Introduction to Probability

Lectures 9: Central Limit Theorem

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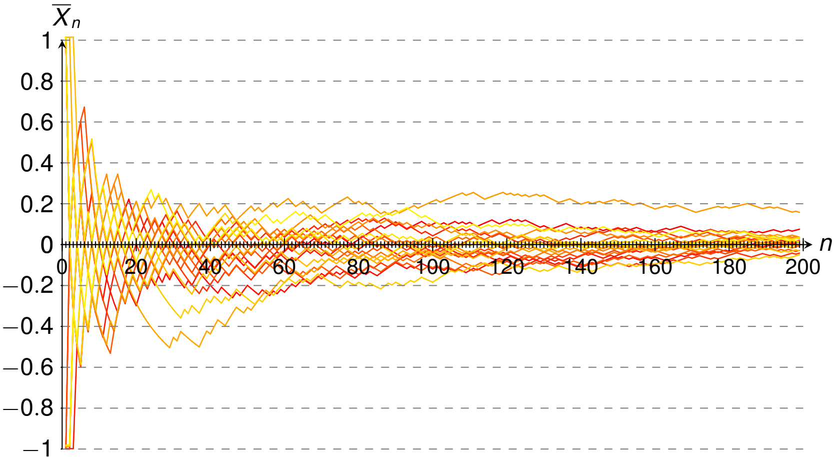
Easter 2024

UNIVERSITY OF CAMBRIDGE

Outline

Weak Law of Large Numbers (4/4)

Weak Law of Large Numbers: For any $\epsilon > 0$, $\lim_{n\to\infty} \mathbf{P}\left[|\overline{X}_n - \mu| > \epsilon \right] = 0$



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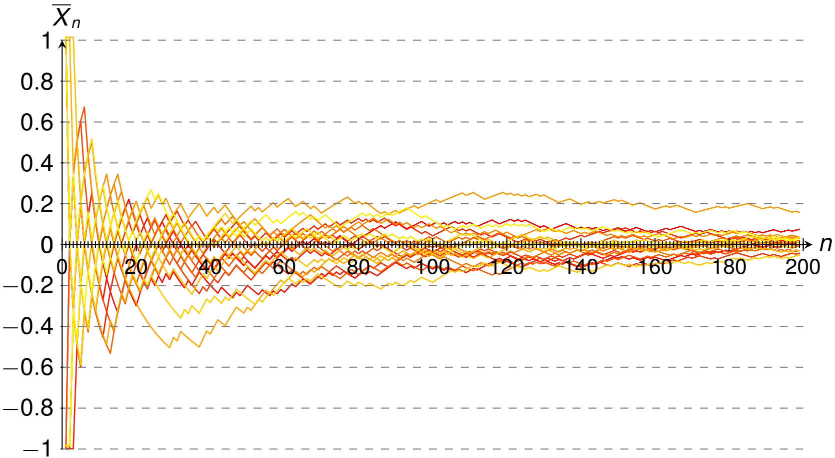

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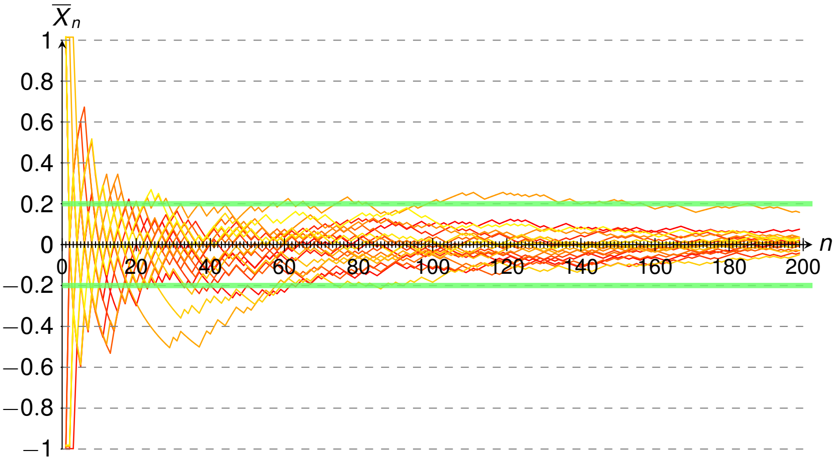

