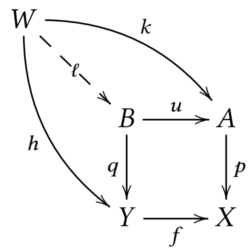
University of Cambridge
2022/23 Part II / Part III / MPhil ACS
Category Theory
Exercise Sheet 4
by Andrew Pitts

1. A *pullback square* in a category **C** is a commutative diagram of the form

$$\begin{array}{ccc} B & \xrightarrow{u} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[f]{} & X \end{array} \qquad p \circ u = f \circ q \tag{1}$$

with the following universal property:

for all **C**-objects W and **C**-morphisms $Y \xleftarrow{h} W \xrightarrow{k} A$ satisfying $f \circ h = p \circ k$, there is a unique **C**-morphism $\ell : W \to B$ satisfying $q \circ \ell = h$ and $u \circ \ell = k$



(a) Let **C** be a category and $f : Y \to X$ a morphism in **C**. Show that f is a monomorphism (see Exercise Sheet 1, question 4) if and only if

$$\begin{array}{ccc} Y & \xrightarrow{\mathrm{id}_Y} & Y \\ {\scriptstyle \mathrm{id}_Y}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[f]{} & X \end{array} \tag{2}$$

is a pullback square in **C**.

(b) If (1) is a pullback square and p is a monomorphism, show that q is a monomorphism.

(c) If (1) is a pullback square and p is a isomorphism, show that q is a isomorphism.

(d) Given an example of a pullback square (1) in the category **Set** of sets and functions, for which q is an isomorphism, but p is not a monomorphism. (Recall that in **Set**, monomorphisms and isomorphisms are given by the functions that are respectively injective and bijective.)

2. (a) Given morphisms $X' \xrightarrow{f} X$ and $Y \xrightarrow{g} Y'$ in a cartesian closed category **C**, show how to define a morphism $Y^X \to (Y')^{X'}$ in **C**.

(b) Given types A', A, B and B' in simply typed lambda calculus (STLC), give a term t satisfying

$$\diamond \vdash t : (A' \to A) \to (B \to B') \to (A \to B) \to (A' \to B')$$

If the semantics in a cartesian closed category of A', A, B and B' are the objects X', X , Y and Y' respectively, what is the semantics of t?

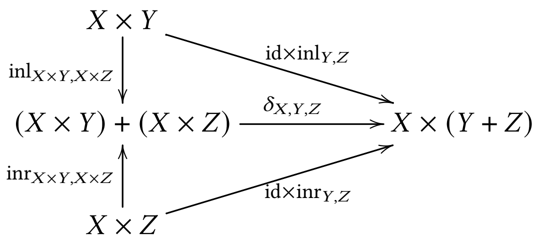
3. Let $\mathbf{C} = \mathbf{Set}^{\mathrm{op}}$ be the opposite category of the category **Set** of sets and functions.

   (a) State, without proof, what is the product in $\mathbf{C}$ of two objects X and Y.

   (b) Show by example that there are objects X and Y in $\mathbf{C}$ for which there is no exponential and hence that $\mathbf{C}$ is not a cartesian closed category.

4. [In this question I use the notation $X \xrightarrow{\mathrm{inl}_{X,Y}} X + Y \xleftarrow{\mathrm{inr}_{X,Y}} Y$ for the coproduct (Lecture 4) of two object X and Y in a category, since it will be clearer to make explicit the objects X and Y in the notation for the associated coproduct injections, $\mathrm{inl}_{X,Y}$ and $\mathrm{inr}_{X,Y}$.]

   A category $\mathbf{C}$ is *distributive* if it has all binary products and binary coproducts, and for all objects $X, Y, Z \in \mathbf{C}$, (using the defining property of the coproduct $X \times Y \xrightarrow{\mathrm{inl}_{X\times Y, X\times Z}} (X \times Y) + (X \times Z) \xleftarrow{\mathrm{inr}_{X\times Y, X\times Z}} X \times Z$), the unique morphism $\delta_{X,Y,Z} : (X \times Y) + (X \times Z) \to X \times (Y + Z)$ that makes the following diagram commute



(3)

   is an isomorphism.

