of each type.

Theorem 4.20. If a monoidal category with duals has uniform deleting, then it is a preorder.

Proof. Let $A \xrightarrow{f,g} B$ be morphisms. Naturality of e gives:

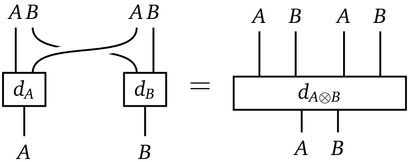
$$\begin{tikzcd} A \otimes B^* \arrow[r, "e_{A\otimes B^*}"] \arrow[d, "\llcorner f \lrcorner"'] & I \arrow[d, "\mathrm{id}_I"] \\ I \arrow[r, "e_I = \mathrm{id}_I"] & I \end{tikzcd}$$

So $\llcorner f \lrcorner = e_{A\otimes B^*}$, and similarly $\llcorner g \lrcorner = e_{A\otimes B^*}$. Hence $f = g$. $\square$

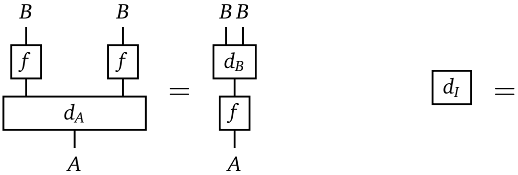
4.2 Uniform deleting and copying

Question: what does it mean to *copy* objects *systematically*?
Answer: copying must respect composition, tensor products.

Definition 4.21. A braided monoidal category has *uniform copying* if there is a natural transformation $A \xrightarrow{d_A} A \otimes A$ with $d_I = \rho_I$, satisfying cocommutativity and coassociativity, and:



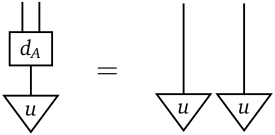
Naturality and $d_I = \rho_I$ look like this for arbitrary $A \xrightarrow{f} B$:



4.2 Uniform deleting and copying

Example 4.22. The monoidal category **Set** has uniform copying, with maps $a \mapsto (a, a)$. We see that $d_1(\bullet) = (\bullet, \bullet) = \rho_1(\bullet)$, and both maps $A \times B \to A \times B \times A \times B$ are $(a, b) \mapsto (a, b, a, b)$.

Definition 4.23. In a braided monoidal category, a state $I \xrightarrow{u} A$ is copyable with respect to a map $A \xrightarrow{d_A} A \otimes A$ when:



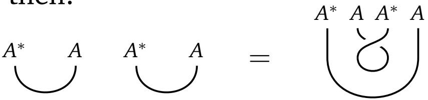
Proposition 4.24. In a braided monoidal category with uniform copying, any state is copyable.

Proof. If there is uniform copying, then, by naturality of the copying maps, we have $d_A \circ u = (u \otimes u) \circ \rho_I$ for each state $I \xrightarrow{u} A$. $\square$

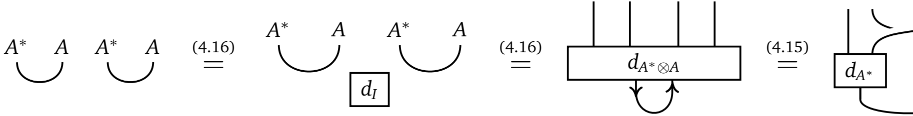
4.2 Uniform deleting and copying

We now investigate braided monoidal categories with duals and uniform copying.

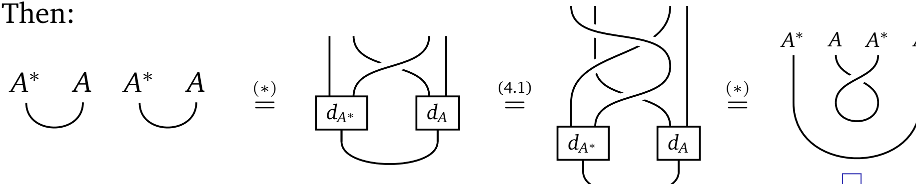
Lemma 4.25. If a braided monoidal category with duals has uniform copying, then:



Proof. First, consider the following equality (∗):

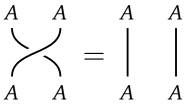


Then:



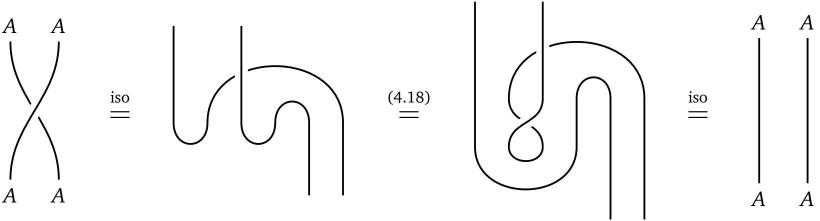
4.2 Uniform deleting and copying

Theorem 4.27. In a braided monoidal category with duals and uniform copying, the braiding is the identity:



(∗)

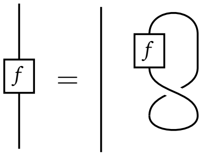
Proof. We show this as follows:



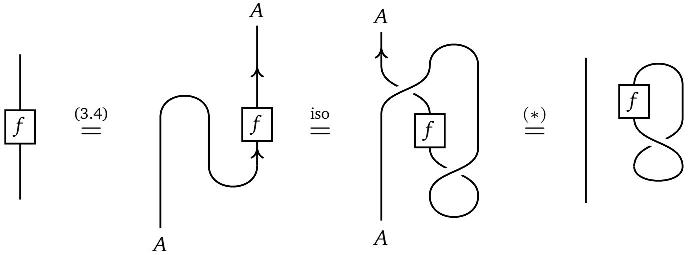
This completes the proof. □

4.2 Uniform deleting and copying

Theorem 4.27. If a braided monoidal category with duals has uniform copying, every endomorphism is a multiple of the identity:



Proof. We perform the following calculation:



This completes the proof. □

4.4 Products

Theorem 4.28. The following are equivalent for a symmetric monoidal category:

- tensor products are products and the tensor unit is terminal;
- it has uniform copying and deleting, satisfying counitality.

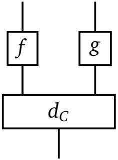
Proof. If category is cartesian, the unique morphism $A \xrightarrow{e_A} I$ and $d_A = \begin{pmatrix} \mathrm{id}_A \\ \mathrm{id}_A \end{pmatrix}$ provide uniform copying and deleting.

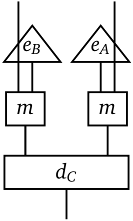
For the converse, need to prove $A \otimes B$ is a product of A, B. For $C \xrightarrow{f} A$ and $C \xrightarrow{g} B$, define

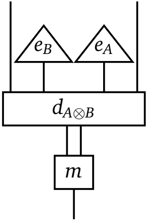
$$\begin{aligned} \begin{pmatrix} f \\ g \end{pmatrix} &= (f \otimes g) \circ d \\ p_A &= \rho_A \circ (\mathrm{id}_A \otimes e_B) \colon A \otimes B \to A \\ p_B &= \lambda_B \circ (e_A \otimes \mathrm{id}_B) \colon A \otimes B \to B \end{aligned}$$

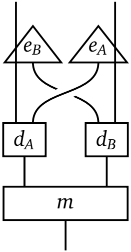
4.4 Products

Proof. (continued) Suppose $C \xrightarrow{m} A \otimes B$ satisfies $p_A \circ m = f$ and $p_B \circ m = g$. Then:

$\begin{pmatrix} f \\ g \end{pmatrix} =$ 

$=$ 

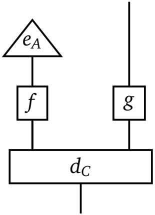
$=$ 

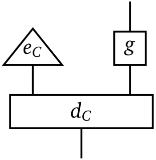
$\overset{(4.15)}{=}$ 

$\overset{(4.3)}{=} m$

Hence mediating morphisms, if they exist, are unique.

Finally, we show the universal morphism has the right properties:

$p_B \circ \begin{pmatrix} f \\ g \end{pmatrix} =$ 

$=$ 

$=$ 

A similar result holds for g. ∎

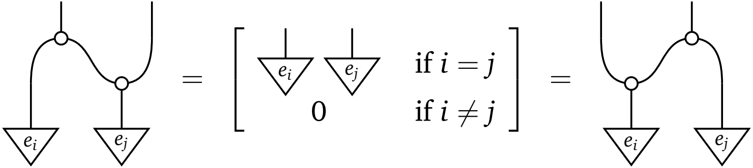
Chapter 5

Frobenius structures

5.1 Frobenius structures

Orthonormal basis $\{e_i\}$ for H in **FHilb** gives comonoid $\curlyvee$: $e_i \mapsto e_i \otimes e_i$.
Its adjoint $\curlywedge$ is *comparison*: $e_i \otimes e_i \mapsto e_i$ and $e_i \otimes e_j \mapsto 0$ if $i \neq j$.

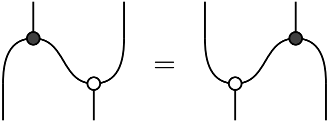
These cooperate:



This monoid/comonoid interaction is called the *Frobenius law*.

5.1 Frobenius structures

Definition 5.1. In a monoidal category, a *Frobenius structure* is a comonoid $(A, ,)$ and monoid $(A, ,)$ satisfying the *Frobenius law*:



If $=$, this is called *dagger Frobenius structure*.

Examples of dagger Frobenius structures:

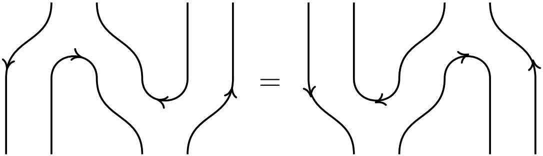
- In **FHilb**: a Hilbert space equipped with an orthogonal basis
- In **FHilb**: let G be finite group, spanning Hilbert space A.
  Define *group algebra* : $g \otimes h \mapsto gh$, and : $z \mapsto z \cdot 1_G$.
  Adjoint: : $\sum_{h \in G} gh^{-1} \otimes h$, and : $1_G \mapsto g$ and $1_G \neq g \mapsto 0$.
  Frobenius law: $\mathrm{LHS}(g \otimes h) = \sum_{k \in G} gk^{-1} \otimes kh = \mathrm{RHS}(g \otimes h)$.
- In **Rel**: let **G** be *groupoid*.
  Monoid in **Rel**: : $(g, h) \sim g \circ h$, and : $\bullet \sim \mathrm{id}_X$.
  Frobenius law: $(g, h) \sim (a, b \circ h)$ for $g = a \circ b$, $\mathrm{t}(h) = \mathrm{s}(b)$.

5.1 Frobenius structures

Lemma 5.9. In a dagger pivotal category, if $A \dashv A^*$, the pair of pants monoid $A^* \otimes A$ carries a dagger Frobenius structure.

Proof. The adjunction properties follow from the graphical calculus for dagger pivotal categories.

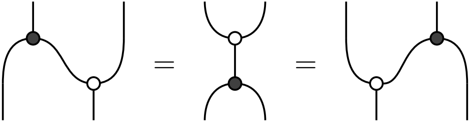
The Frobenius law is verified as follows:



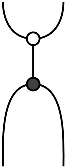
□

5.1 Frobenius structures

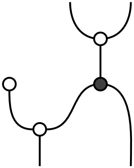
Lemma 5.4. Any Frobenius structure satisfies:



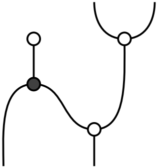
Proof. Let's prove the first equality:



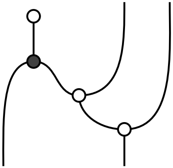
$\overset{(4.3)}{=}$



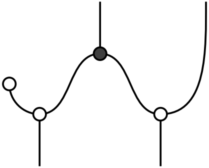
$\overset{(5.1)}{=}$



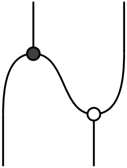
$\overset{(4.2)}{=}$



$\overset{(5.1)}{=}$



$\overset{(4.3)}{=}$

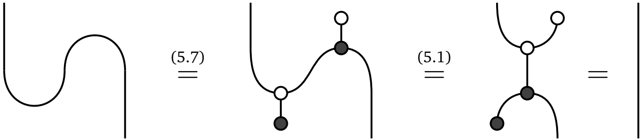


□

5.1 Frobenius structures

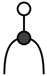
Theorem 5.15. If $(A, \curlyvee, \circ, \curlywedge, \bullet)$ Frobenius structure in monoidal category, then $A \dashv A$ is self-dual with:

Proof. Snake equation:



5.1 Frobenius structures

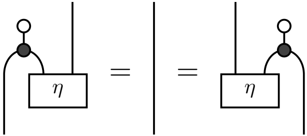
Proposition 5.16. Monoid $(A, ,)$ forms Frobenius structure with comonoid $(A, ,)$ iff allows *nondegenerate form*: map $: A \to I$ with



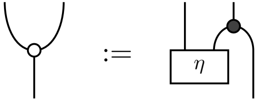
part of self-duality $A \dashv A$.

Proof. One direction is the previous theorem.

Conversely, suppose $I \xrightarrow{\eta} A \otimes A$ satisfies:



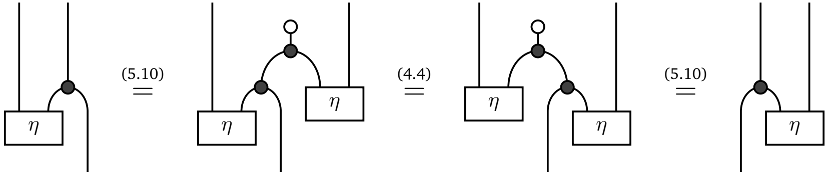
Then define the comultiplication as follows:



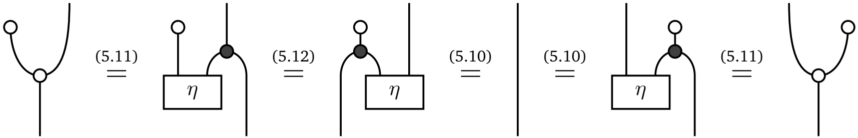
5.1 Frobenius structures

Proof (continued.)

Could have defined the comultiplication with η left or right:



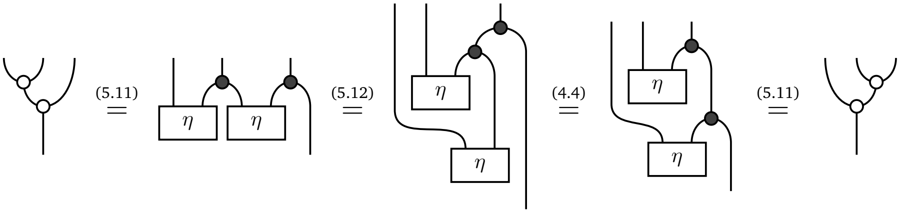
We can verify counitality:



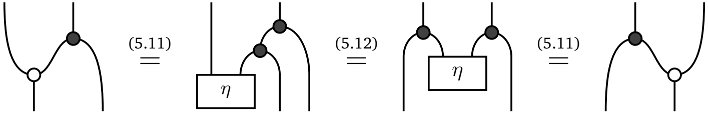
5.1 Frobenius structures

Proof (continued.)

Coassociativity is verified as follows:



Finally, we can verify the Frobenius law:



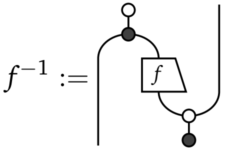
This completes the proof. □

5.1 Frobenius structures

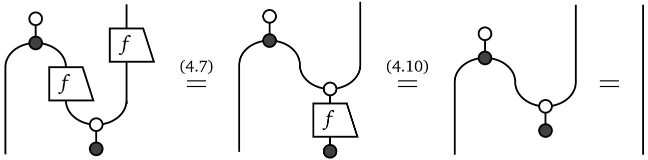
Definition 5.18. In a monoidal category, a *homomorphism of Frobenius structures* is morphism which is both a monoid homomorphism and a comonoid homomorphism.

Lemma 5.19. In a monoidal category, a homomorphism of Frobenius structures is invertible.

Proof. Given homomorphism $A \xrightarrow{f} B$, construct inverse as follows:



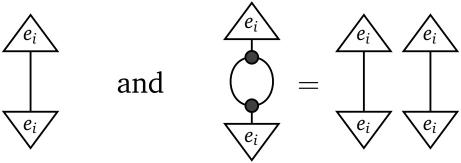
Let's verify that this is the inverse of f:



□

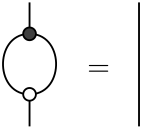
5.1 Frobenius structures

If  copies orthogonal basis $\{e_i\}$, can find (squared) norm of e_i:



So can characterize orthonormality via Frobenius structure.

Definition 5.5. In a monoidal category, a Frobenius structure is *special* when the following equation holds:



We can consider this for the dagger Frobenius structures we know:

- Group algebra in **FHilb** is only special for trivial group
- Orthogonal basis in **FHilb** is special just when basis is orthonormal
- Groupoid Frobenius structure in **Rel** is always special

5.1 Frobenius structures

Definition 5.10. In a braided monoidal dagger category, a *classical structure* is a special commutative dagger Frobenius structure.