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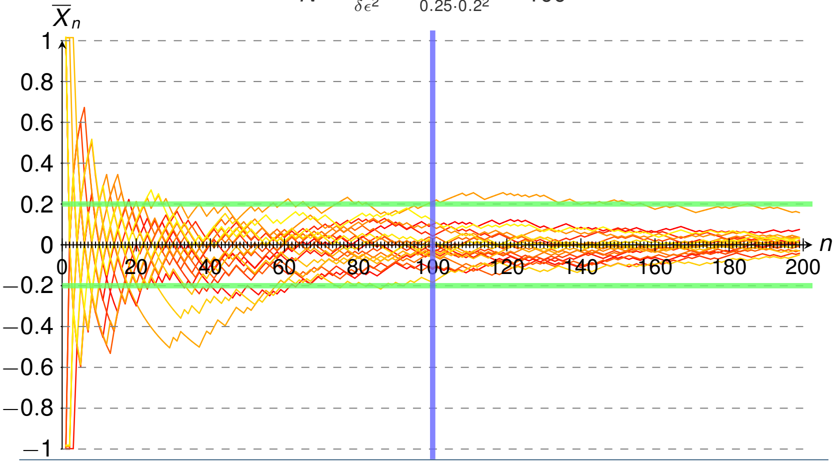

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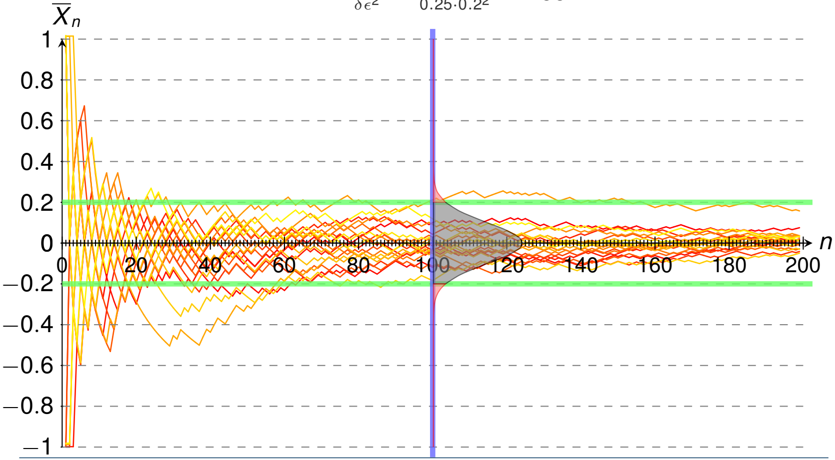

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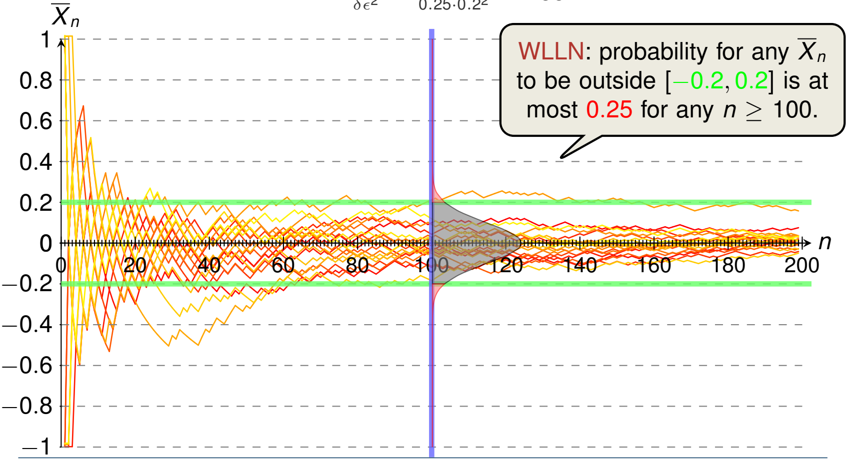