   (a) Using the usual product and coproduct constructs in the category **Set** of sets and functions, show that it is a distributive category.

   (b) Give, with justification, an example of a category with binary products and coproducts that is not distributive.

   (c) If $\mathbf{C}$ is a distributive category and 0 is an initial object in $\mathbf{C}$, prove that for all $X \in \mathbf{C}$, the unique morphism $0 \to X \times 0$ is an isomorphism.

5. A category $\mathbf{C}$ is called *locally finite* if for all $X, Y \in \mathrm{obj}\,\mathbf{C}$, the set of morphisms $\mathbf{C}(X, Y)$ is finite. $\mathbf{C}$ is said to be *finite* if it is both locally finite and $\mathrm{obj}\,\mathbf{C}$ is finite.

   (a) Prove that any finite category with binary products is a pre-order, that is, there is at most one morphism between any pair of objects. [Hint: if $f, g : X \to Y$ were distinct, use them to construct too large a number of morphisms from X to the product Y^n of Y with itself n (> 0) times, for some suitable some number n.]

   (b) Is every locally finite category with binary products a pre-order? (Either prove it, or give a counterexample.)

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Question 1

(a) Suppose (2) is a pullback square. To see that f is a monomorphism, suppose $h, k : W \longrightarrow Y$ satisfy $f \circ h = f \circ k$; then by the universal property of (2), there is some (unique) $\ell : W \longrightarrow Y$ satisfying $\mathrm{id}_Y \circ \ell = h$ and $\mathrm{id}_Y \circ \ell = k$, so that $h = \ell = k$. Thus f is a monomorphism.

Conversely, suppose f is a monomorphism. Then the pullback property holds for (2). For if $Y \xleftarrow{h} W \xrightarrow{k} Y$ satisfy $f \circ h = f \circ k$, then since f is a monomorphism, we have $h = k$. Therefore, there is some $\ell : W \longrightarrow Y$ with $\mathrm{id}_Y \circ \ell = h$ and $\mathrm{id}_Y \circ \ell = k$, namely $\ell = h = k$; and clearly ℓ is unique with this property.

(b) Suppose p is a monomorphism in the pullback square (1) and that $\ell_1, \ell_2 : W \longrightarrow B$ satisfy $q \circ \ell_1 = q \circ \ell_2$. We have to show $\ell_1 = \ell_2$. But note that $p \circ (u \circ \ell_1) = f \circ q \circ \ell_1 = f \circ q \circ \ell_2 = p \circ (u \circ \ell_2)$; and since p is a monomorphism, this implies that $u \circ \ell_1 = u \circ \ell_2$. Therefore, by the uniqueness part of the universal property of the pullback square (1), we do indeed have $\ell_1 = \ell_2$.

(c) Suppose p is a isomorphism in the pullback square (1). Applying the universal property with $W = Y$, $h = \mathrm{id}_Y$ and $k = p^{-1} \circ f$ (for which we do indeed have $f \circ h = f = p \circ (p^{-1} \circ f) = p \circ k$), there is a unique $\ell : Y \longrightarrow B$ with $q \circ \ell = \mathrm{id}_Y$ and $u \circ \ell = p^{-1} \circ f$. To see that q is an isomorphism it thus suffices to prove that $\ell \circ q = \mathrm{id}_B$. But $q \circ (\ell \circ q) = id_Y \circ q = q \circ \mathrm{id}_B$ and $u \circ (\ell \circ q) = p^{-1} \circ f \circ q = p^{-1} \circ p \circ u = u = u \circ \mathrm{id}_B$; so by the uniqueness part of the universal property of pullbacks, we do indeed have $\ell \circ q = \mathrm{id}_B$.