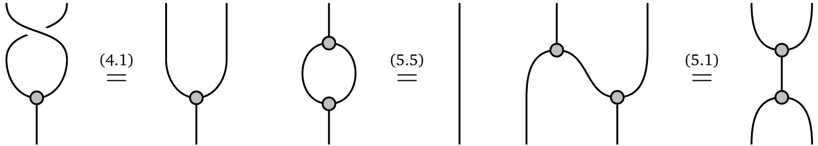
Examples:

- In **FHilb**: an orthonormal basis
- In **Rel**: abelian group

Definition of classical structure redundant:

- (Co)commutativity implies half of (co)unitality
- Speciality and Frobenius law imply (co)associativity
- Dual object and Frobenius law imply (co)unitality

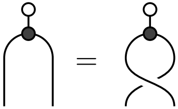
To check that $(A, \curlywedge, \circ)$ is classical structure, only need:



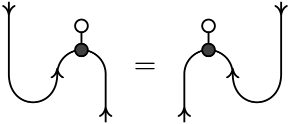
5.1 Frobenius structures

Pair of pants hardly ever commutative. However:

Definition 5.12. In a braided monoidal category, a Frobenius structure is *symmetric* when:



In a compact category, this is equivalent to the following:

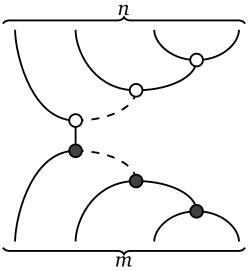


Examples:

- Pair of pants: in **FHilb** this says $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$
- Group algebras: inverses in groups are two-sided inverses
- Groupoid Frobenius structure: inverses are two-sided

5.2 Normal forms

Lemma 5.20. In a monoidal category, let $(A, , , ,)$ be a special Frobenius structure. Any connected morphism $A^{\otimes m} \to A^{\otimes n}$ built out of finitely many pieces , , , , and id, using $\circ$ and $\otimes$, equals:



Proof. Strategy is induction on the number of dots.

5.2 Normal forms

Proof. (continued.)

Base case. Trivial, as the diagram must be one of , , , .

Induction step. Assume all diagrams with at most n dots can be brought in normal form, and consider a diagram with $n+1$ dots.

Use naturality to write the diagram in a form where there is a topmost dot.

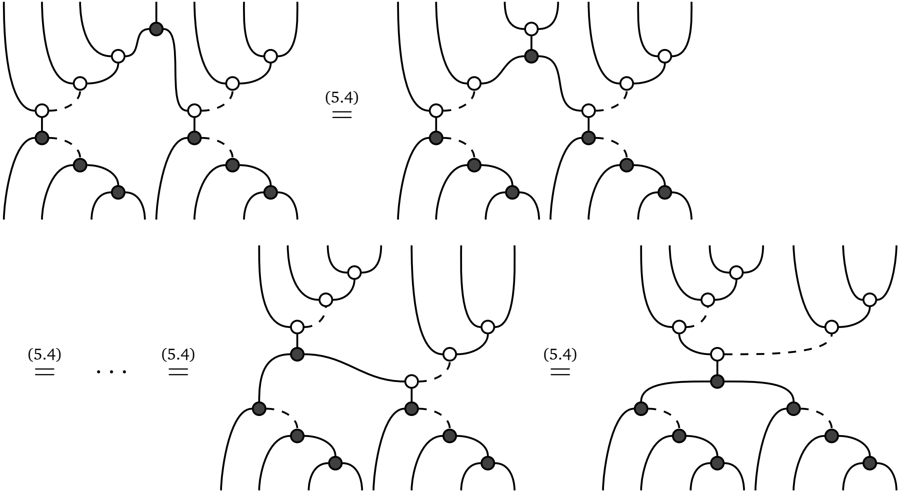
- Topmost dot is : use counitality to eliminate it.
- Topmost dot is : use coassociativity to reach normal form.
- Topmost dot is : impossible by connectedness.
- Topmost dot is : the most interesting case.

  Is the diagram underneath the connected?
  If so, use coassociativity and speciality.

5.2 Normal forms

Proof. (continued.)

Suppose instead the rest of the diagram is disconnected:

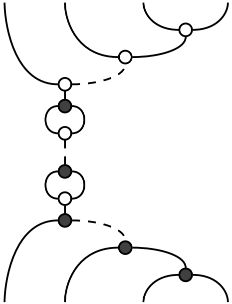


This completes the proof. □

5.2 Normal forms

There are normal forms for other sorts of Frobenius structures.

Theorem 5.21. In a monoidal category, let $(A, \ , \ , \ , \ )$ be a Frobenius structure. Any connected morphism $A^{\otimes m} \to A^{\otimes n}$ built out of finitely many pieces , , , , and id, using $\circ$ and $\otimes$, equals $(*)$.



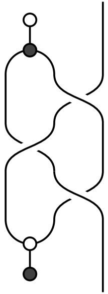
$(*)$

Theorem 5.22. In a symmetric monoidal category, let $(A, \ , \ , \ , \ )$ be a commutative Frobenius structure. Any connected morphism $A^{\otimes m} \to A^{\otimes n}$ built out of finitely many pieces , , , , id, and , using $\circ$ and $\otimes$, equals $(*)$.

5.2 Normal forms

Proposition 5.23. In a braided non-symmetric monoidal category, there is no normal form for commutative Frobenius structures.

Proof. Regard the following diagram as a piece of string on which an overhand knot is tied:

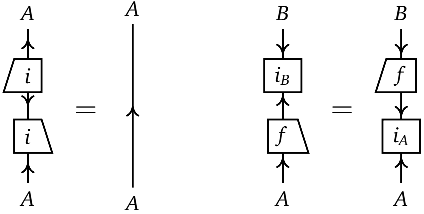


The Frobenius structure axioms induce homotopy equivalences ('deformations') of the corresponding graph. Such moves are clearly not able to untie the knot. □

5.3 Involutive monoids

Lemma 5.24. In a dagger pivotal category, if (A, m, u) is a monoid, then (A^*, m_*, u_*) is monoid. $\square$

Definition 5.25. In a dagger pivotal category, an *involution* for a monoid $(A, \curlywedge, \multimap)$ is a monoid homomorphism $A \xrightarrow{i} A^*$ satisfying $i_* \circ i = \mathrm{id}_A$.



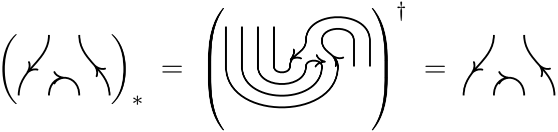
A *morphism of involutive monoids* is monoid homomorphism $A \xrightarrow{f} B$ satisfying $i_B \circ f = f_* \circ i_A$.

5.3 Involutive monoids

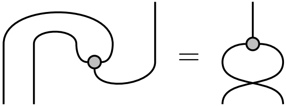
Examples:

- *Matrix algebra.* $\mathbb{M}_n$ is an involutive monoid in **FHilb**.
  Opposite monoid $\mathbb{M}_n^*$: multiplication ab in $\mathbb{M}_n^*$ is ba in $\mathbb{M}_n$.
  Canonical involution $\mathbb{M}_n \to \mathbb{M}_n^*$ given by $f \mapsto f^\dagger$.

- *Pair of pants.* $A^* \otimes A$ involutive in a dagger pivotal category.
  Identity map as involution, because of conventions:



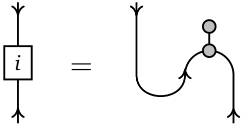
- *Groupoid Frobenius structure.* **G** in **Rel** is involutive.
  Opposite monoid: induced by opposite groupoid $\mathbf{G}^{\mathrm{op}}$



  Canonical involution $G \to G^*$ given by $g \sim g^{-1}$.

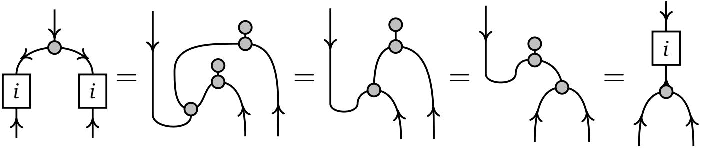
5.3 Involutive monoids

Theorem 5.28. In a dagger pivotal category, a monoid (A, ,) is dagger Frobenius if and only if i is an involution:

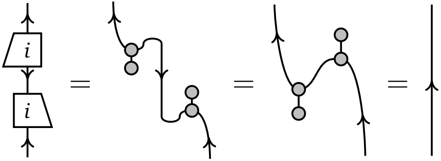


Proof. Assume dagger Frobenius.

- i preserves multiplication:



- i preserves units: easy.
- i is involution:



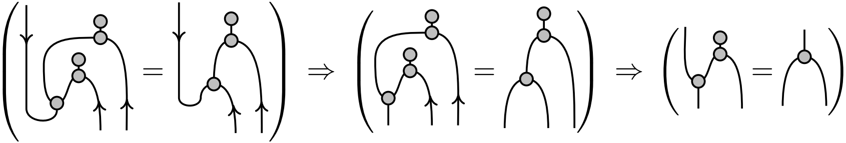
5.3 Involutive monoids

Proof. (continued.) Conversely, suppose $i_* \circ i = \mathrm{id}$. Then:

So we have a Frobenius structure, defined by a nondegenerate form.

Is it a dagger Frobenius structure?

The condition that i preserves multiplication gives:



So the form definition gives rise to the correct comultiplication. □

5.4 Classification

In **FHilb**, Frobenius structures cannot be classified in general.

Here is a 'wild' Frobenius structure on $\mathbb{C}[1,X]$, with unit u, $m : \mathbb{C}[1,X] \otimes \mathbb{C}[1,X] \to \mathbb{C}[1,X]$ and $f : \mathbb{C}[1,X] \to \mathbb{C}$:

$$\begin{aligned} m(1,1) &= 1 & u &= 1 \\ m(1,X) &= X & & \\ m(X,1) &= X & f(1) &= 0 \\ m(X,X) &= 0 & f(X) &= 1 \end{aligned}$$

However, we can classify them in various cases, when we add sufficient adjectives.

5.4 Classification

Theorem. In **FHilb**, special commutative Frobenius structures correspond to Hilbert spaces equipped with a basis.

Proof. The specialness property implies that the algebra structure is strongly separable.

The Artin-Wedderburn theorem says that a strongly separable algebra over $\mathbb{C}$ is a direct sum of matrix algebras over $\mathbb{C}$.

If the algebra is commutative, these must be 1-by-1 matrix algebras.

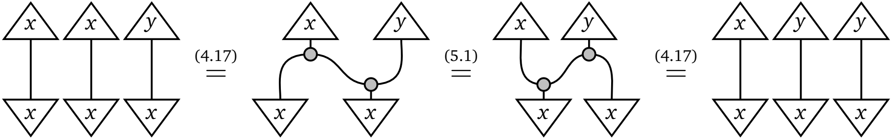
So if the underlying Hilbert space is H, we have $H \simeq \mathbb{C} \oplus \cdots \oplus \mathbb{C}$, which is exactly the choice of a basis.

The Frobenius laws then follow, choosing the comultiplication to copy this chosen basis. $\square$

5.4 Classification

Lemma. Given a basis for a finite-dimensional Hilbert space, its comonoid in **FHilb** is dagger Frobenius just when the basis is orthogonal.

Proof. Let x, y be nonzero copyable states, then:



If $\langle x|y\rangle = 0$, then this is satisfied.

If $\langle x|y\rangle \neq 0$, this implies $\langle x|x\rangle = \langle x|y\rangle$. Similarly $\langle y|x\rangle = \langle y|y\rangle$.

Hence $\langle x-y|x-y\rangle = \langle x|x\rangle - \langle x|y\rangle - \langle y|x\rangle + \langle y|y\rangle = 0$, so $x = y$. $\square$

5.4 Classification

Theorem. In **FHilb**, commutative dagger Frobenius structures correspond to Hilbert spaces equipped with an orthogonal basis.

Proof. We have seen that a dagger Frobenius structure on H has an involution-preserving homomorphism into $\mathrm{Hom}(H, H)$.

This is a finite-dimensional C*-algebra, and involution-closed subalgebras of f.d. C*-algebras are again C*-algebras.

By the spectral theorem, the copyable states form a basis—so if we know what happens to these states, we know the whole algebra.

By the previous lemma, the only restriction on these states is that they are orthogonal. $\square$

5.4 Classification

Theorem. In **FHilb**, classical structures correspond to Hilbert spaces equipped with a choice of orthonormal basis.

Proof. Classical structures are special commutative dagger Frobenius structures.

By the previous theorem, they must correspond to orthogonal bases with some additional property.

The specialness condition says exactly that the basis elements are normalized. □

5.4 Classification

We can compare these classification theorems:

Commutative Frobenius structure	**Basis**
Special	Arbitrary
Dagger	Orthogonal
Special dagger	Orthonormal

How can this make sense?

The comultiplications are different.

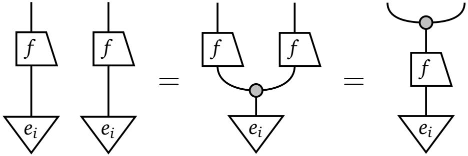
For an arbitrary basis, the dagger structure plays no role.

For the other bases, the comultiplication is the adjoint of the multiplication.

5.4 Classification

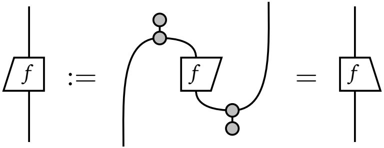
Corollary 5.37. In **FHilb**, a morphism between two commutative dagger Frobenius structures acts as a function on copyable states if and only if it is a comonoid homomorphism.

Proof. Suffices to see about basis of copyable states $\{e_i\}$.



Hence $f(e_i)$ copyable. □

Lemma 5.38. In **FHilb**, comonoid homomorphisms between commutative dagger Frobenius structures are self-conjugate:



Proof. Verify they have the same matrix entries. □

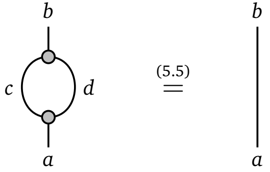
5.4 Classification

We now consider the classification in **Rel**.

Theorem 5.41. Special dagger Frobenius structures in **Rel** correspond exactly to groupoids.

Proof. Write $A \times A \xrightarrow{M} A$ for multiplication, $U \subseteq A$ for unit.

M is single-valued: by speciality $a(M \circ M^{\dagger})b$ iff $a = b$:

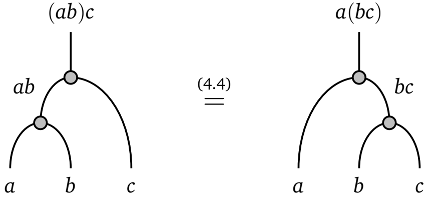


So: if $(c, d)Ma$ and $(c, d)Mb$, must have $a = b$.
May simply write ab for unique c with $(a, b)Mc$.
Remember: ab not always defined!

5.4 Classification

Proof. (continued)

Associativity:

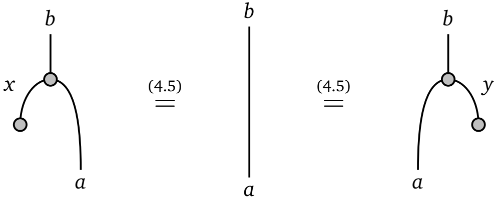


So ab and $(ab)c$ defined exactly when bc and $a(bc)$ are defined, and then $(ab)c = a(bc)$.

5.4 Classification

Proof. (continued)

Unitality: for units $x, y \in U$



So: a, b allow $x \in U$ with $xa = b$ iff $a = b$.
And: a, b allow $y \in U$ with $ay = b$ iff $a = b$.
If $z \in U$ then $xz = x$ for some $x \in U$. But then $x = z$!
Units idempotent; multiplication of different ones undefined.

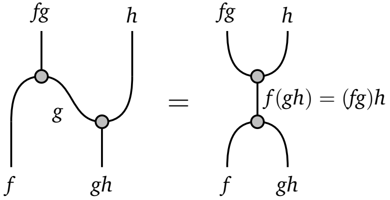
If $xa = a = x'a$, then $a = xa = x(x'a) = (xx')a$, so $x = x'$.
So every element has unique left/right identity.

5.4 Classification

Proof. (continued)

Category: U set of objects, A set of morphisms.

If fg defined and gh defined, want $(fg)h = f(gh)$ defined too:

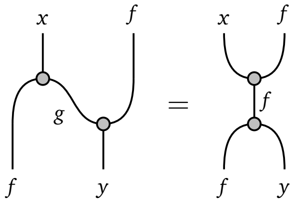


If fg and gh defined then LHS defined, so RHS defined too.

5.4 Classification

Proof. (continued)

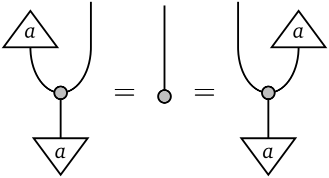
Inverses: for $f \in A$ with left unit x and right unit y:



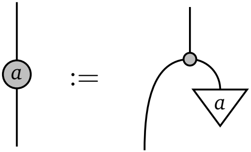
That completes the proof. □

5.5 Phases

Definition 5.44. Let $(A, \cdot, \cdot)$ be a Frobenius structure in a monoidal dagger category. A state $I \xrightarrow{a} A$ is called a *phase* when:



Its *(right) phase shift* is the following morphism $A \to A$:



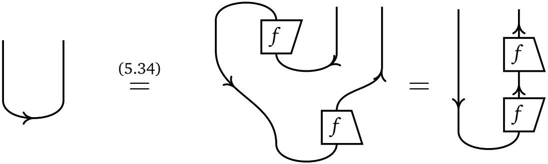
5.5 Phases

Examples:

- For classical structure in **FHilb** copying basis $\{e_i\}$, vector $a = a_1 e_1 + \cdots a_n e_n$ is phase iff each a_i on unit circle: $|a_i|^2 = 1$.
- The unit $\circ$ of a Frobenius structure is always a phase.

Lemma 5.46. In a dagger pivotal category, phases for a pair of pants structure $(A^* \otimes A, \wedge, \smile)$ correspond to unitary morphisms.