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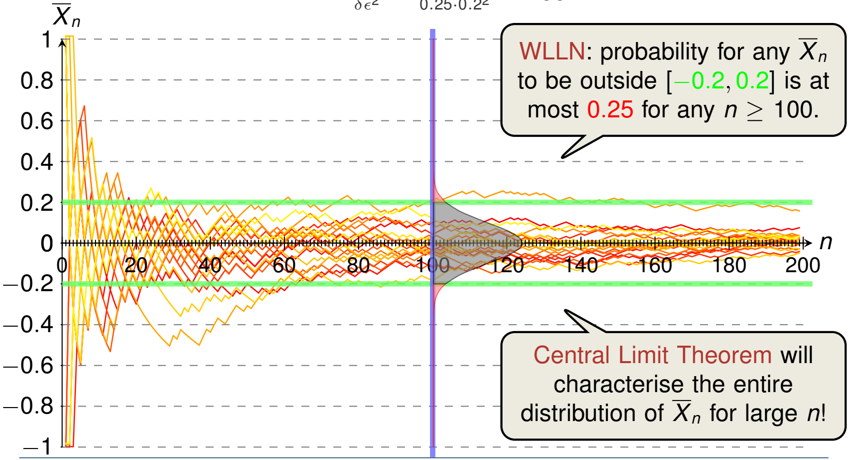

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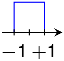

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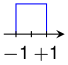

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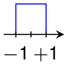

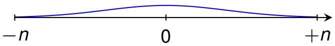

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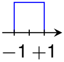

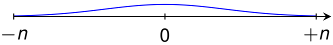

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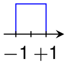

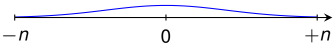

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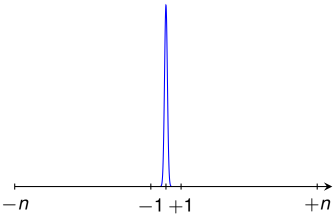

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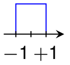

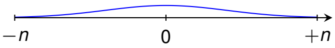

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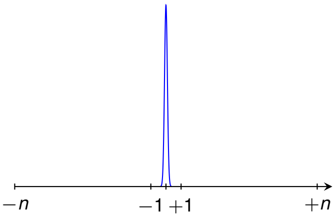

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- Let $Z_n := \frac{1}{\sqrt{n}\cdot\sigma} \cdot \sum_{i=1}^{n} X_i$
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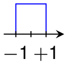

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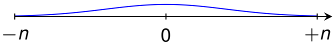

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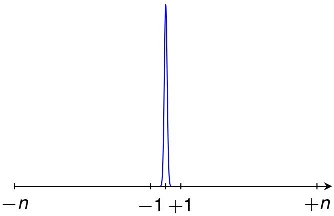

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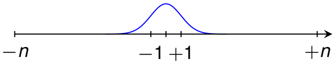

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P.-S. de Laplace (1749-1827)

C. Gauss (1777-1855)

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Then for any number $a \in \mathbb{R}$, it holds that

$$\lim_{n \to \infty} F_{Z_n}(a) = \Phi(a)$$

where Φ is the distribution function of the $\mathcal{N}(0, 1)$ distribution.

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In words: the distribution of Z_n **always** converges to the distribution function Φ of the standard normal distribution.

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- one of the most remarkable results in probability/statistics
- extremely useful tool in data analysis or physical measurements
  - we may not know the actual distribution in real-world, and CLT says we don't have to(!)
  - adding up independent noises in measurements leads to an error following the Normal distribution
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When is the approximation good?