(d) In the category of sets, monomorphisms are injective functions (see Exercise Sheet 1, question 4(d)); and isomorphisms are bijections (see Lecture 2). Take $A = \{0, 1\}$, $B = \emptyset$, $X = \{0\}$ and $Y = \emptyset$. The functions f, p, q, u are uniquely determined and $f \circ q = p \circ u$. Note that $p : \{0, 1\} \longrightarrow \{0\}$ is not injective and hence not a monomorphism; and $q : \emptyset \longrightarrow \emptyset$ is trivially a bijection, hence an isomorphism. The square is a pullback, since given and W, h, k as in the universal property, since h is a function from W to $\emptyset$, it must be the case that $W = \emptyset$, from which the unique existence of $\ell : W \longrightarrow B$ satisfying $q \circ \ell = h$ and $u \circ \ell = k$ follows trivially.

Question 2

(a) One can define $g^f : Y^X \longrightarrow (Y')^{X'}$ by

$$g^f \triangleq \mathrm{cur}\left(Y^X \times X' \xrightarrow{id_{Y^X} \times f} Y^X \times X \xrightarrow{\mathrm{app}} Y \xrightarrow{g} Y'\right)$$

(b) $t \triangleq \lambda f : A' \to A.\, \lambda g : B \to B'.\, \lambda h : A \to B.\, \lambda x' : A'.\, g(h(f\, x'))$

$[\![\diamond \vdash t : (A' \to A) \to (B \to B') \to (A \to B) \to (A' \to B')]\!] =$

$\mathrm{cur}(\mathrm{cur}(\mathrm{cur}(\mathrm{cur}(\mathrm{app}\langle \pi_2 \circ \pi_1 \circ \pi_1 \,, \mathrm{app}\langle \pi_2 \circ \pi_1 \,, \mathrm{app}\langle \pi_2 \circ \pi_1 \circ \pi_1 \circ \pi_1 \,, \pi_2 \rangle\rangle\rangle$

Question 3

(a) The product of X and Y in **C** is their coproduct in **Set**, which is the disjoint union

$$X \uplus Y = \{(0,x) \mid x \in X\} \cup \{(1,y) \mid y \in Y\}$$

together with the functions $\mathrm{inl} \in \mathbf{Set}(X, X \uplus Y)$ and $\mathrm{inr} \in \mathbf{Set}(Y, X \uplus Y)$ that respectively map $x \in X$ to $(0,x) \in X \uplus Y$ and $y \in Y$ to $(1,y) \in X \uplus Y$.

(b) Consider the one-element set $1 = \{0\}$ as an object of **C**. If the exponential 1^1 existed in **C**, there would be a bijection $\mathbf{C}(1 \times 1, 1) \cong \mathbf{C}(1, 1^1)$. But from part (a)

$$\mathbf{C}(1 \times 1, 1) \triangleq \mathbf{Set}(1, 1 \uplus 1)$$

is a two-element set, whereas

$$\mathbf{C}(1, 1^1) \triangleq \mathbf{Set}(1^1, 1)$$

has exactly one element no matter what set 1^1 is. Thus for any set X, the sets $\mathbf{C}(1 \times 1, 1)$ and $\mathbf{C}(1, X)$ cannot be in bijection and therefore the exponential 1^1 of 1 and 1 in **C** cannot exist.

Question 4

(a) Product in **Set** is given by Cartesian product (x) of sets and coproduct by disjoint union ($\uplus$) of sets. Thus given $X, Y, Z \in$ **Set**, $\delta_{X,Y,Z}$ is the function $(X \mathrm{x} Y) \uplus (X \mathrm{x} Z) \to X \mathrm{x} (Y \uplus Z)$ satisfying for all $x \in X$, $y \in Y$ and $z \in Z$

$$\delta_{X,Y,Z}(0, (x,y)) = (x, (0,y))$$
$$\delta_{X,Y,Z}(1, (x,z)) = (x, (1,z))$$

and clearly this has a two-sided inverse $\delta^{-1}_{X,Y,Z} : X \mathrm{x} (Y \uplus Z) \to (X \mathrm{x} Y) \uplus (X \mathrm{x} Z)$ given by:

$$\delta^{-1}_{X,Y,Z}(x, (0,y)) = (0, (x,y))$$
$$\delta^{-1}_{X,Y,Z}(x, (1,z)) = (1, (x,z))$$

Alternative proof: use the fact that **Set** is a cartesian closed category and then appeal to Exercise Sheet 3 question 6.

