Introduction to Probability

Lecture 7: Independence, Covariance and Correlation

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Independence of Random Variables

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Two random variables X and Y are independent if for all values a, b:

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Using the joint probability distribution, the above is equivalent to for all a, b,

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All these definitions extend in the natural way to more than two variables!

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The definition of independence of X and Y implies the following factorisation formula: for any "suitable" sets A and B,

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Example

Let X and Y be two independent variables. Let $I = (a, b]$ be any interval and define $U := \mathbf{1}_{X \in I}$ and $V := \mathbf{1}_{Y \in I}$. Prove U and V are independent.

Answer

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Georges-Louis Leclerc de Buffon 1707–1788 (Source Wikipedia)

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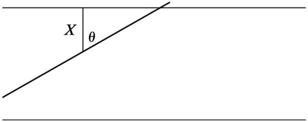

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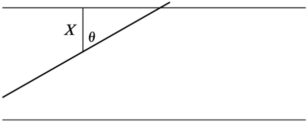

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Let X be the distance of the middle point of the needle to the closest parallel line. Needle intersects a line if hypotenuse of the triangle is less than $L/2$, i.e.,

$$\frac{X}{\cos(\theta)} < \frac{L}{2} \qquad \Leftrightarrow \qquad X < \frac{L}{2}\cos(\theta).$$

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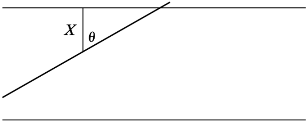

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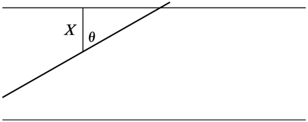

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Can be thought of as: 1. Sample the middle point of needle, 2. Sample the angle.

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This gives us a method to estimate π!

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- Two variables X, Y with $\mathbf{Cov}[X, Y] = 0$ are uncorrelated.

Illustration of 3 Cases for Cov $[X, Y]$

500 outcomes of randomly generated pairs of RVs (X, Y) with different joint distributions

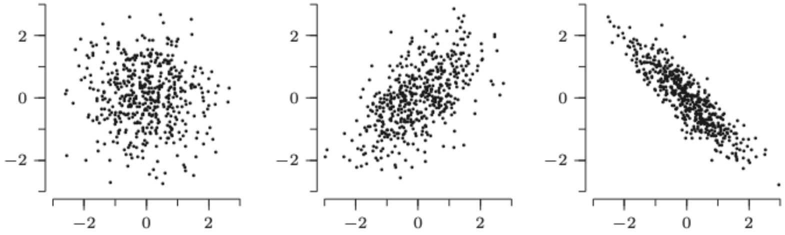

Fig. 10.1. Some scatterplots.
Source: Textbook by Dekking

1. What is the covariance (positive, negative, neutral)?

2. Where is the covariance the largest (in magnitude)?

Independence implies Uncorrelated

Example

Let X and Y be two independent random variables. Then X and Y are uncorrelated, i.e., $\mathbf{Cov}[X, Y] = 0$.

Answer

We give a proof for the discrete case:

Uncorrelated may not imply Independence

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$$1 = \mathbf{P}[X \cdot Y = 0] > \mathbf{P}[X = 0] \cdot \mathbf{P}[Y = 0] = 2/9.$$

Variance of Sums and Covariances

Variance of Sum Formula

- For any two random variables X, Y,

$$\mathbf{V}[X + Y] = \mathbf{V}[X] + \mathbf{V}[Y] + 2 \cdot \mathbf{Cov}[X, Y].$$

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Generalisation of the case where X and Y are even independent!

- For any random variables $X_1, X_2, \ldots, X_n$:

$$\mathbf{V}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{V}[X_i] + 2 \cdot \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathbf{Cov}[X_i, X_j].$$

Computing Variances of Sums of Uncorrelated Variables

Example

Recall the example where $X \in \{-1, 0, +1\}$ uniformly and $Y := \mathbf{1}_{X=0}$. Compute $\mathbf{V}[X + Y]$.

Answer

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If $\mathbf{V}[X] = 0$ or $\mathbf{V}[Y] = 0$, then it is defined as 0.

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Properties:

1. The correlation coefficient is scaling-invariant, i.e., $\rho(X, Y) = \rho(\alpha \cdot X, \beta \cdot Y)$ for any $\alpha, \beta > 0$.

Correlation Coefficient: Normalising the Covariance

The definition of covariance is not scaling invariant:

- If X increases by a factor of α, then $\mathbf{Cov}[X, Y]$ increases by a factor of α.

$\Rightarrow$ Even if X and Y both increase by α, then $\mathbf{Cov}[X, Y]$ will change. (Exercise: It changes by?)

Correlation Coefficient

Let X and Y be two random variables. The correlation coefficient $\rho(X, Y)$ is defined as:

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Properties:

1. The correlation coefficient is scaling-invariant, i.e., $\rho(X, Y) = \rho(\alpha \cdot X, \beta \cdot Y)$ for any $\alpha, \beta > 0$.
2. For any two random variables X, Y, $\rho(X, Y) \in [-1, 1]$.

Range of the Correlation Coefficient

Example

Verify that the correlation coefficients' range satisfies $\rho(X, Y) \in [-1, 1]$.

Answer

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Answer

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&= 2 \cdot (1 + \rho(X, Y)).
\end{aligned}$$