- usually $n \geq 10$ or $n \geq 15$ is sufficient in practice
- approximation tends to be worse when threshold a is far from 0, distribution of X_i's asymmetric, bimodal or discrete
- (for a result quantifying the approximation error: Berry-Esseen-Theorem)

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Recap: Weak Law of Large Numbers

Central Limit Theorem

Illustrations

Examples

Bonus Material (non-examinable)

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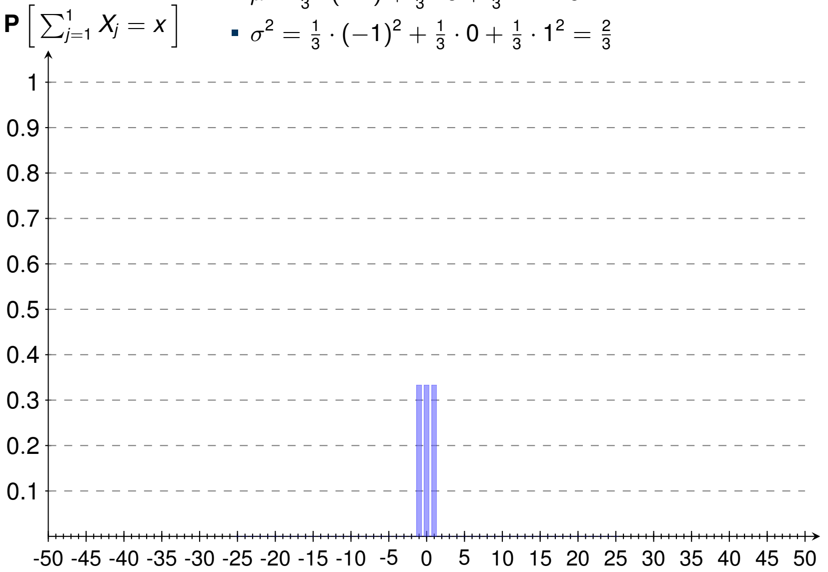

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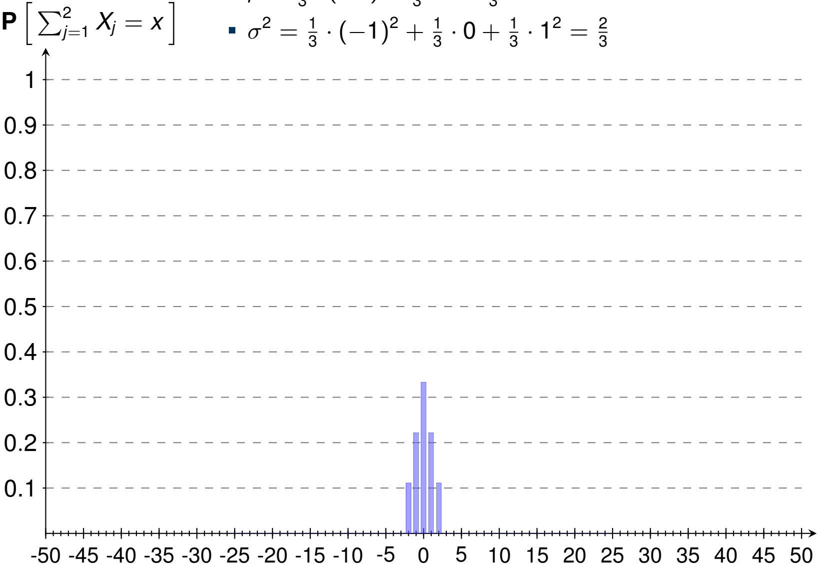