Proof. The name of an morphism $A \xrightarrow{f} A$ is a phase when:



But this means $f \circ f^\dagger = \mathrm{id}_A$; similarly $f^\dagger \circ f = \mathrm{id}_A$. □

5.5 Phases

Example 5.47. Phases of Frobenius structure $\mathbb{M}_n$ in **FHilb** form set $U(n)$ of n-by-n unitary matrices. Hence phases of $\mathbb{M}_{k_1} \oplus \cdots \oplus \mathbb{M}_{k_n}$ range over $U(k_1) \times \cdots \times U(k_n)$.

Special case: classical structure $\mathbb{C}^n$ copying basis $\{e_1, \ldots, e_n\}$.
Phases are elements of $U(1) \times \cdots \times U(1)$;
phase shift $\mathbb{C}^n \to \mathbb{C}^n$ is accompanying unitary matrix.

Example 5.48. The phases of a Frobenius structure in **Rel** induced by a group G are elements of that group G itself.

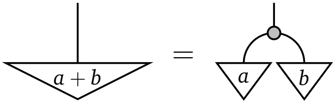
Proof. For a subset $a \subseteq G$, equation defining phases reads

$$\{g^{-1}h \mid g, h \in a\} = \{1_G\} = \{hg^{-1} \mid g, h \in a\}.$$

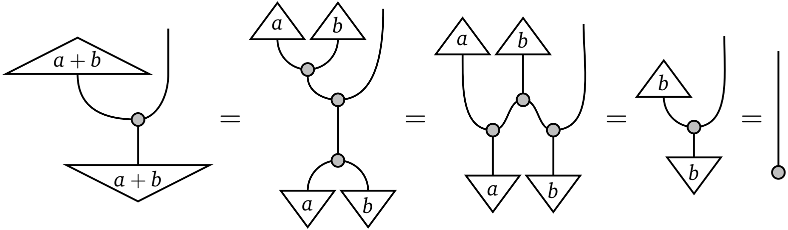
So if $g \in G$, then $a = \{g\}$ is a phase. But if a contains two distinct elements $g \neq h$ of G, then it cannot be a phase. Similarly, $a = \emptyset$ is not a phase. Hence a is a phase precisely when it is a singleton $\{g\}$.

5.5 Phases

Proposition 5.49. In a monoidal dagger category, the phases for a dagger Frobenius structure form a group, with unit ⫯ and:



Proof. This is again a well-defined phase:



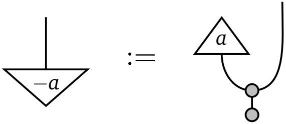
The flipped equation follows similarly.

Associativity is clear, hence phases form a monoid.

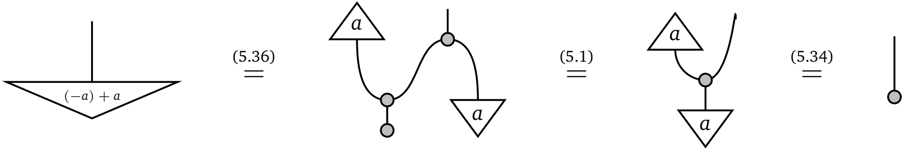
5.5 Phases

Proof. (continued)

Left-inverse of phase a is:



Left-inverse of a is $-a$:



Similarly there is right-inverse. But in monoids, left and right inverses are equal: $l = l(xr) = (lx)r = r$. □

5.5 Phases

This group is called the *phase group*.

Examples:

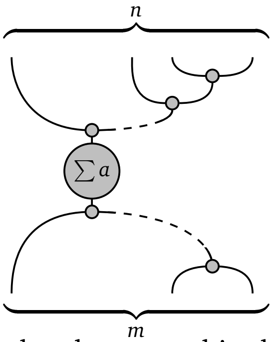
- In **FHilb**, the phase group for the pair of pants Frobenius structure is the unitary group.

- Phase addition in the Frobenius structure $\mathbb{M}_{k_1} \oplus \cdots \oplus \mathbb{M}_{k_n}$ in **FHilb** is entrywise multiplication in $U(k_1) \times \cdots \times U(k_n)$.
  In particular, phase addition in a classical structure in **FHilb** is multiplication of diagonal matrices.

- In **Rel**, the phase group induced by a group G is the group itself.

5.5 Phases

Corollary 5.51. Let $(A, ⅄, ⫯)$ be classical structure in braided monoidal dagger category. Any connected morphism $A^{\otimes m} \to A^{\otimes n}$ built of finitely many ⅄, ⫯, id, σ and phases using $\circ$, $\otimes$, and $\dagger$, equals



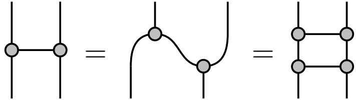
where a ranges over all the phases used in the diagram.

Proof. Using braidings to have all phases dangle at the bottom. Apply Spider Theorem. Use phase addition to reduce to single phase $\sum a$ on bottom right. Apply Spider Theorem again. □

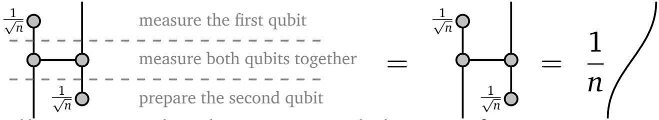
5.5 Phases

Quantum state transfer protocol: transfer state of Hilbert space H from one system to another, with success probability $1/\dim(H)^2$.

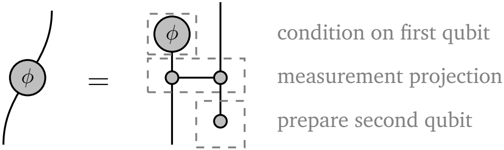
May be lax in drawing, e.g. projection $H \otimes H \to H \otimes H$:



The procedure looks like this:



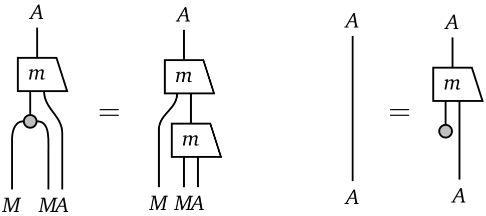
Extra challenge: apply phase gate while transferring state



5.6 Modules

Modules give us a more sophisticated way to model measurement.

Definition 5.52. In a monoidal category, a *module* for a monoid $(M, \curlywedge, \circ)$ is an object A equipped with $M \otimes A \xrightarrow{m} A$ satisfying:

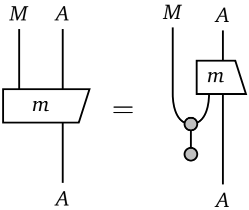


The morphism m is called an *action* of the monoid on the object A.

We will only consider *left modules*.

5.6 Modules

Definition 5.55. *Dagger module* for dagger Frobenius structure $(M, ,)$ in monoidal dagger category is module $M \otimes A \xrightarrow{m} A$ with:



Examples:

- Multiplication : $M \otimes M \to M$ of a dagger Frobenius structure is the action of a dagger module on itself.
- Let group G induce group algebra A in **FHilb**.

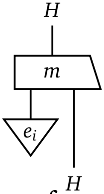
  Modules $A \otimes \mathbb{C}^n \to \mathbb{C}^n$ are *representations* of G.

  Dagger modules $A \otimes \mathbb{C}^n \to \mathbb{C}^n$ are *unitary representations* of G.

5.6 Modules

Lemma 5.57. Dagger modules for classical structure $(M, \curlywedge, \circ)$ acting on H in **FHilb** correspond to *projection-valued measure* on H with $\dim(M)$ outcomes.

Proof. Module $M \otimes H \xrightarrow{m} H$ determined by following morphisms p_i:



for copyable states $e_i \in M$. These form a PVM:

- Associativity, speciality, copyability: $p_i \circ p_i = p_i$, and $p_i \circ p_j = 0$.
- Dagger module axiom: $p_i = p_i^\dagger$.
- Since $\circ = \sum_i e_i$, also $\sum_i p_i = \mathrm{id}_H$.

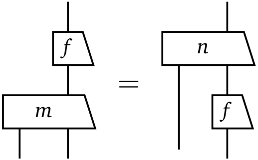
Conversely: if $\{p_I\}$ is PVM, get a module action $M \otimes H \to H$. $\square$

Special case $m = \curlywedge$ gives a *nondegenerate* measurement.

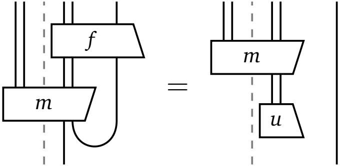
5.6 Modules

After measurement, only allowed *controlled operations*: unitary maps that do not affect the measurement result.

Definition 5.60. Given monoid $(M, \text{ mult}, \text{ unit})$ in monoidal category and module actions $M \otimes A \xrightarrow{m} A$ and $M \otimes B \xrightarrow{n} B$, a *module homomorphism* $m \xrightarrow{f} n$ is a morphism $A \xrightarrow{f} B$ satisfying the following condition:



We can use this to formalize quantum teleportation:

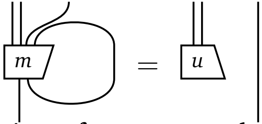


Here $(A \otimes A^*, m, u)$ is a classical structure, f is module homomorphism.

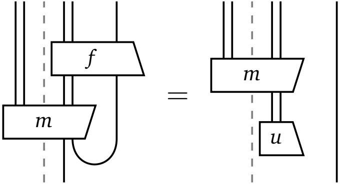
5.6 Modules

Can now treat teleportation without biproducts, purely graphically.

Proposition 5.64. In a dagger monoidal category, a classical structure $(A \otimes A^*, m, u)$ describes measurement in a teleportation protocol if and only if:

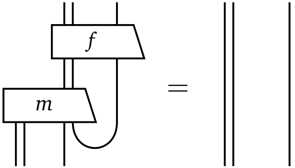


Proof. Successful execution of quantum teleportation means:

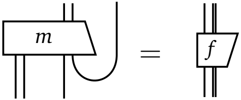


5.6 Modules

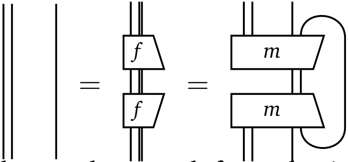
Proof. (continued.) Bend down the top-left $A \otimes A$ wires:



Compose both sides with $f^\dagger$ at the top:



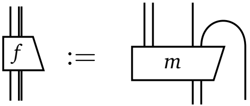
Using this description of $f^\dagger$:



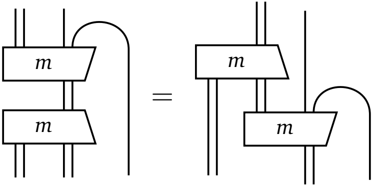
Finally, compose with u on bottom-left to obtain desired formula.

5.6 Modules

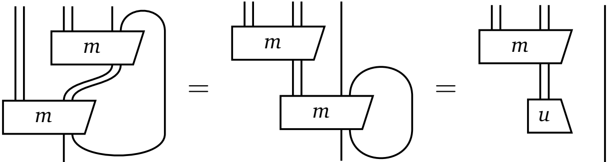
Proof. (continued.) Conversely, suppose classical structure m satisfies the condition. Then define f as follows:



This f is unitary, and a module homomorphism:



It correctly implements quantum teleportation:



□

Chapter 6

Complementarity

6.1 Complementarity

Measure qubit in basis $\{\binom{1}{0}, \binom{0}{1}\}$, then in $\{\frac{1}{\sqrt{2}}\binom{1}{1}, \frac{1}{\sqrt{2}}\binom{1}{-1}\}$.

After first measurement, qubit collapses to either $\binom{1}{0}$ or $\binom{0}{1}$.

Either way, second measurement has probability $1/2$ for outcomes.

The first measurement provides no information about the second.

This is a simple form of Heisenberg's *uncertainty principle*.

We formalize this as follows.

Definition 6.1. For a finite-dimensional Hilbert space H, two orthogonal bases $\{a_i\}$ and $\{b_j\}$ are *complementary*, or *unbiased*, when there is some constant $c \in \mathbb{C}$ such that the following holds:

$$\langle a_i | b_j \rangle \langle b_j | a_i \rangle = c$$

That is, the inner products have constant absolute value.

6.1 Complementarity

We can prove a simple lemma about complementary bases.

Lemma 6.2. For a pair of complementary bases $\{a_i\}$ and $\{b_j\}$, within each basis, the elements have constant norm.

Proof. We perform the following computation:

$$\langle b_j | b_j \rangle = \sum_i \frac{\langle b_j | a_i \rangle \langle a_i | b_j \rangle}{\langle a_i | a_i \rangle} \stackrel{(6.1)}{=} \sum_i \frac{c}{\langle a_i | a_i \rangle}$$

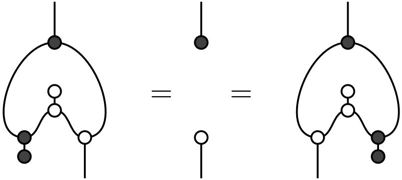
In the first equality, we insert the identity as a sum over the complete family of projectors $|a_i\rangle\langle a_i| / \langle a_i | a_i \rangle$.

The final expression is independent of j as required.

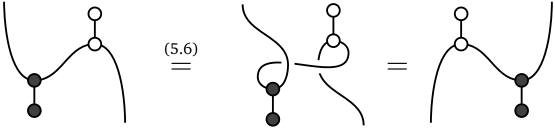
A similar argument holds for the $\{a_i\}$ basis. $\square$

6.1 Complementarity

Definition 6.3. In a braided monoidal dagger category, two symmetric dagger Frobenius structures  and  on the same object are *complementary* when the following equations hold:



Black and white not obviously interchangeable. But by symmetry:



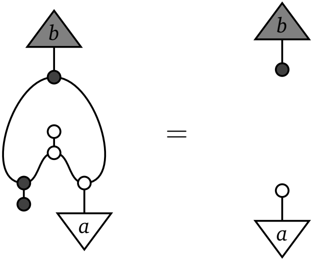
So could have added two more equalities.

6.1 Complementarity

Proposition 6.4. In **FHilb**, the following are equivalent for two commutative dagger Frobenius structures on the same object:

- as Frobenius structures, they are complementary;
- as bases, they are complementary with constant $c = 1$.

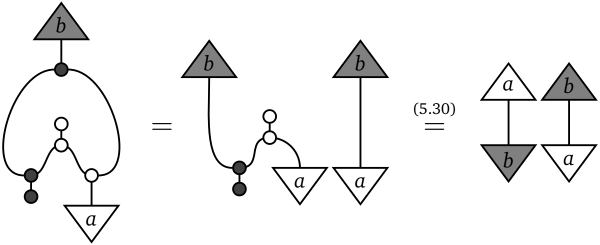
Proof. The complementarity equation (6.4) holds if and only if the following equation holds for all a in the white basis, and b in the black basis:



6.1 Complementarity

Proof. (continued.)

The left-hand side can be simplified as follows:

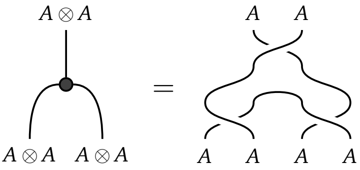


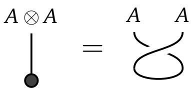
The right-hand side expands to 1. □

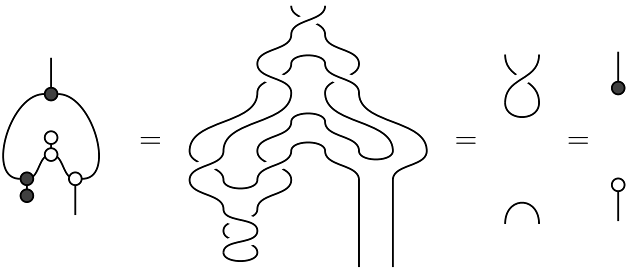
6.1 Complementarity

Lemma 6.6. In a braided dagger pivotal category, if A is self-dual, then the following Frobenius structures on $A \otimes A$ are complementary: pair of pants, and transport across braiding.

Proof. Draw pair of pants white, transport across braiding black:







□

6.1 Complementarity

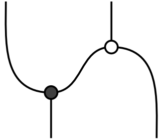
Example 6.5. Three mutually complementary bases of $\mathbb{C}^2$:

$$\begin{array}{ll}
X \text{ basis} & \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \right\} \\
Y \text{ basis} & \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} \right\} \\
Z \text{ basis} & \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}
\end{array}$$

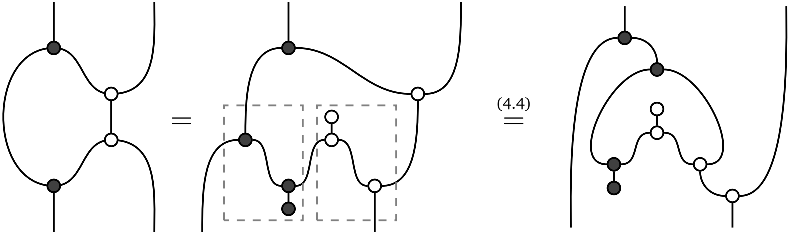
- Largest family of complementary bases for $\mathbb{C}^2$: no four bases all mutually unbiased.
- What is the maximum number of mutually complementary bases in a given dimension?
- Only known for prime power dimensions p^n.