(b) Since **Set** has binary products and coproducts, by duality, $\mathbf{Set}^{\mathrm{op}}$ has binary coproducts and products. It is not distributive because, for example, if we take $X = Y = Z = 1 = \{0\}$ then $(X \times Y) + (X \times Z)$ in $\mathbf{Set}^{\mathrm{op}}$ is given by a set $(1 \uplus 1) \mathrm{x} (1 \uplus 1)$ with four elements, whereas $X \times (Y + Z)$ in $\mathbf{Set}^{\mathrm{op}}$ is given by a set $1 \uplus (1 \mathrm{x} 1)$ with only two elements; these cannot be isomorphic in $\mathbf{Set}^{\mathrm{op}}$ because isomorphism is a self-dual concept and we know that isomorphism in **Set** is given by bijection.

(c) Write i for the unique morphism $0 \to X \times 0$. We will show that $\pi_2 : X \times 0 \to 0$ is its two-sided inverse. It is certainly the case that $\pi_2 \circ i = id : 0 \to 0$, because 0 is initial. So it just remains to show that $i \circ \pi_2 = id : X \times 0 \to X \times 0$.

Consider the unique morphism $[\mathrm{id}, i \circ \pi_2] : (X \times 0) + (X \times 0) \to X \times 0$ whose compositions with $\mathrm{inl}_{X \times 0, X \times 0}$ and $\mathrm{inr}_{X \times 0, X \times 0}$ are id and $i \circ \pi_2$ respectively. If suffices to prove

$$\mathrm{inl}_{X \times 0, X \times 0} = \mathrm{inr}_{X \times 0, X \times 0} \tag{4}$$

since then $\mathrm{id} = [\mathrm{id}, i \circ \pi_2] \circ \mathrm{inl}_{X\times 0, X\times 0} = [\mathrm{id}, i \circ \pi_2] \circ \mathrm{inr}_{X\times 0, X\times 0} = i \circ \pi_2$. To see (4), take $Y = Z = 0$ in (3) to deduce that

$$\mathrm{inl}_{X\times 0, X\times 0} = (\mathrm{id} \times \mathrm{inl}_{0,0}) \circ \delta^{-1}_{X,0,0}$$
$$\mathrm{inr}_{X\times 0, X\times 0} = (\mathrm{id} \times \mathrm{inr}_{0,0}) \circ \delta^{-1}_{X,0,0}$$

But since 0 is initial we have $\mathrm{inl}_{0,0} = \mathrm{inr}_{0,0} : 0 \to 0 + 0$ and therefore $\mathrm{inl}_{X\times 0, X\times 0} = \mathrm{inr}_{X\times 0, X\times 0}$.

Question 5

(a) Given any $Y \in \mathrm{obj}\,\mathbf{C}$, for each natural number $n > 0$, by iterating the binary product we can form the product of n copies of Y; this is an object Y^n together with morphisms $\pi^n_i : Y^n \to Y$ ($i = 1, \ldots, n$) that have the universal property that for each n-tuple of morphisms ($f_i : Y \to Y \mid i = 1, \ldots, n$), there is a unique morphism $h : Y \to Y^n$ satisfying $\forall j = 1, \ldots, n,\ \pi^n_j \circ h = f_j$. Therefore given any $f, g : X \to Y$ in $\mathbf{C}$, for each $n > 0$ there are morphisms $h^n_i : X \to Y^n$ ($i = 1, \ldots, n$) where for each $j = 1, \ldots, n$

$$\pi^n_j \circ h^n_i = \begin{cases} f & \text{if } i = j \\ g & \text{if } i \neq j \end{cases}$$

Since $\mathbf{C}$ is finite, we can pick n sufficiently large that for some $i \neq j$ we have $h^n_i = h^n_j$; and then $f = \pi^n_j \circ h^n_j = \pi^n_j \circ h^n_i = g$. So $\mathbf{C}$ is a pre-order.

(b) Clearly the category whose objects are *finite* sets and whose morphisms are functions (with composition and identities as for **Set**) is locally finite (there are only finitely many diferent functions from one finite set to another), but is not a pre-order. It has binary products, given as in **Set** by Cartesian product of sets.