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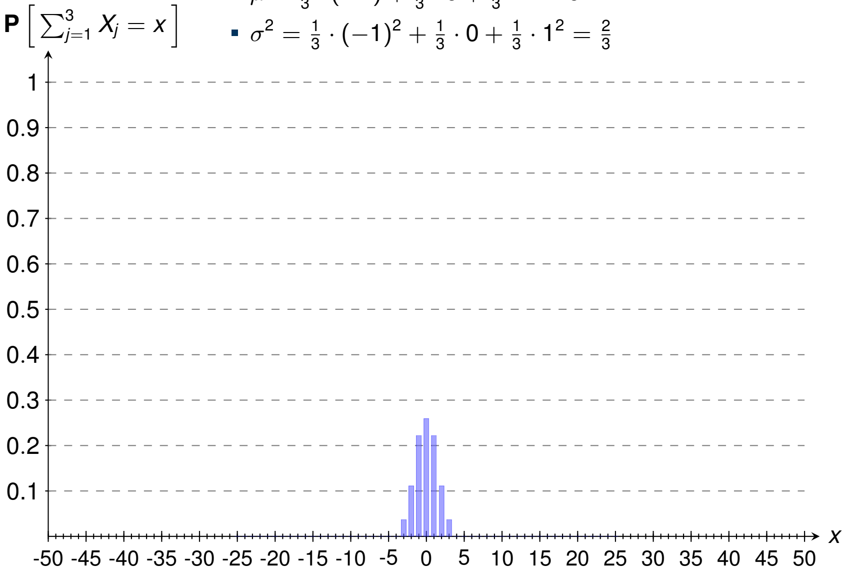

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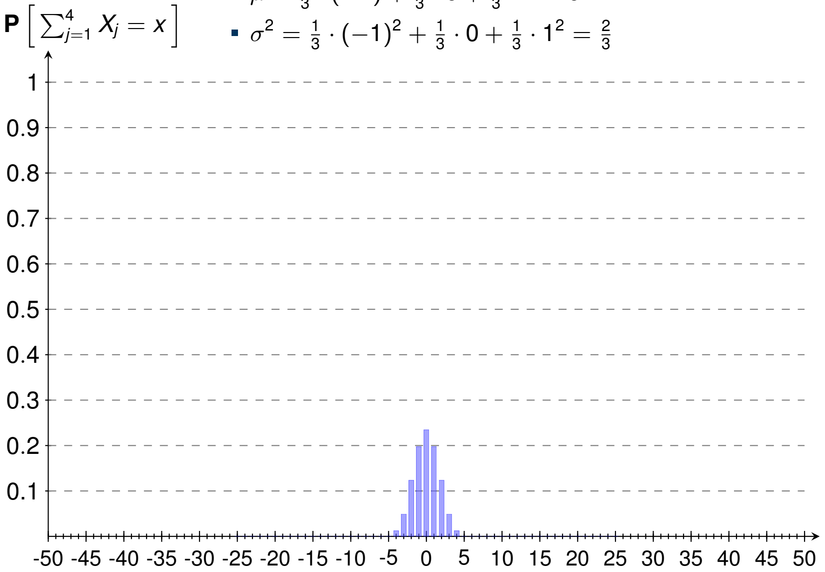

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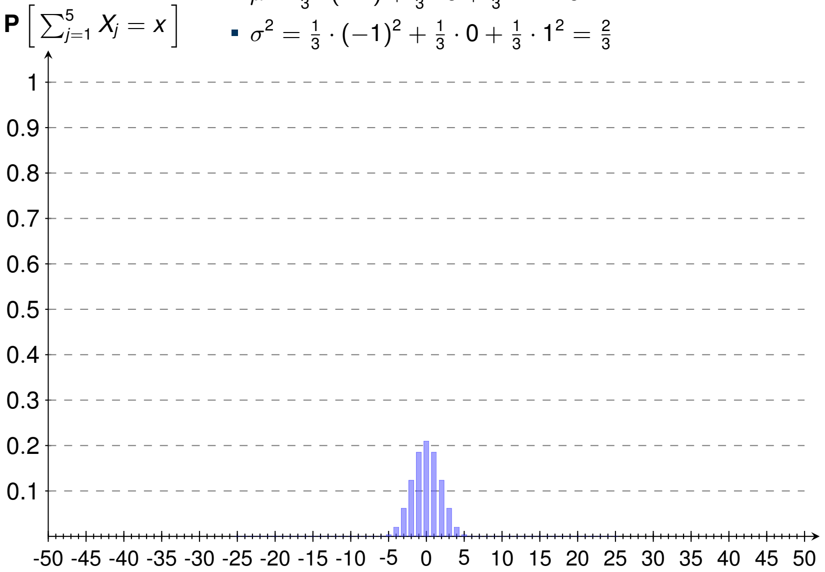