6.1 Complementarity

Proposition 6.7. Two symmetric dagger Frobenius structures in a braided monoidal dagger category are complementary if and only if the following endomorphism is unitary:



Proof. Compose with adjoint:



Conversely, if is identity, compose with white counit on top right, black unit on bottom left, to get complementarity. □

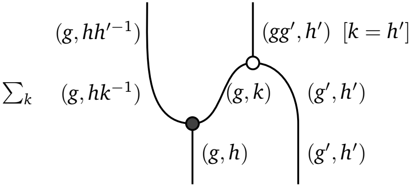
6.1 Complementarity

Example 6.8. Let G and H be nontrivial groups, and define:

- groupoid ● with objects $g \in G$, morphisms $g \xrightarrow{(g,h)} g$, composition $g \xrightarrow{(g,h)} g \xrightarrow{(g,h')} g = g \xrightarrow{(g,hh')} g$;
- groupoid ○ with objects $h \in H$, morphisms $h \xrightarrow{(g,h)} h$, composition $h \xrightarrow{(g,h)} h \xrightarrow{(g',h)} h = h \xrightarrow{(gh',h)} h$;

Then **G** and **H** are complementary Frobenius structures.

Proof. Let's consider the following composite:



Every input element is related to a unique output element, so the structures are complementary by Proposition 6.7. □

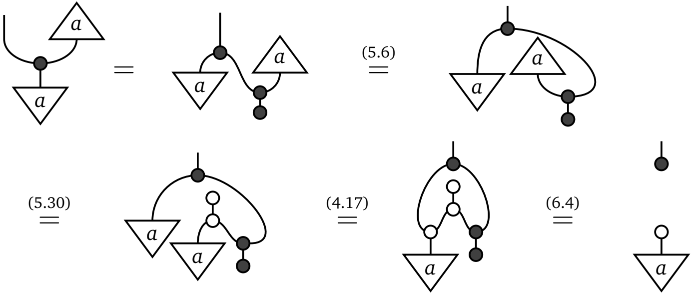
Proposition 6.10. In **Rel**, a groupoid allows a complementary one just when every object has the same number of morphisms out of it.

6.1 Complementarity

Complementary bases: copyable states for one *unbiased* for other.
Abstractly: state is unbiased phase shift is unitary.

Lemma 6.11. In a braided monoidal dagger category, if symmetric dagger Frobenius structures are complementary, then up to scalar, state that is self-conjugate and copyable for one is phase for other, up to an idempotent scalar.

Proof.



□

6.2 The Deutsch-Jozsa algorithm

Deutsch–Jozsa solves certain problem faster in quantum case than possible the classical case.

- Typical of quantum algorithms that decide on a solution without relying on approximation.
- Solves artificial problem, but other important algorithms have a similar structure:
  - Shor's factoring algorithm
  - Grover's search algorithm
  - the hidden subgroup problem
- 'All or nothing' nature of Deutsch-Jozsa makes it amenable to categorical modelling.

6.2 The Deutsch-Jozsa algorithm

Problem:

- Given 2-valued function $A \xrightarrow{f} \{0,1\}$ on a finite set A.
- *Constant* if takes just a single value on every element of A.
- *Balanced* if takes value 0 on exactly half the elements of A.
- You are promised that f is either constant or balanced.
  You must decide which.

Best classical strategy:

- Sample f on $\frac{1}{2}|A| + 1$ elements of A.
  If different values balanced, otherwise constant.

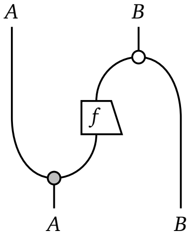
6.2 The Deutsch-Jozsa algorithm

Quantum Deutsch–Jozsa uses f only *once*!

How to access f? Can only apply unitary operators.

Must embed $A \xrightarrow{f} \{0, 1\}$ into an *oracle*.

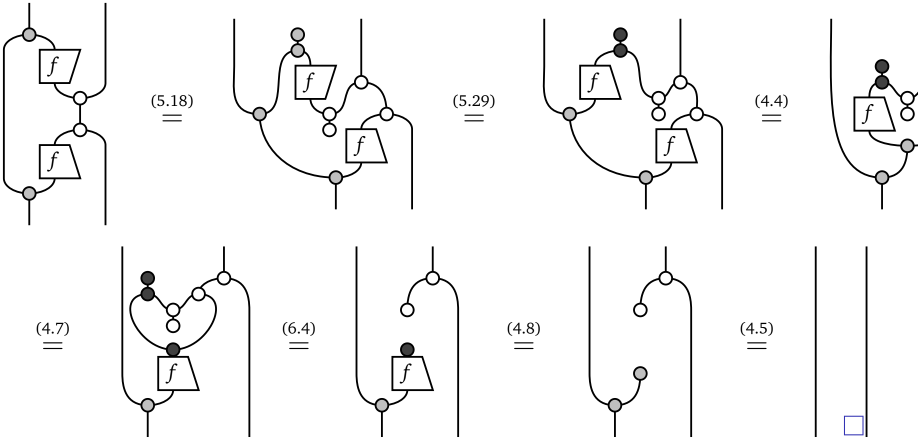
Definition 6.12. In a monoidal dagger category, given Frobenius structures $(A, \curlywedge, \bullet)$ and $(B, \curlywedge, \circ)$, an *oracle* is a morphism $A \xrightarrow{f} B$ such that the following morphism is unitary:



6.2 The Deutsch-Jozsa algorithm

Proposition 6.14. In a braided monoidal dagger category, let $(A, \tinymult[graydot])$, $(B, \tinymult[blackdot])$ and $(B, \tinymult[whitedot])$ be symmetric dagger Frobenius structures. Then if $\tinymult[blackdot]$, $\tinymult[whitedot]$ are complementary, a self-conjugate comonoid homomorphism $(A, \tinymult[graydot]) \xrightarrow{f} (B, \tinymult[blackdot])$ gives an oracle.

Proof. Suppose $\tinymult[blackdot]$, $\tinymult[whitedot]$ complementary, compose with adjoint:



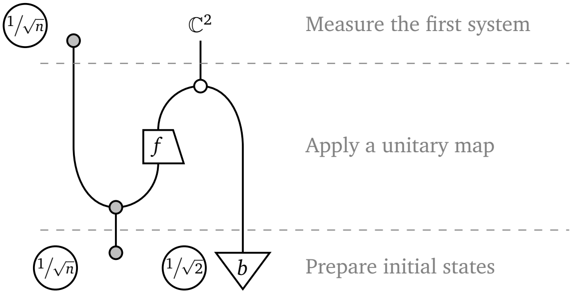
6.2 The Deutsch-Jozsa algorithm

Suppose $|A| = n$, and let $A \xrightarrow{f} \{0, 1\}$ be the given function.

Choose complementary bases $\bullet = \mathbb{C}^2$, $\circ = \mathbb{C}[\mathbb{Z}_2]$.

Let $b = \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$, a copyable state of $\circ$.

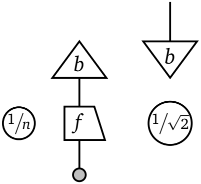
Definition 6.15. The *Deutsch–Jozsa algorithm* is this morphism:



It describes a particular quantum history.

6.2 The Deutsch-Jozsa algorithm

Lemma 6.16. The Deutsch–Jozsa algorithm (6.11) simplifies to:



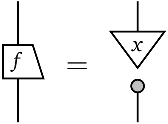
Proof. Duplicate copyable state b through white dot, and apply noncommutative spider theorem to cluster of gray dots. □

6.2 The Deutsch-Jozsa algorithm

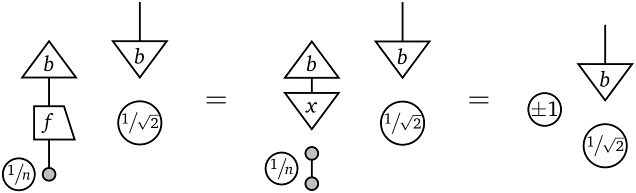
To prove correctness, distinguish two cases.

Lemma 6.17 (The constant case). If $A \xrightarrow{f} \{0,1\}$ is constant, the Deutsch–Jozsa history is certain.

Proof. If $f(a) = x$ for all $a \in A$, oracle $H \xrightarrow{f} \mathbb{C}^2$ decomposes as:



Hence we can express our history as follows:

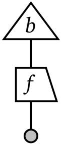


This has norm 1, so the history is certain. □

6.2 The Deutsch-Jozsa algorithm

Lemma 6.18 (The balanced case). If $A \xrightarrow{f} \{0, 1\}$ is balanced, the Deutsch–Jozsa history is impossible.

Proof. The function f is balanced just when the following holds:



$$= \quad 0$$

Recall $b = \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$.

Hence the final history equals 0. $\square$

6.3 Bialgebras

Complementary classical structures in **FHilb** are mutually unbiased bases. How to build them?

One standard way: let G be finite group, and consider Hilbert space with basis $\{g \in G\}$, with

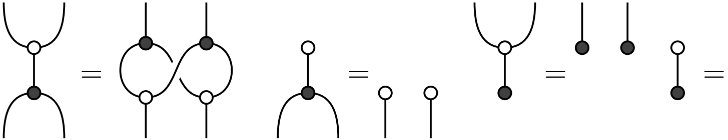
$$: g \mapsto g \otimes g \qquad : g \mapsto 1$$

$$: g \otimes h \mapsto gh \qquad : 1 \mapsto \sum_{g \in G} g$$

Some nice relationships emerge between and .

6.3 Bialgebras

Definition 6.20. In a braided monoidal category, a *bialgebra* consists of a monoid $(A, \,\cdot\,, \,\cdot\,)$ and a comonoid $(A, \,\cdot\,, \,\cdot\,)$ satisfying the following equations:



A bialgebra is *commutative* when the underlying monoid and comonoid are commutative. In a braided monoidal dagger category, a *dagger bialgebra* is a bialgebra for which $\cdot = \cdot$.

In the commutative case, interpretation in terms of counting paths. Leads to normal form.

6.3 Bialgebras

Example 6.21.

- In any category with biproducts, any object A has bialgebra:

$$A \xrightarrow{\begin{pmatrix}\mathrm{id}_A \\ \mathrm{id}_A\end{pmatrix}} A \oplus A \qquad 0 \xrightarrow{0_{0,A}} A \qquad A \oplus A \xrightarrow{\begin{pmatrix}\mathrm{id}_A & \mathrm{id}_A\end{pmatrix}} A \qquad A \xrightarrow{0_{A,0}} 0$$

- Any monoid M is a bialgebra in **Set**:

$$: m \mapsto (m,m) \qquad : m \mapsto \bullet \qquad : (m,n) \mapsto mn \qquad : \bullet \mapsto 1_M.$$

- Symmetric monoidal functors **FSet** $\to$ **FHilb**, **Set** $\to$ **Rel** extend these examples to other categories.

6.3 Bialgebras

Here is a nice characterization of the bialgebra laws.

Lemma 6.22. In a braided monoidal category, the following are equivalent:

- a comonoid $(A, ,)$ and monoid $(A, ,)$ form a bialgebra;
- and are comonoid homomorphisms;
- and are monoid homomorphisms.

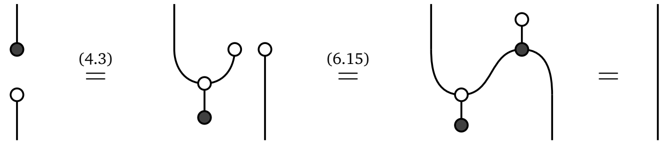
Proof. Unfold what it means for to be a comonoid homomorphism: comultiplication preservation gives the first of the bialgebra laws; counit preservation gives the second; and the last two come from requiring that is a comonoid homomorphism. The case of monoid homomorphisms is analogous. □

6.3 Bialgebras

Frobenius structures and bialgebras are not compatible.

Theorem 6.23. In a braided monoidal category, if a monoid $(A, \,, \,)$ and comonoid $(A, \,, \,)$ form a Frobenius structure and a bialgebra, then $A \simeq I$.

Proof. Will show and are inverses. The bialgebra laws already require $\circ = \mathrm{id}_I$. For the other composite:

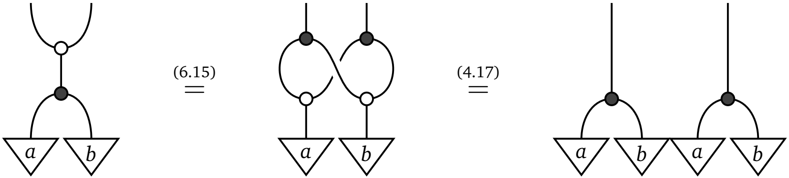


$\square$

6.3 Bialgebras

Lemma 6.24. In a braided monoidal category, if a monoid and comonoid interact as a bialgebra, then the copyable states for are a monoid under .

Proof. Associativity is immediate. Unitality comes down to third bialgebra law: is copyable for . Have to prove well-definedness. Let a and b be copyable states for .



Hence -copyable states are indeed closed under . □

6.3 Bialgebras

Example 6.27. Consider $\mathbb{C}^2$ in **FHilb**. Computational basis $\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$ gives dagger Frobenius structure . Orthogonal basis $\{\begin{pmatrix}e^{i\varphi}\\e^{i\theta}\end{pmatrix}, \begin{pmatrix}e^{i\varphi}\\-e^{i\theta}\end{pmatrix}\}$ gives dagger Frobenius structure .
Complementary, but only a bialgebra if $\varphi = \theta = 0$.

Definition 6.28. In a braided monoidal dagger category, two dagger symmetric Frobenius structures are *strongly complementary* when they are complementary, and also form a bialgebra.

Strongly complementary pairs have extra nice properties.

6.3 Bialgebras

Theorem 6.29. In a braided monoidal dagger category, given strongly complementary symmetric dagger Frobenius structures, the states that are self-conjugate, copyable and deletable for $(\curlyvee, \circ)$ form a group under $\curlywedge$.

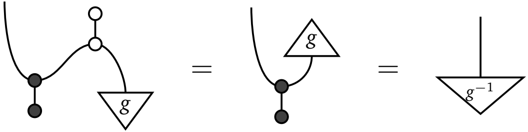
Proof. By Theorem 6.24 they form a monoid, and by Lemma 6.11 every element of this monoid has a left and right inverse. □

Theorem 6.30. In **FHilb**, strongly complementary symmetric dagger Frobenius structures, one of which is commutative, correspond to finite groups.

Proof. Suppose $\curlyvee$ is commutative. By Theorem 6.29 the states which are self-conjugate, copyable and deletable for $(\curlyvee, \circ)$ form a group for $\curlywedge$. But by the classification theorem for commutative dagger Frobenius structures, there is an entire basis of such states for $\curlyvee$. □

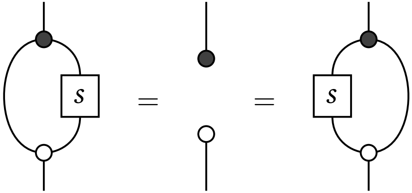
6.3 Bialgebras

For symmetric dagger Frobenius structures in **FHilb**, one of which is commutative, the 'black-white snake' is linear extension of $g \mapsto g^{-1}$:



Same calculation for complementary Frobenius structures in **Rel**.

Definition 6.31. An *antipode* for a monoid $(A, \ , \ )$ and comonoid $(A, \ , \ )$ in a monoidal category is a morphism $A \xrightarrow{s} A$ satisfying

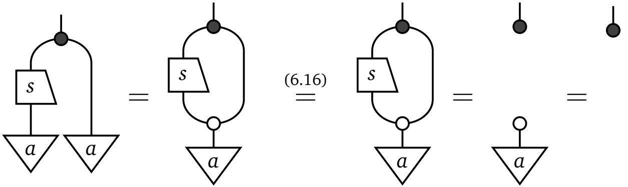


A *Hopf algebra* is a bialgebra with an antipode.

6.3 Bialgebras

Theorem ??. In a braided monoidal category, given a Hopf algebra, the states which are copied by the comultiplication and deleted by the counit form a group under the multiplication.

Proof. The states which are copied by the comultiplication form a monoid. Acting on an element by the antipode gives a left inverse:



(4)

Similarly, acting by the antipode also gives a right inverse. □

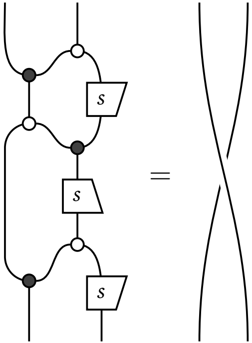
Corollary 6.34. In **Set**, Hopf algebras are exactly groups.

Proof. The only comonoids in **Set** are built from the diagonal and terminal morphisms, which copy and delete every element of the underlying set.

6.4 Qubit gates

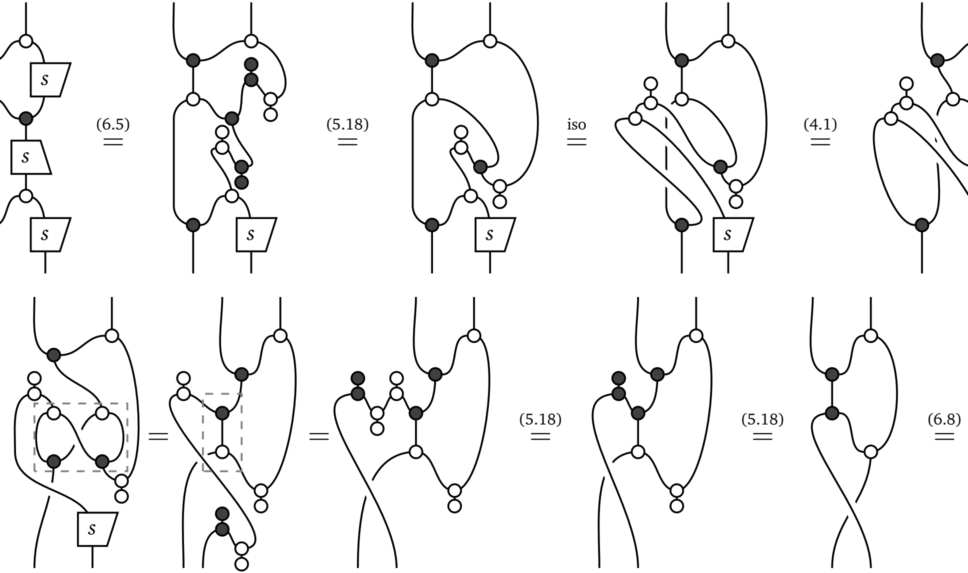
Graphical calculus can describe useful gates in quantum computing.

Theorem 6.35. In a braided monoidal dagger category, let ($\curlywedge$, $\bullet$) and ($\curlyvee$, $\circ$) be complementary classical structures. Then the following holds, if an only if the first bialgebra law holds:



6.4 Qubit gates

Proof. We use the following graphical argument:



6.4 Qubit gates

Example 6.36. In **FHilb**, fix A to be qubit $\mathbb{C}^2$; let $(\curlywedge, \bullet)$ copy computational basis $\{|0\rangle, |1\rangle\}$, and $(\curlyvee, \circ)$ copy the X basis.
Then the three antipodes s become identities.

The three unitaries indeed reduce to three CNOT gates: negate second qubit if the first (control) qubit is $|1\rangle$, do nothing otherwise.

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

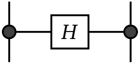
Fix these two classical structures for the rest of this chapter. The relationship between them is $|+\rangle = |0\rangle + |1\rangle$, and $|-\rangle = |0\rangle - |1\rangle$.
Hence they are transported into each other by the *Hadamard gate*:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \boxed{H}$$

6.4 Qubit gates

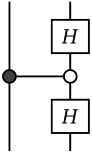
Lemma 6.37. The CZ gate in **FHilb** can be defined as follows.

CZ :=



Proof. Rewrite as:

CZ $\overset{(5.13)}{=}$



Hence

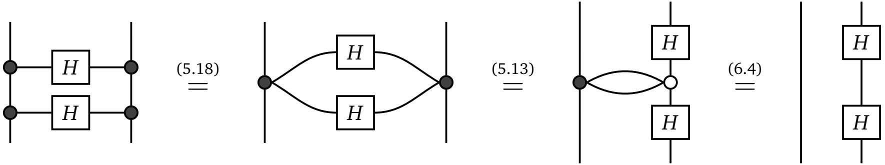
$$\mathrm{CZ} = (\mathrm{id} \otimes H) \circ \mathrm{CNOT} \circ (\mathrm{id} \otimes H) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

□

6.4 Qubit gates

Proposition 6.39. If $(A, \bullet)$ and $(A, \circ)$ complementary classical structures in braided monoidal dagger category, and $A \xrightarrow{H} A$ satisfies $H \circ H = \mathrm{id}_A$, then CZ makes sense and satisfies $\mathrm{CZ} \circ \mathrm{CZ} = \mathrm{id}$.

Proof. Easy graphical manipulation:

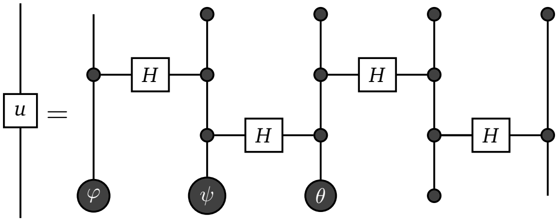


$\square$

6.4 Qubit gates

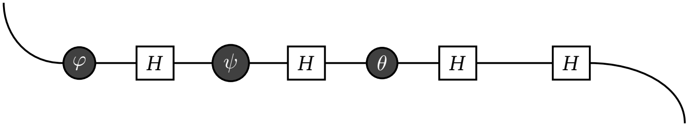
Single-qubit unitaries can be implemented via *Euler angles*: unitary $\mathbb{C}^2 \xrightarrow{u} \mathbb{C}^2$ allows phases φ, ψ, θ with $u = Z_\theta \circ X_\psi \circ Z_\varphi$, where Z_θ is rotation in Z basis over angle θ, and X_φ in X basis over angle φ.

Theorem 6.40. If unitary $\mathbb{C}^2 \xrightarrow{u} \mathbb{C}^2$ in **FHilb** has Euler angles φ, ψ, θ,

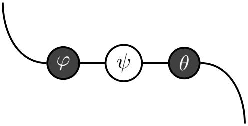


6.4 Qubit gates

Proof. Use phased spider theorem to reduce to:



But by transport lemma, this is just:



which equals u, by definition of the Euler angles. $\square$

Chapter 7

Complete positivity

7.1 Completely positive maps

Suppose machine produces quantum systems with Hilbert space H.

Two buttons: one produces state $v \in H$, another state $w \in H$.
You receive the system, but can't see machine operating.
All you know is, a coin is flipped to decide which button to press.

Taking this into account, the state of the system you receive can't be described by an element of H. The system is in a *mixed state*.

Definition 0.65. A *density matrix* on a Hilbert space H is a positive map $H \xrightarrow{\rho} H$. It is *normalized* when $\mathrm{Tr}(\rho) = 1$. It is *pure* when $\rho = |\psi\rangle\langle\psi|$ for some $\psi \in H$; otherwise, it is *mixed*.

Set of density matrices is *convex*.

Definition 0.71. For Hilbert spaces H and K, the *partial trace over K* is the unique linear map $\mathrm{Tr}_K : \mathbf{Hilb}(H \otimes K, H \otimes K) \to \mathbf{Hilb}(H, H)$ satisfying $\mathrm{Tr}_K(\rho \otimes \sigma) = \mathrm{Tr}(\sigma) \cdot \rho$.

Partial trace of pure state can be mixed.

7.1 Completely positive maps

Mixed version of measurement:

Definition 0.69. A *positive operator-valued measure (POVM)* on a Hilbert space H is a family of positive maps $H \xrightarrow{f_i} H$ satisfying

$$\sum_i f_i = \mathrm{id}_H.$$

Every projection-valued measure $\{p_i\}$ gives rise to a positive operator–valued measure in a canonical way, by choosing $f_i = p_i$.

Definition 0.63 (Born rule). For a positive operator–valued measure $\{f_i\}$ on a system with normalized density matrix $H \xrightarrow{\rho} H$, the *probability of outcome i* is $\langle\psi|f_i|\psi\rangle$.

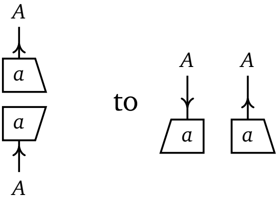
7.1 Completely positive maps

Will now develop mixed states *categorically*, in 4 steps.
So far have defined *pure state* as morphism $I \xrightarrow{a} A$.

Step 1: consider $p = a \circ a^\dagger : A \to A$ instead of $I \xrightarrow{a} A$.
This is really just a switch of perspective: we can recover a from p up to a phase, which is physically unimportant.

7.1 Completely positive maps

Step 2: switch from



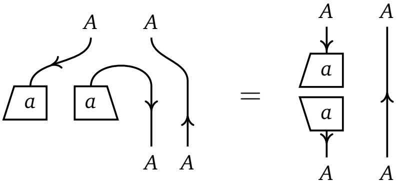
Instead of $A \to A$, may take names $I \to A^* \otimes A$, so no information lost.

Definition 7.1. A *positive matrix* is a morphism $I \xrightarrow{m} A^* \otimes A$ that is the name $\ulcorner f^\dagger \circ f \urcorner$ of a positive morphism for some $A \xrightarrow{f} B$. If we can choose $B = I$, we call m a *pure state*.

Will sometimes write $\sqrt{m}$ for f to indicate that m has a 'square root' and is hence positive. However, $\sqrt{m}$ is by no means unique.

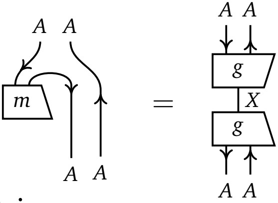
7.1 Completely positive maps

Step 3: move from positive matrix $I \xrightarrow{m} A^* \otimes A$ to multiplication $A^* \otimes A \to A^* \otimes A$ on left with m; compare Cayley embedding.



Loses no information:

Lemma 7.3. In **FHilb**, if a morphism $I \xrightarrow{m} A^* \otimes A$ satisfies

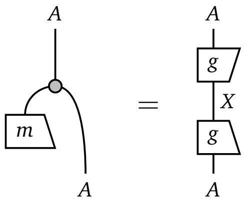


then it is a positive matrix. □

7.1 Completely positive maps

Step 4: Recognize pants, upgrade to arbitrary Frobenius structure.

Definition 7.4. A *mixed state* of a dagger Frobenius structure $(A, \text{⅄}, \text{ǫ})$ in a monoidal dagger category is a morphism $I \xrightarrow{m} A$ with



for some object X and some morphism $A \xrightarrow{g} X$.

Will sometimes write $\sqrt[\circ]{m}$ instead of g, even though not unique.

7.1 Completely positive maps

Example 7.5. Mixed states in our example categories:

- Recall pair of pants on $A = \mathbb{C}^n$ in **FHilb** is n-by-n matrices. Mixed states are n-by-n matrices m satisfying $m = \sqrt{m}^\dagger \circ \sqrt{m}$ for some n-by-m matrix $\sqrt{m}$: precisely *density matrices*.
- Dagger Frobenius structures in **FHilb** are finite-dimensional C*-algebras A. Mixed states $I \to A$ are elements $a \in A$ satisfying $a = b^* b$ for some $b \in A$; usually called the *positive* elements.
- Special dagger Frobenius structure in **Rel** correspond to groupoids $\mathbf{G}$. Mixed states are subsets R closed under inverses, and such that $g \in R$ implies $\mathrm{id}_{\mathrm{dom}(g)} \in R$.

7.1 Completely positive maps

What is the accompanying notion of morphism?

Individual morphisms are physical processes; free or controlled time evolution, preparation, or measurement. So should take (mixed) states to (mixed) states, and be determined by behaviour on (mixed) states.

Definition 7.6. Let $(A, \curlywedge, \circ)$ and $(B, \curlywedge, \bullet)$ be dagger Frobenius structures in dagger monoidal category. A *positive map* is morphism $A \xrightarrow{f} B$ such that $I \xrightarrow{f \circ m} B$ is mixed state when $I \xrightarrow{m} A$ is mixed state.

Warning: different from *positive-semidefinite* morphisms $f = g^\dagger \circ g$, abbreviated to *positive morphisms*.

7.1 Completely positive maps

Not yet the 'right' morphisms: forgot compound systems!
If f and g are physical channels, then so is $f \otimes g$.

Specifically, $f \otimes \mathrm{id}_E$ should be positive map for any Frobenius structure E and any positive map $A \xrightarrow{f} B$. Might only be interested in A, but can never be sure it's isolated from environment E.

Definition 7.7. Let $(A, \;, \;)$ and $(B, \;, \;)$ be dagger Frobenius structures in a dagger monoidal category. A *completely positive map* is a morphism $A \xrightarrow{f} B$ such that $f \otimes \mathrm{id}_E$ is a positive map for any dagger Frobenius structure $(E, \;, \;)$.

7.1 Completely positive maps

Example 7.8. Completely positive maps in **FHilb**:

- *Unitary evolution*: letting an n-by-n matrix m evolve freely along unitary u to $u^\dagger \circ m \circ u$; can phrase it as $A^* \otimes A \xrightarrow{u_* \otimes u} A^* \otimes A$ for $A = \mathbb{C}^n$.
- *Measurement*: if $A \xrightarrow{p_1, \ldots, p_n} A$ is a POVM, then $|i\rangle \mapsto p_i$ is completely positive $\mathbb{C}^n \xrightarrow{p} A^* \otimes A$. Conversely, if p completely positive map preserving units, $\{p(|1\rangle), \ldots, p(|n\rangle)\}$ is POVM.

Definition 7.9. Let G and H be the sets of morphisms of groupoids **G** and **H**. A relation $G \to H$ is said to *respect inverses* when $g \sim h$ implies $g^{-1} \sim h^{-1}$ and $\mathrm{id}_{\mathrm{dom}(g)} \sim \mathrm{id}_{\mathrm{dom}(h)}$.

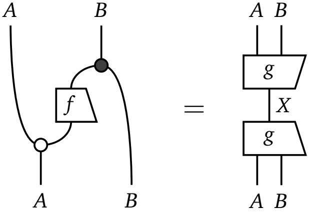
Proposition 7.10. A morphism $\mathbf{G} \xrightarrow{R} \mathbf{H}$ in **Rel** is completely positive if and only if it respects inverses.

7.2 Categories of completely positive maps

Definition of completely positive map was *operational*,
will now reformulate in *structural* form.

Need category to be *positive monoidal*: $f \otimes \mathrm{id}_E \geq 0 \implies f \geq 0$.

Lemma 7.14. In a positively monoidal braided dagger category, if $f : (A, \curlywedge_{\circ}, \circ) \to (B, \curlywedge_{\bullet}, \bullet)$ is completely positive, then

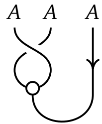


for some object X and some morphism $A \otimes B \xrightarrow{g} X$.

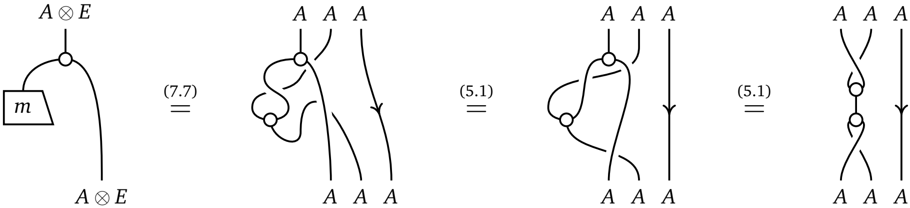
This is called the *CP–condition*.

7.2 Categories of completely positive maps

Proof. Let $E = A \otimes A^*$ be pair of pants, define $I \xrightarrow{m} A \otimes E$ as:

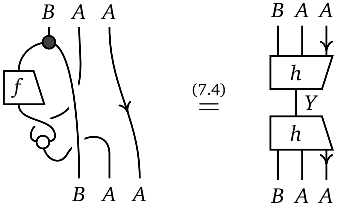


Then m is a mixed state:

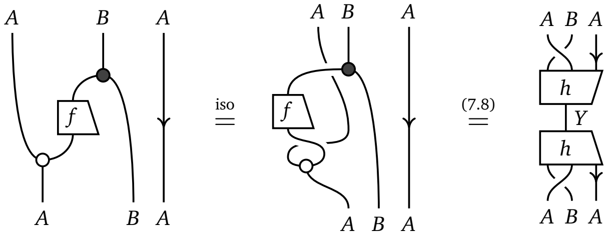


7.2 Categories of completely positive maps

Since f is completely positive, so $(f \otimes \mathrm{id}_E) \circ m$ is a mixed state:



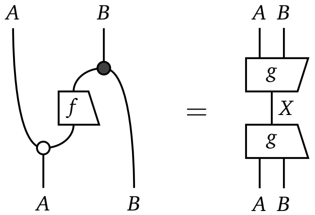
for some object Y and morphism h. Hence:



CP–condition then follows from positively monoidal. □

7.2 Categories of completely positive maps

CP–condition:



Striking similarity to oracles, Frobenius law.

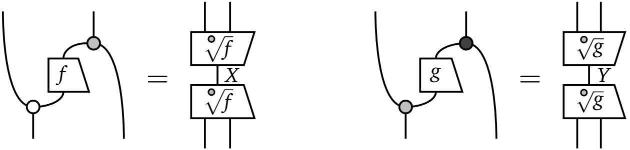
Object X is also called the *ancilla system*.
Map g is called a *Kraus morphism*, written $\sqrt[\circ]{f}$ although not unique.

Will now prove converse; need to show CP–condition well-behaved.

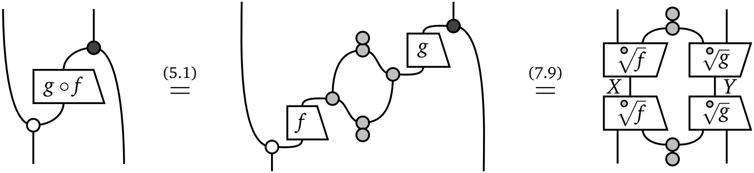
7.2 Categories of completely positive maps

Lemma 7.16 (CP maps compose). In a monoidal dagger category, let $(A, \curlywedge, \circ)$, $(B, \curlywedge, \circ)$, and $(C, \curlywedge, \bullet)$ be special dagger Frobenius structures. If $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ satisfy the CP condition, so does $g \circ f$.

Proof. Since f and g satisfy the CP condition:



Then we perform the following calculation:

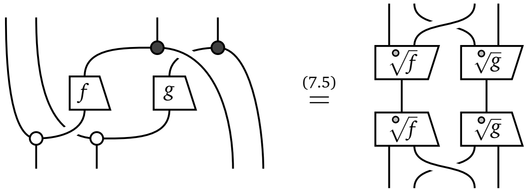


This uses the special law to insert a "handle" $d \bullet \diamond$. □

7.2 Categories of completely positive maps

Lemma 7.17 (Product CP maps).If $(A, \curlywedge_{\circ}, \circ) \xrightarrow{f} (B, \curlywedge_{\bullet}, \bullet)$ and $(C, \curlywedge_{\circ}, \circ) \xrightarrow{g} (D, \curlywedge_{\bullet}, \bullet)$ are maps between dagger Frobenius structures in a braided monoidal dagger category that satisfy CP–condition, then so is $(A, \curlywedge_{\circ}, \circ) \otimes (C, \curlywedge_{\circ}, \circ) \xrightarrow{f\otimes g} (B, \curlywedge_{\bullet}, \bullet) \otimes (D, \curlywedge_{\bullet}, \bullet)$.