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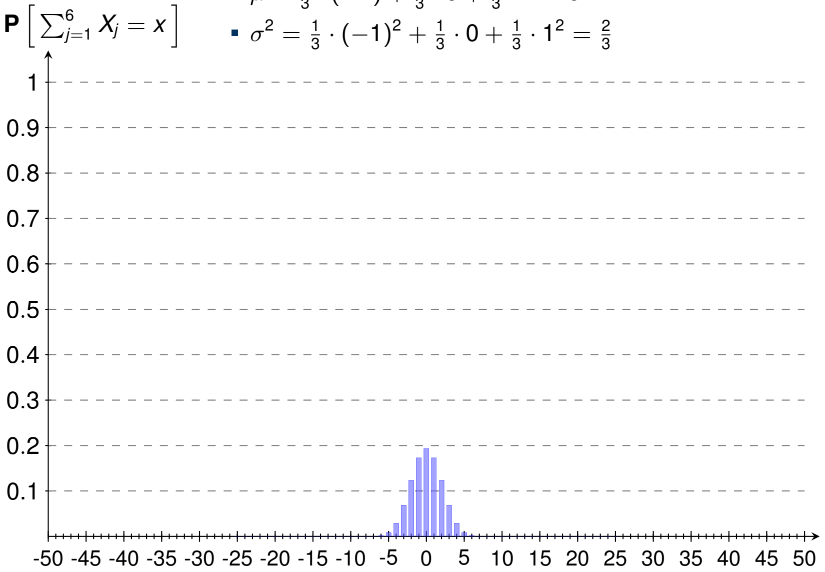

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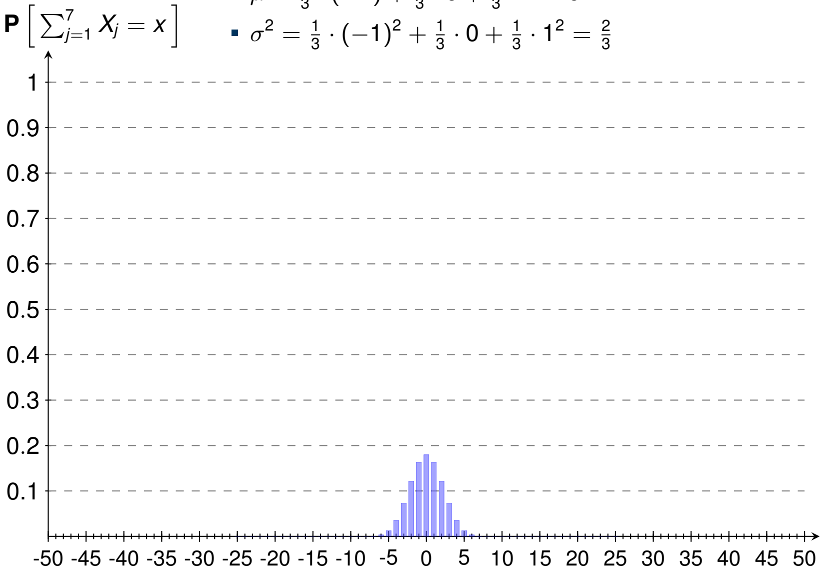

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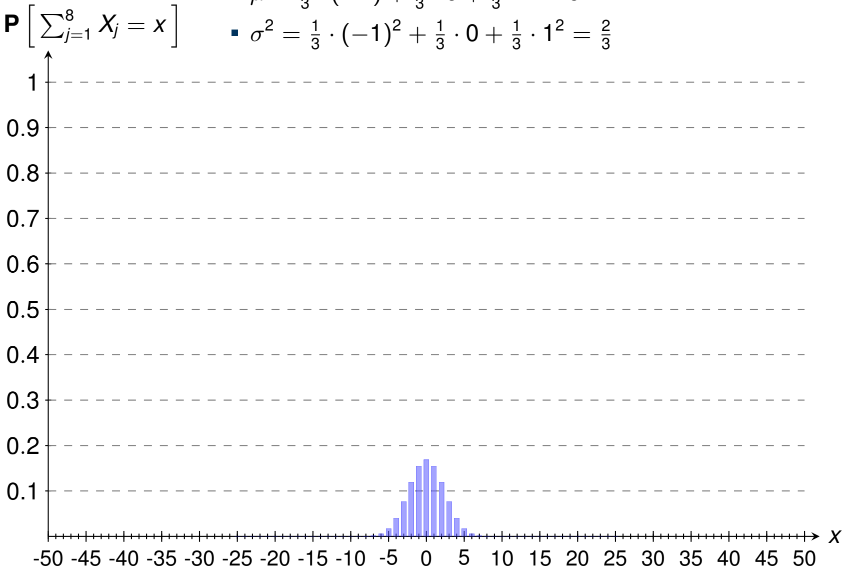