Proof. Suppose $\sqrt[\circ]{f}$ and $\sqrt[\circ]{g}$ are Kraus morphisms for f and g. Then:



□

7.2 Categories of completely positive maps

Can now show that the CP–condition characterizes completely positive maps.

Theorem 7.18. Let $(A, \quad, \quad)$ and $(B, \quad, \quad)$ be special dagger Frobenius structures, $A \xrightarrow{f} B$ morphism in braided monoidal dagger category that is positively monoidal. The following are equivalent:

(a) f is completely positive;

(b) $f \otimes \mathrm{id}_E$ is positive map for all $E = (X^* \otimes X, \quad, \quad)$;

(c) f satisfies the CP–condition.

Proof. (a) $\Rightarrow$ (b) clear; (b) $\Rightarrow$ (c) already shown; (c) $\Rightarrow$ (a) follows from previous two lemmas. $\square$

7.2 Categories of completely positive maps

Main construction: turn compact dagger category **C** modeling pure states into new compact dagger category $\mathrm{CP}[\mathbf{C}]$ of mixed states.

Definition ??. Let **C** be a monoidal dagger category. Define a new category $\mathrm{CP}[\mathbf{C}]$ as follows: objects are special dagger Frobenius structures in **C**, and morphisms are completely positive maps.

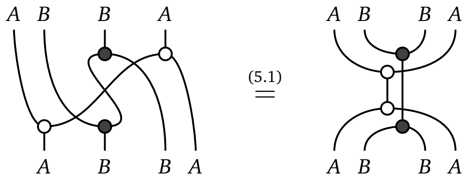
7.2 Categories of completely positive maps

Proposition 7.22 (CP preserves tensors). If **C** is a braided monoidal dagger category, then $\mathrm{CP}[\mathbf{C}]$ is a monoidal category:

- the tensor product of objects is product comonoid;
- the tensor product of morphisms is well-defined by lemma;
- the tensor unit is I with multiplication $I \otimes I \xrightarrow{\rho_I} I$ and unit $I \xrightarrow{\mathrm{id}_I} I$;
- the coherence isomorphisms α, λ, and ρ are inherited from **C**.

If **C** is a symmetric monoidal category, then so is $\mathrm{CP}[\mathbf{C}]$.

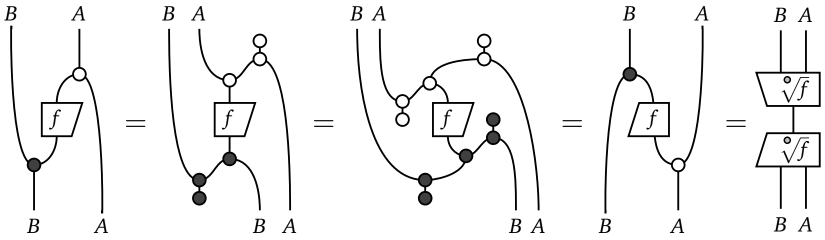
Proof. If **C** symmetric, swap maps are CP by Frobenius:



Hence, in that case, $\mathrm{CP}[\mathbf{C}]$ is symmetric monoidal. □

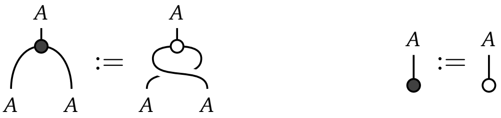
Lemma 7.25 (CP preserves daggers). Let $(A, ,)$ and $(B, ,)$ be special dagger Frobenius structures in a braided monoidal dagger category. If $A \xrightarrow{f} B$ satisfies CP–condition, so does $B \xrightarrow{f^\dagger} A$.

Proof.



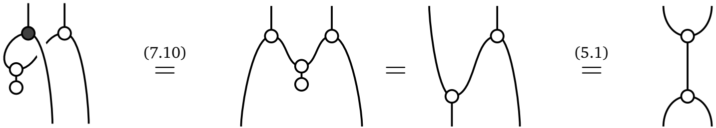
7.2 Categories of completely positive maps

Lemma 7.24 (CP preserves duals). Let $(A, \;, \;)$ be a special dagger Frobenius structure in a braided monoidal dagger category **C**, and:



Then $(A, \;, \;) \dashv (A, \;, \;)$ in $\mathrm{CP}[\mathbf{C}]$.
If **C** symmetric monoidal, both objects are dagger dual in $\mathrm{CP}[\mathbf{C}]$.

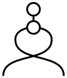
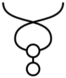
Proof. Define $\;:=\;\colon I \to R \otimes L$.



Also $\;:=\;\colon L \otimes R \to I$ is CP.
Because composition in $\mathrm{CP}[\mathbf{C}]$ is as in **C**, snake equations come down precisely to the Frobenius law. Thus $L \dashv R$ in $\mathrm{CP}[\mathbf{C}]$. □

7.2 Categories of completely positive maps

If **C** symmetric,

 $:=$  $: L \otimes R \longrightarrow I$  $:=$  $: R \otimes L \longrightarrow I$

are CP: composition of CP swap map and adjoint of CP map.
So L and R dagger dual objects in $\mathrm{CP}[\mathbf{C}]$. □

7.2 Categories of completely positive maps

Summary:

Theorem 7.26 (CP is compact).
If **C** braided monoidal dagger, $\mathrm{CP}[\mathbf{C}]$ monoidal dagger with duals.
If **C** symmetric monoidal dagger, $\mathrm{CP}[\mathbf{C}]$ compact dagger. □

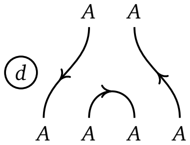
Duals fabricated out of thin air?
No: Frobenius structures have duals, so $\mathrm{CP}[\mathbf{C}_{\text{duals}}] = \mathrm{CP}[\mathbf{C}]$.

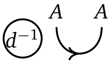
- $\mathrm{CP}[\mathbf{FHilb}]$: fin-dim C*-algebras and completely positive maps
- $\mathrm{CP}[\mathbf{Rel}]$: groupoids and inverse-respecting relations

Next: look at subcategories of quantum/classical structures.

7.3 Quantum structures

Definition 7.34. A *quantum structure* is a dagger Frobenius structure on $A^* \otimes A$ in a monoidal dagger category of the form





for an object A and an invertible scalar $I \xrightarrow{d} I$.

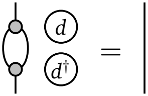
As far away from classical structures as possible:

- In **FHilb**: *matrix algebras* $\mathbb{M}_n$;
  normalizing scalar is (necessarily) $d = \frac{1}{\sqrt{n}}$.
- In **Rel**: *indiscrete groupoids*;
  normalizing scalar is (necessarily) $d = 1$.

7.3 Quantum structures

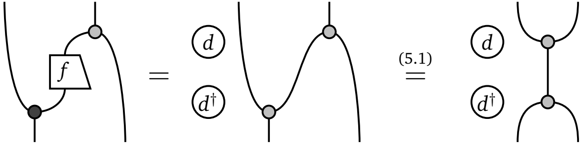
Remark 7.36. Not quite pair of pants; normalizing scalar bit ugly. But can pass to *monoidally equivalent* category without it.

arrows: completely positive maps,
objects: *normalizable* dagger Frobenius structures



for some invertible scalar $I \xrightarrow{d} I$.

Proof. Rescale normalizable Frobenius structure $(A, \ , \ , d)$ to special one $(A, d \bullet \ , d^{-1} \bullet \ )$. Isomorphism $A \xrightarrow{d \bullet \mathrm{id}_A} A$.



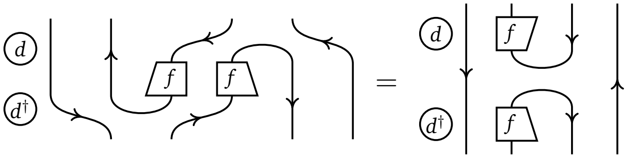
So can pretend all Frobenius structures are special as long as A *positive-dimensional*:

7.3 Quantum structures

Pure is special case of mixed.

Proposition 7.37 (CP embeds **C**). Let **C** be braided monoidal dagger category that is positive-dimensional. There is functor $\overline{P}\colon \mathbf{C} \to \overline{\mathrm{CP}}[\mathbf{C}]$ defined by letting $\overline{P}(A)$ be the quantum structure on $A^* \otimes A$, and $\overline{P}(f) = f_* \otimes f$ on morphisms. It is a monoidal functor that preserves daggers.

Proof. Let $A \xrightarrow{f} B$ in **C**. Check $\overline{P}(f)$ is completely positive.



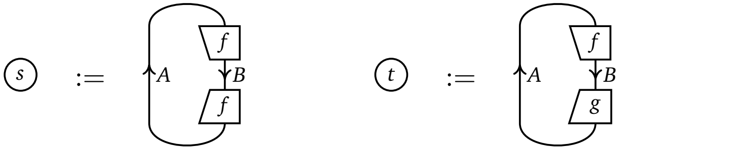
Daggers and tensor products in $\overline{\mathrm{CP}}[\mathbf{C}]$ are by definition as in **C**. The only other subtlety is that we have to fix a choice of scalar d for each object A. □

7.3 Quantum structures

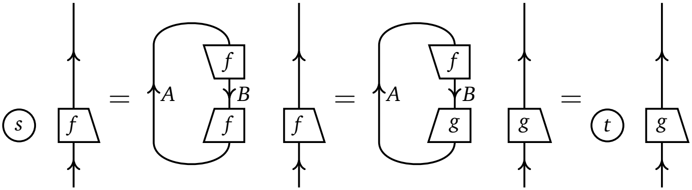
Well, embedding not quite faithful: only up to phase.

Lemma 7.38 (CP kills phases). If $\overline{P}(f) = \overline{P}(g)$ for $A \xrightarrow{f,g} B$, there are $I \xrightarrow{s,t} I$ with $s \bullet f = t \bullet g$ and $s^\dagger \bullet s = t^\dagger \bullet t$.

Proof. Define:



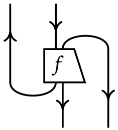
Then:



And:

7.3 Quantum structures

Definition 7.39. Let $\mathrm{CP_q}[\mathbf{C}]$ be subcategory of $\mathrm{CP}[\mathbf{C}]$ of quantum structures. Can abbreviate objects $A^* \otimes A$ to just A itself; CP–condition simplifies to positivity of



As before: if $\mathbf{C}$ is compact dagger category, so is $\mathrm{CP_q}[\mathbf{C}]$.

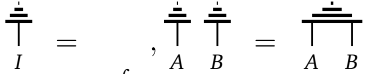
- $\mathrm{CP_q}[\mathbf{FHilb}]$: finite-dimensional Hilbert spaces H, completely positive maps $H^* \otimes H \to K^* \otimes K$.
- $\mathrm{CP_q}[\mathbf{Rel}]$: sets A, relations $A \times A \to B \times B$ with $(a,a) \sim (b,b)$ and $(a',a) \sim (b',b)$ when $(a,a') \sim (b,b')$.

7.3 Quantum structures

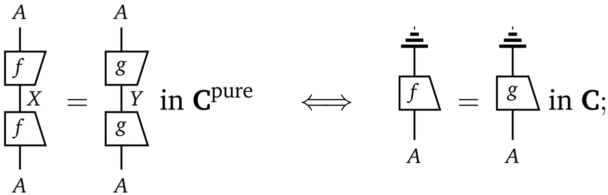
Any object A in $\mathrm{CP_q}[\mathbf{C}]$ has 'discarding' map $A \to I$, namely ;
in $\mathrm{CP_q}[\mathbf{FHilb}]$ this is the trace. Leads to axiomatization of $\mathrm{CP_q}[\mathbf{C}]$.

Definition 7.41. *Environment structure* for compact dagger $\mathbf{C}^{\text{pure}}$ is:

- a compact dagger category $\mathbf{C}$ of which $\mathbf{C}^{\text{pure}}$ is a compact dagger subcategory with the same objects;
- for each object A, a morphism : $A \to I$ in $\mathbf{C}$; such that:

(a) 

(b) for all $A \xrightarrow{f} X$ and $A \xrightarrow{g} Y$ in $\mathbf{C}^{\text{pure}}$:

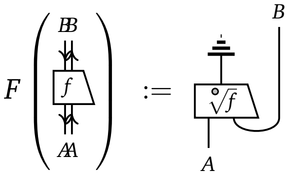


(c) for each $A \xrightarrow{f} B$ in $\mathbf{C}$ there is $A \xrightarrow{g} X \otimes B$ in $\mathbf{C}^{\text{pure}}$ such that:

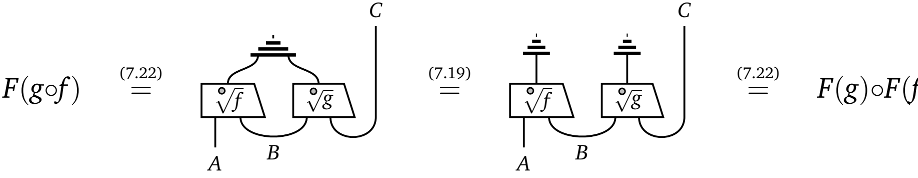
7.3 Quantum structures

Theorem 7.42. If compact dagger category $\mathbf{C}^{\mathrm{pure}}$ comes with environment structure, there is invertible functor $\mathrm{CP}_{\mathrm{q}}[\mathbf{C}^{\mathrm{pure}}] \xrightarrow{F} \mathbf{C}$ with $F(A) = A$ on objects and $F(f \otimes g) = F(f) \otimes F(g)$ on morphisms.

Proof. Define $F(A) = A$ on objects, and on morphisms:

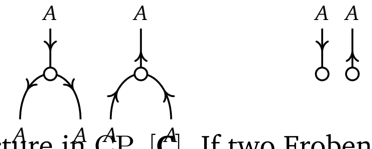


Functoriality:



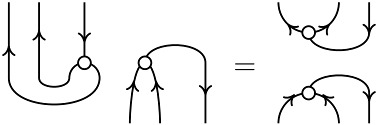
7.3 Quantum structures

Lemma 7.45. If $(A, ⅄, \circ)$ is a Frobenius structures in a braided monoidal category **C**, then

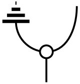


is a Frobenius structure in $\mathrm{CP_q}[\mathbf{C}]$. If two Frobenius structures in **C** are complementary, so are the two Frobenius structures in $\mathrm{CP_q}[\mathbf{C}]$.

Proof. CP–condition:



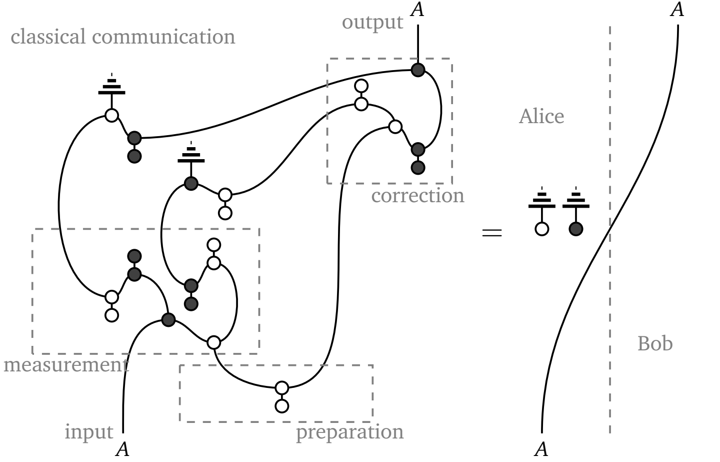
Classical communication:



is channel that carries classical information encoded in ⅄

7.3 Quantum structures

Theorem 7.46 (Quantum teleportation of mixed states). If $(A, \text{white multiplication}, \text{white unit})$ and $(A, \text{black multiplication}, \text{black unit})$ are complementary symmetric dagger Frobenius structures in a braided monoidal dagger category **C**, of which the white multiplication is commutative, then the following equation holds in $\mathrm{CP_q}[\mathbf{C}]$:



7.4 Classical structures

Definition 7.28. Let **C** be a braided monoidal dagger category. The category $\mathrm{CP}_c[\mathbf{C}]$ has as objects classical structures in **C**. Its morphisms are completely positive maps.

Again, if **C** is compact, so is $\mathrm{CP}_c[\mathbf{C}]$.
In fact, any object in $\mathrm{CP}_c[\mathbf{C}]$ is self-dual.

7.4 Classical structures

If **C** models pure state quantum mechanics, and $\mathrm{CP}[\mathbf{C}]$ mixed state quantum mechanics, then $\mathrm{CP}_{\mathrm{c}}[\mathbf{C}]$ models *statistical mechanics*.

Example 7.29. The category $\mathrm{CP}_{\mathrm{c}}[\mathbf{FHilb}]$ is monoidally equivalent to: objects are natural numbers, morphisms are m-by-n matrices with nonnegative real entries. The maps that preserve counit correspond to those matrices whose rows sum up to one, *i.e. stochastic matrices*.

Consistent with morphisms of classical structures of Chapter 5:

- Comonoid homomorphisms between classical structures: every column has single entry 1 and 0s elsewhere
- These are *deterministic* maps within stochastic setting
- These are *self-conjugate*: matrix entries are real numbers.

7.4 Classical structures

Compact dagger categories have no uniform copying/deleting. However, doesn't yet mean they model quantum mechanics. Classical mechanics might have copying, and quantum mechanics might not, but statistical mechanics has no copying either; rather: impossibility of *broadcasting* unknown mixed states.

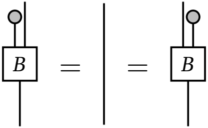
First make sure that there exist 'discarding' maps $A \to I$ in $\mathrm{CP}[\mathbf{C}]$:

Lemma 7.30. Let $(A, \;, \;)$ be a dagger Frobenius structure in a braided monoidal dagger category **C**. Then is completely positive. If $(A, \;, \;)$ is a classical structure, then is completely positive.

Proof. Verifying CP–condition for is easy. CP–condition for commutative can be rewritten into positive form easily using noncommutative spider theorem. □

7.4 Classical structures

Definition 7.31. let **C** be a braided monoidal dagger category. A *broadcasting map* for an object $(A, \;, \;)$ of $\mathrm{CP}[\mathbf{C}]$ is a morphism $A \xrightarrow{B} A \otimes A$ in $\mathrm{CP}[\mathbf{C}]$ satisfying:



Object $(A, \;, \;)$ is *broadcastable* if it allows a broadcasting map.

Note: concerns just single object, so weaker than uniform copying.

7.4 Classical structures

Lemma 7.32. Let **C** be a braided monoidal dagger category. Classical structures are broadcastable objects in $\mathrm{CP}[\mathbf{C}]$.

Proof. satisfies CP–condition. $\square$

In **FHilb** converse holds: *no-broadcasting theorem*.
So dagger Frobenius structure broadcastable iff classical structure.

Not so in **Rel**! Call category *skeletal* when only morphisms are endomorphisms.

7.4 Classical structures

Lemma 7.33. Broadcastable objects in CP[**Rel**] are precisely skeletal groupoids.

Proof. If **G** is skeletal, then $G \xrightarrow{B} G \times G$ given by

$$B = \{\big(g, (\mathrm{id}_{\mathrm{dom}(g)}, g)\big) \mid g \in G)\} \cup \{\big(g, (g, \mathrm{id}_{\mathrm{dom}(g)})\big) \mid g \in G\}$$

is broadcasting map.

Converse: use that broadcasting means

$$\begin{aligned} \{(g,g) \mid g \in G\} &= \{(g,h) \mid (g, (\mathrm{id}_{\mathrm{cod}(h)}, h)) \in B\} \\ &= \{(g,h) \mid (g, (h, \mathrm{id}_{\mathrm{dom}(h)})) \in B\}. \end{aligned}$$

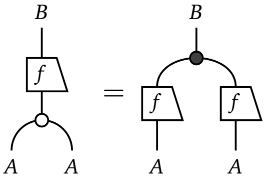
$\square$

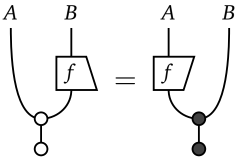
7.5 Interaction with linear structure

Theorem 7.51. If a braided monoidal dagger category **C** with duals has biproducts, then so does CP[**C**].

Proof. Main idea: show that $A \xrightarrow{i_A} A \oplus B$, $B \xrightarrow{i_B} A \oplus B$, $A \oplus B \xrightarrow{p_A} A$, and $A \oplus B \xrightarrow{p_B} B$ are *homomorphisms of involutive monoids*. □

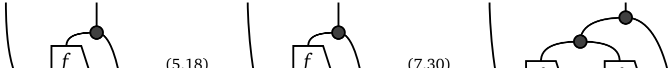
Definition 7.48. An *involutive homomorphism* is a morphism $(A, \curlywedge, \circ) \xrightarrow{f} (B, \curlywedge, \bullet)$ between dagger Frobenius structures in a monoidal dagger category satisfying:





Lemma 7.49. Involutive homomorphisms are completely positive.

Proof. Verify CP–condition:



7 Complete positivity

- Completely positive maps:
  pure states/evolutions vs mixed ones
- Categories of completely positive maps:
  everything happily in one category
- Quantum structures:
  axiomatization, teleportation
- Classical structures:
  operational view, broadcasting
- Interaction with linear structure

Chapter 8

Monoidal 2-categories

8.1 Monoidal 2-categories

Definition 8.1. A *2-category* $\mathbf{C}$ consists of the following data:

- a collection $\mathrm{Ob}(\mathbf{C})$ of *objects*;
- for any two objects A, B, a category $\mathbf{C}(A, B)$, with objects called *1-morphisms* drawn as $A \xrightarrow{f} B$, and morphisms μ called *2-morphisms* drawn as $f \overset{\mu}{\Longrightarrow} g$, or in full form as follows:

$$\begin{tikzcd} B & & A \arrow[ll, bend right=50, "g"'] \arrow[ll, bend left=50, "f"] \arrow[ll, phantom, "\Uparrow \mu"] \end{tikzcd}$$

8.1 Monoidal 2-categories

- for 2-morphisms $f \overset{\mu}{\Longrightarrow} g$ and $g \overset{\nu}{\Longrightarrow} h$, an operation called *vertical composition* given by their composite as morphisms in $\mathbf{C}(A,B)$:

$$
\begin{tikzcd}
B & A \arrow[l, "g"'] \arrow[l, bend right=60, "h"'] \arrow[l, bend left=60, "f"] \arrow[l, phantom, shift right=4, "\Uparrow \nu"] \arrow[l, phantom, shift left=4, "\Uparrow \mu"]
\end{tikzcd}
$$

- for any triple of objects A, B, C a *horizontal composition* functor:

$$\circ : \mathbf{C}(A,B) \times \mathbf{C}(B,C) \to \mathbf{C}(A,C)$$

$$
\begin{tikzcd}
C & A \arrow[l, bend right=40, "j \circ g"'] \arrow[l, bend left=40, "h \circ f"] \arrow[l, phantom, "\Uparrow \nu \circ \mu"]
\end{tikzcd}
\equiv
\begin{tikzcd}
C & B \arrow[l, bend right=40, "j"'] \arrow[l, bend left=40, "h"] \arrow[l, phantom, "\Uparrow \nu"] & A \arrow[l, bend right=40, "g"'] \arrow[l, bend left=40, "f"] \arrow[l, phantom, "\Uparrow \mu"]
\end{tikzcd}
$$

8.1 Monoidal 2-categories

- for any object A, a 1-morphism $A \xrightarrow{\mathrm{id}_A} A$ called the *identity 1-morphism*;
- a natural family of invertible 2-morphisms $f \circ \mathrm{id}_A \overset{\rho_f}{\Longrightarrow} f$ and $\mathrm{id}_B \circ f \overset{\lambda_f}{\Longrightarrow} f$ called the *left and right unitors*;
- a natural family of invertible 2-morphisms $(h \circ g) \circ f \xRightarrow{\alpha_{h,g,f}} h \circ (g \circ f)$ called the *associators*.

This structure is required to be *coherent*, meaning that any well-formed diagram built from the components of α, λ, ρ and their inverses under horizontal and vertical composition must commute.

As for monoidal categories, coherence follows just from the triangle and pentagon equations.

A 2-category is *strict* just when every λ_f, ρ_f, $\alpha_{h,g,f}$ is an identity.

8.1 Monoidal 2-categories

Theorem. *A monoidal category is a 2-category with one object.*

Proof. We sketch the correspondence with this table:

Monoidal category	**One-object 2-category**
Objects	1-morphisms
Morphisms	2-morphisms
Composition	Vertical composition
Tensor product	Horizontal composition
Unit object	Identity 1-morphism

The transformations α, λ and ρ are the same for both structures.

8.1 Monoidal 2-categories

Cat, the 2-category of categories, functors and natural transformations, is an important motivating example.

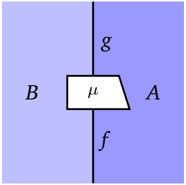
Definition. The 2-category **Cat** is defined as follows:

- **objects** are categories;
- **1-morphisms** are functors;
- **2-morphisms** are natural transformations;
- **vertical composition** is componentwise composition of natural transformations, with $(\mu \cdot \nu)_A := \mu_A \circ \nu_A$;
- **horizontal composition** is composition of functors.

8.1 Monoidal 2-categories

In this more general graphical calculus, objects are represented by regions, 1-morphisms by vertically-oriented lines, and 2-morphisms by vertices:

$$\begin{tikzcd} B & A \arrow[l, bend right=50, "g"'] \arrow[l, bend left=50, "f"] \arrow[l, phantom, "\Uparrow\mu"] \end{tikzcd} \quad \rightsquigarrow$$

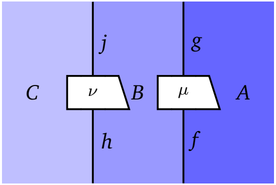


The graphical calculus is the *dual* of the pasting diagram notation.

8.1 Monoidal 2-categories

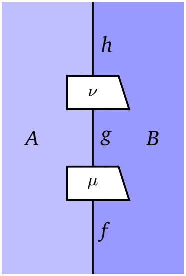
Horizontal and vertical composition is represented like this:

$$\begin{tikzcd} C & B \arrow[l, bend right, "j"'] \arrow[l, bend left, "h"] \arrow[l, phantom, "\Uparrow \nu"] & A \arrow[l, bend right, "g"'] \arrow[l, bend left, "f"] \arrow[l, phantom, "\Uparrow \mu"] \end{tikzcd} \quad \rightsquigarrow$$



$$= \quad \nu \circ \mu$$

$$\begin{tikzcd} A & & B \arrow[ll, bend right=60, "h"'] \arrow[ll, "g" description] \arrow[ll, bend left=60, "f"] \arrow[ll, phantom, shift right=3, "\Uparrow \nu"] \arrow[ll, phantom, shift left=3, "\Uparrow \mu"] \end{tikzcd} \quad \rightsquigarrow$$



$$= \quad \nu \cdot \mu$$

8.1 Monoidal 2-categories

When using the graphical notation, as for monoidal categories, the structures λ, ρ and α are not depicted.

There is also a correctness theorem, as we would expect.

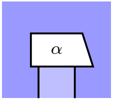
Theorem. (Correctness of the graphical calculus for a 2-category) *A well-formed equation between 2-morphisms in a 2-category follows from the axioms if and only if it holds in the graphical language up to planar isotopy.*

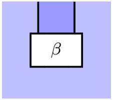
If we have only a single object A, which we may as well denote by a region coloured white, then the graphical calculus is identical to that of a monoidal category.

8.1 Monoidal 2-categories

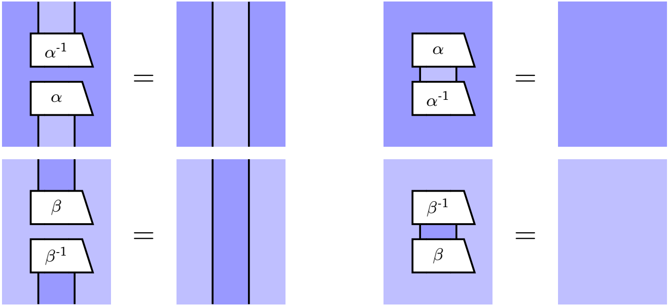
We can use the graphical calculus to define equivalence.

Definition. In a 2-category, an *equivalence* is a pair of 1-morphisms $A \xrightarrow{F} B$ and $B \xrightarrow{G} A$, and 2-morphisms $G \circ F \overset{\alpha}{\Longrightarrow} \mathrm{id}_A$ and $\mathrm{id}_B \overset{\beta}{\Longrightarrow} F \circ G$:



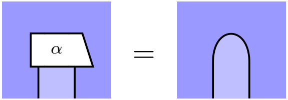


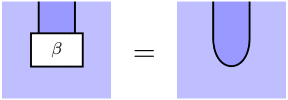
They must satisfy the following equations:



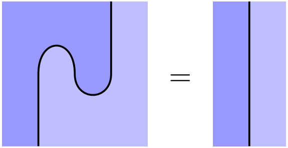
8.1 Monoidal 2-categories

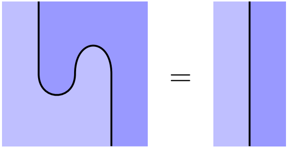
Definition. In a 2-category, a 1-morphism $A \xrightarrow{L} B$ has a *right dual* $B \xrightarrow{R} A$ when there are 2-morphisms $G \circ F \overset{\alpha}{\Longrightarrow} \mathrm{id}_A$ and $\mathrm{id}_B \overset{\beta}{\Longrightarrow} F \circ G$





satisfying the snake equations:





Theorem. In **Cat**, a duality $F \dashv G$ is exactly an adjunction $F \dashv G$ between F and G as functors.

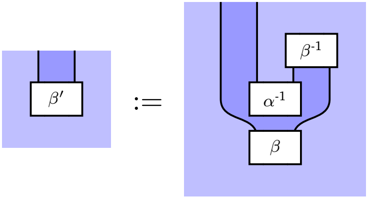
It may seem that adjunctions have largely been absent from this course. But now we see they have been everywhere!

8.1 Monoidal 2-categories

We now prove a nontrivial theorem relating equivalences and duals.

Theorem. In a 2-category, every equivalence gives rise to a dual equivalence.

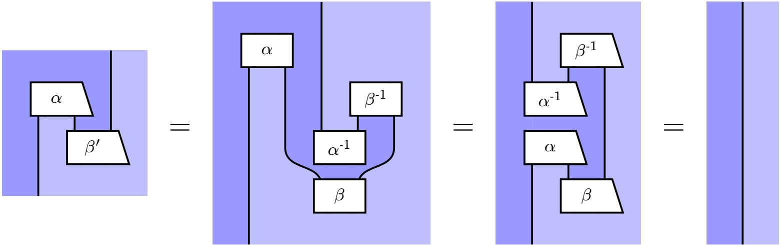
Proof. Suppose we have an equivalence in a 2-category, witnessed by invertible 2-morphisms α and β. Then we will build a new equivalence witnessed by α and β', with β' defined like this:



Since α' is composed from invertible 2-morphisms it must itself be invertible, and so it is clear that α' and β still give an equivalence.

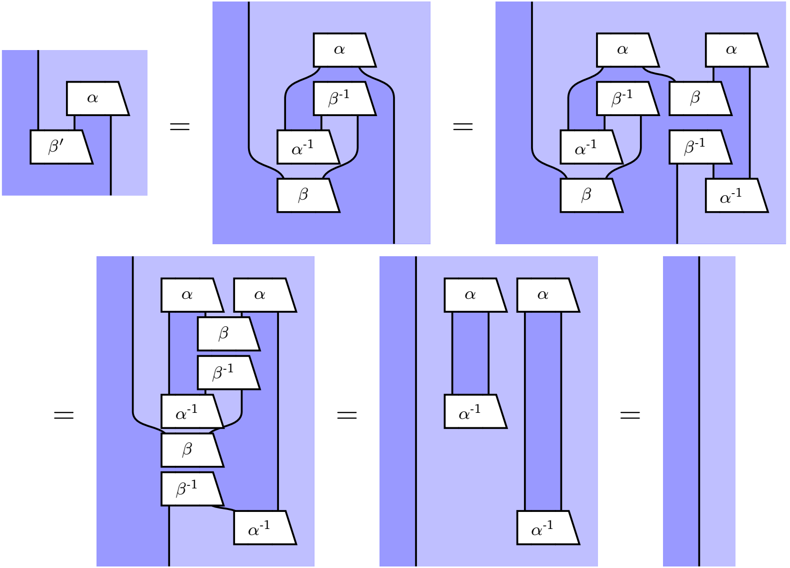
We now demonstrate that the adjunction equations are satisfied.

The first adjunction equation takes following form:



8.1 Monoidal 2-categories

The second is demonstrated as follows:



□

8.1 Monoidal 2-categories

Since monoidal categories are just 2-categories with one object, we immediately have the following corollary.

Corollary. In a monoidal category, if $A \otimes B \simeq B \otimes A \simeq I$, then $A \dashv B$ and $B \dashv A$.

8.1 Monoidal 2-categories

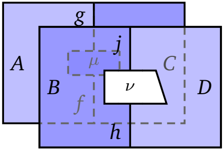
Monoidal 2-categories are hard to define. The definition is known, but it is long and complex. This is a big problem in the field!

Remember the 2d graphical calculus for 2-categories:

- objects correspond to planes;
- 1-morphisms correspond to wires;
- 2-morphisms correspond to vertices.

For monoidal 2-categories, we simply extend this into 3d.

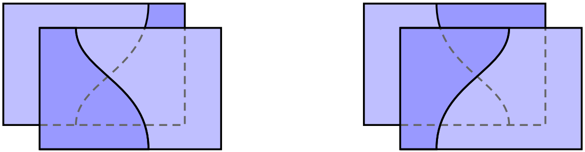
Tensor product. Given 2-morphisms $f \overset{\mu}{\Longrightarrow} g$ and $h \overset{\nu}{\Longrightarrow} j$, the their *tensor product* 2-morphism $\mu \boxtimes \nu$ is given like this:



8.1 Monoidal 2-categories

Interchange. Components can move freely in their separate layers.

The order of 1-morphisms in separate sheets can be *interchanged*:



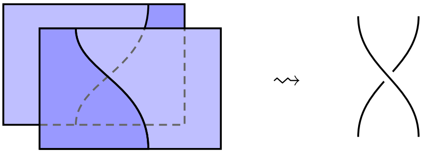
This process itself gives a 2-morphism, which is called an *interchanger*.

These two interchangers are inverse to each other.

Unit object. A monoidal 2-category has a *unit object* I, represented by a 'blank' region.

8.1 Monoidal 2-categories

Something interesting happens when we combine interchangers and the unit object. Consider the interchanger diagram, but with all 4 planar regions labelled by the unit object:



We obtain the graphical representation of a *braiding*.

8.1 Monoidal 2-categories

Recall the following result which we saw earlier.

Theorem. *A monoidal category is a 2-category with one object.*

We can now extend this as follows.

Theorem. *A braided monoidal category is a monoidal 2-category with one object.*

We can put this into context with notions of higher category.

Theorem. *A monoidal 2-category is a 3-category with one object.*

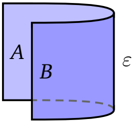
Corollary. *A braided monoidal category is a 3-category with one object and one 1-morphism.*

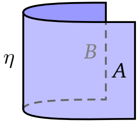
Conjecture. *A symmetric monoidal category is a 4-category with one object, one 1-morphism and one 2-morphism.*

The emerging pattern here is called the *periodic table*, and was predicted by Baez and Dolan in 1995.

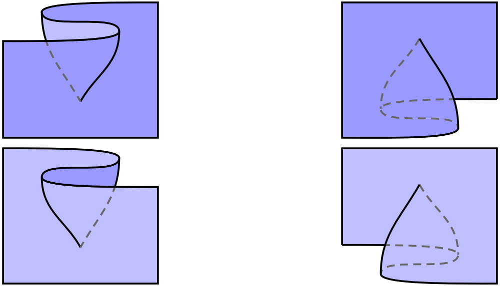
8.1 Monoidal 2-categories

Definition. In a monoidal 2-category, an object A has a *right dual* B when it can be equipped with 1-morphisms called *folds*



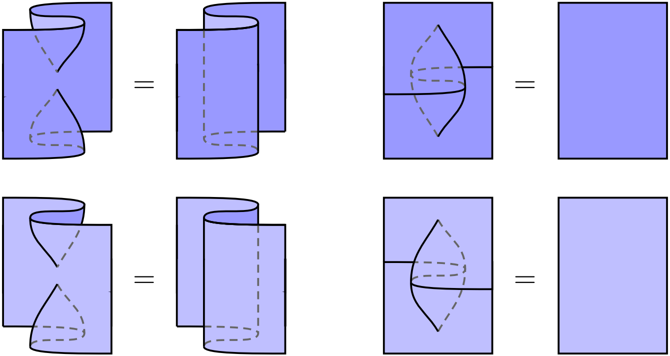


and invertible 2-morphisms called *cusps*:



8.1 Monoidal 2-categories

The invertibility equations look like this:

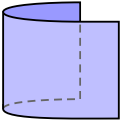
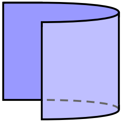


It's just like deforming a piece of fabric!

8.1 Monoidal 2-categories

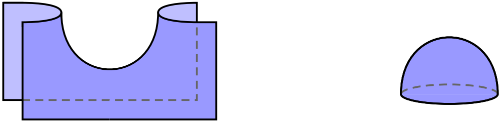
To capture all the structure of oriented manifolds, we must require that our fold morphisms *themselves* have duals.

To see what happens, let's investigate this duality:

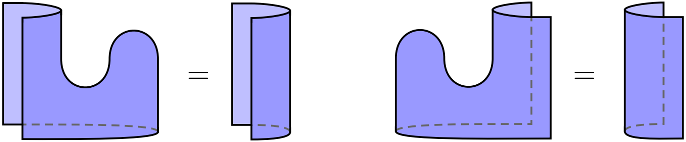

$\dashv$


8.1 Monoidal 2-categories

It has a unit and counit, which we draw like this:



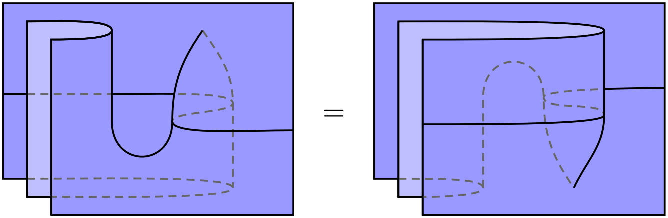
The snake equations for the duality then look like this:



Again, this makes sense in terms of deformations of surfaces!

8.1 Monoidal 2-categories

There is only one set of equations left to completely specify the behaviour of oriented surfaces. They look like this:



These are called the *cusp-flip equations*.

The *Cobordism Hypothesis* says that you can describe n-dimensional manifolds in a similar way.

8.2 2–Hilbert spaces

8.2 2–Hilbert spaces

Definition. A 2–Hilbert space is a **FHilb**-enriched dagger category which is Cauchy complete.

This categorifies the definition of an ordinary Hilbert space, as a Cauchy-complete inner product space.

Definition 8.23. For a 2–Hilbert space **H**, a *basis* is a set of objects of **H**, such that every object in **H** is a biproduct of elements of the basis in an essentially unique way.

Definition. A 2–Hilbert space is *finite-dimensional* when it has a finite basis.

8.2 2–Hilbert spaces

There are many analogies between Hilbert spaces and 2–Hilbert spaces.

- every finite-dimensional Hilbert space is of the form $\mathbb{C}^n$ up to isomorphism, while every finite-dimensional Hilbert space is of the form $\mathbf{FHilb}^n$ up to equivalence;
- Hilbert spaces have zero elements, while 2–Hilbert spaces have zero objects;
- Hilbert spaces have sums of elements $v + w$, while 2–Hilbert spaces have biproducts $A \oplus B$;
- in a Hilbert space we can multiply an element by any complex number, while in a 2–Hilbert space we can multiply an object by any Hilbert space;
- Hilbert spaces have an equality $\overline{\langle v|w\rangle} = \langle w|v\rangle$, while 2–Hilbert spaces have an isomorphism $\mathbf{H}(A,B)^* \simeq H(B,A)$;

8.2 2–Hilbert spaces

Definition. The symmetric monoidal 2-category **2Hilb** is built from the following structures:

- 0-cells are finite-dimensional 2–Hilbert spaces;
- 1-cells are linear functors, meaning $F(f+g) = F(f) + F(g)$;
- 2-cells are natural transformations.

This is a standard structure in higher representation theory.

There is a matrix calculus, just as for ordinary Hilbert spaces.

Definition. The symmetric monoidal 2-category $\mathbf{Mat}(\mathbf{FHilb})$ is built from the following structures:

- 0-cells are natural numbers;
- 1-cells are matrices of Hilbert spaces;
- 2-cells are matrices of linear maps.

8.3 Modelling quantum procedures

We can arrange cobordisms into monoidal categories.

Definition. The symmetric monoidal category $\mathbf{Cob_{12}}$ has objects given by compact oriented 1-manifolds, and morphisms given by diffeomorphism classes of compact oriented 2-manifolds with boundary.

Definition. The symmetric monoidal category $\mathbf{Cob_{012}}$ has objects given by compact oriented 0-manifolds, 1-morphisms given by compact oriented 1-manifolds with boundary, and 2-morphisms given by compact oriented 2-manifolds with boundary.

Definition. A *2d TQFT* is a symmetric monoidal functor:

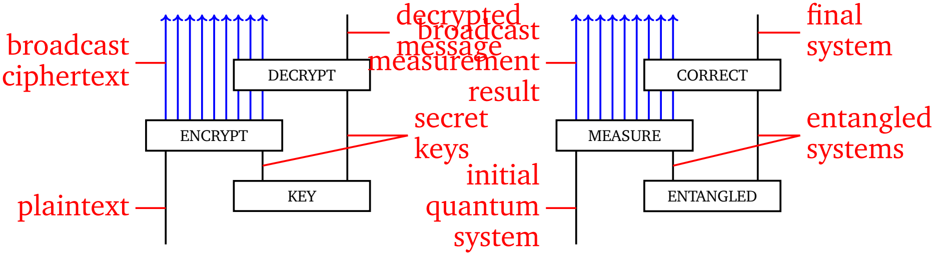
$$Z : \mathbf{Cob_{12}} \to \mathbf{FHilb}$$

Definition. An *extended 2d TQFT* is a symmetric monoidal functor:

$$Z : \mathbf{Cob_{012}} \to \mathbf{2Hilb}$$

8.3 Modelling quantum procedures

We will now consider a new perspective on quantum teleportation.



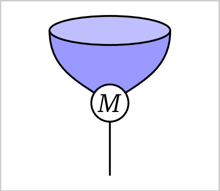
Encrypted communication   Quantum teleportation

New idea. We can make this precise using *defects* between topological quantum field theories.

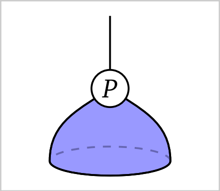
8.3 Modelling quantum procedures

Surfaces carry a commutative dagger Frobenius structure, so they describe the behaviour of classical information.

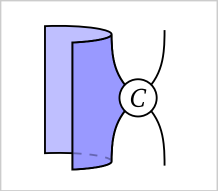
We now consider interactions between TQFTs.



Measurement



Preparation



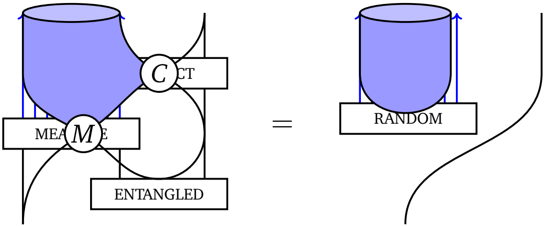
Controlled operation

We require these to be unitary, because *all* processes in physics and computer science are (arguably) unitary at a fundamental level.

This is a **123 TQFT with defects**.

8.3 Modelling quantum procedures

Here is the heuristic quantum teleportation diagram:

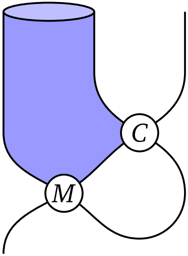


We make it rigorous with this equation between topological defects.

8.3 Modelling quantum procedures

We can use the topological formalism to prove interesting things.

We begin with the definition of quantum teleportation:



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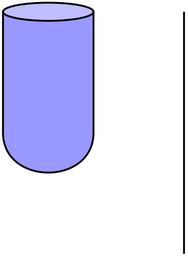
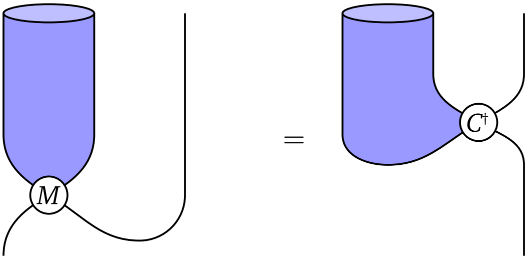


8.3 Modelling quantum procedures

We can use the topological formalism to prove interesting things.

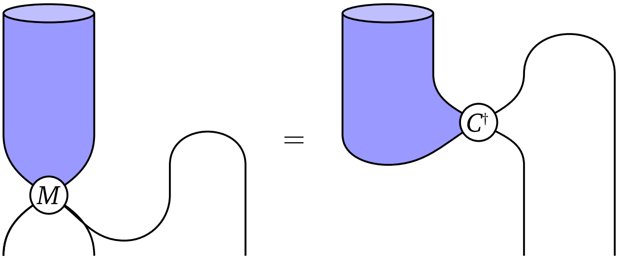
Apply $C^\dagger$:



8.3 Modelling quantum procedures

We can use the topological formalism to prove interesting things.

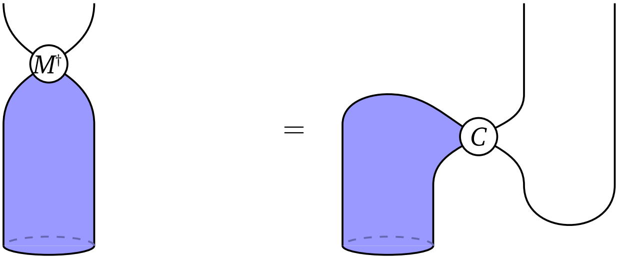
Bend down a wire:



8.3 Modelling quantum procedures

We can use the topological formalism to prove interesting things.

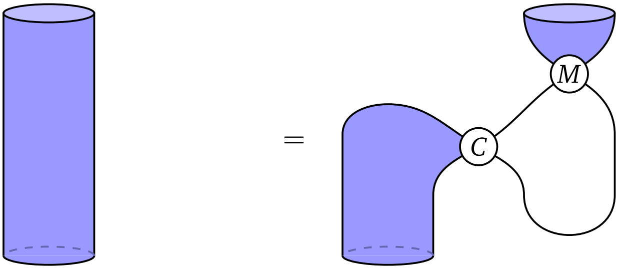
Take adjoints:



8.3 Modelling quantum procedures

We can use the topological formalism to prove interesting things.

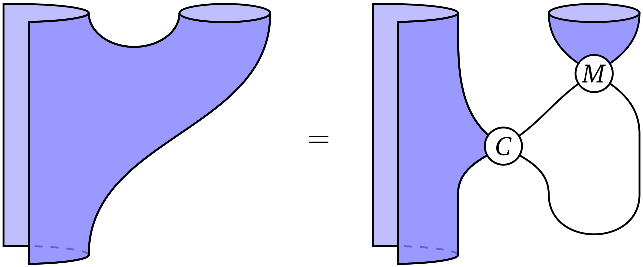
Apply M:



8.3 Modelling quantum procedures

We can use the topological formalism to prove interesting things.

Bend up the surface:

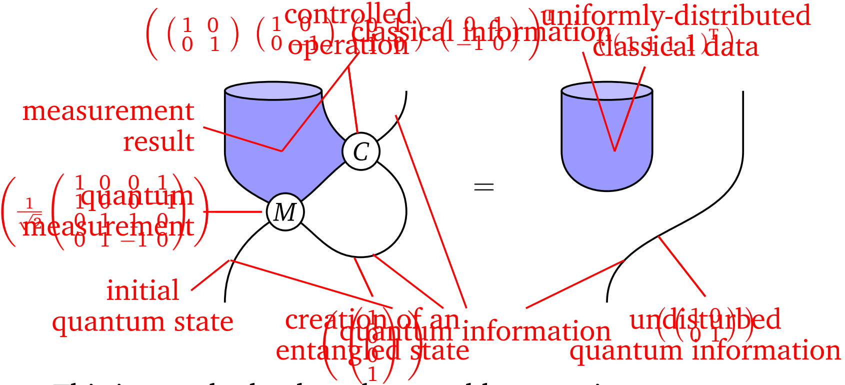


This is *dense coding*, another famous quantum procedure.

We have a *topological* proof of equivalence with teleportation, independent of the Hilbert space formalism.

8.3 Modelling quantum procedures

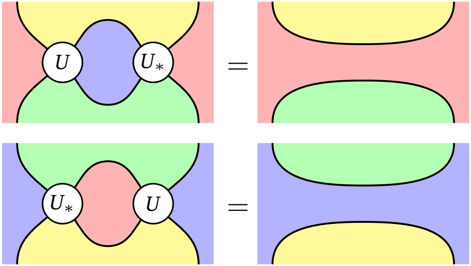
Theorem. Solutions to the teleportation equation in **2Hilb** correspond exactly to quantum teleportation schemes.



This is exactly the data that would appear in a quantum information textbook.

8.3 Modelling quantum procedures

Definition. In a pivotal dagger 2-category, a 4-valent vertex is *horizontally unitary* when the following equations hold:

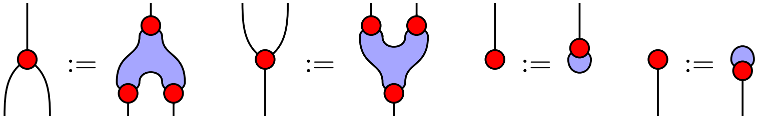


Warning: from here onwards we are dropping some scalar factors.

Theorem. A measurement vertex forms part of a teleportation protocol if and only if it is horizontally unitary.

8.3 Modelling quantum procedures

Given a measurement 2-morphism, we can define these composites:

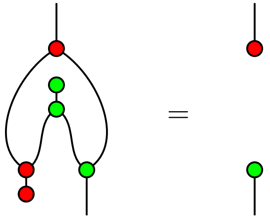


These form a commutative dagger Frobenius structure, since they are the transport of the pair of pants across a unitary.

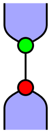
8.3 Modelling quantum procedures

Theorem. Given a pair of measurement defects on the same wire, the following properties are equivalent:

- The complementarity condition holds:

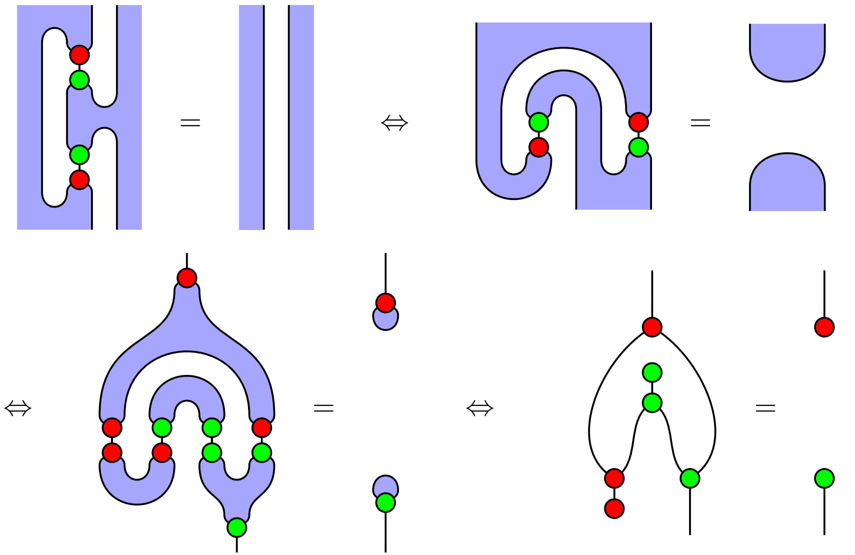


- This is horizontally unitary:



8.3 Modelling quantum procedures

Proof.



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