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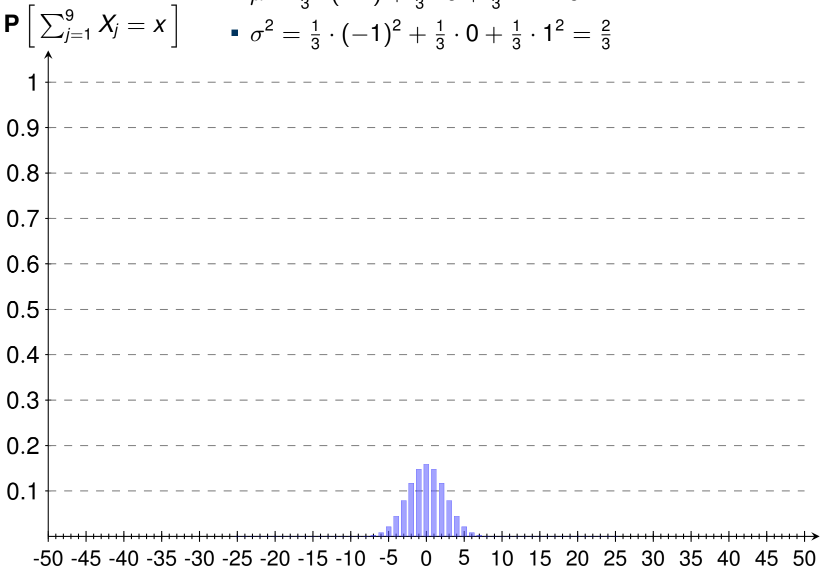

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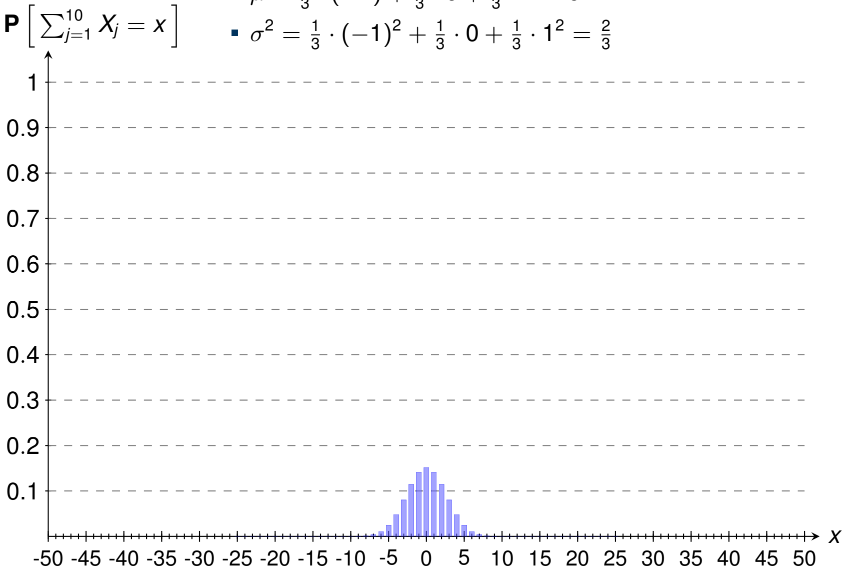

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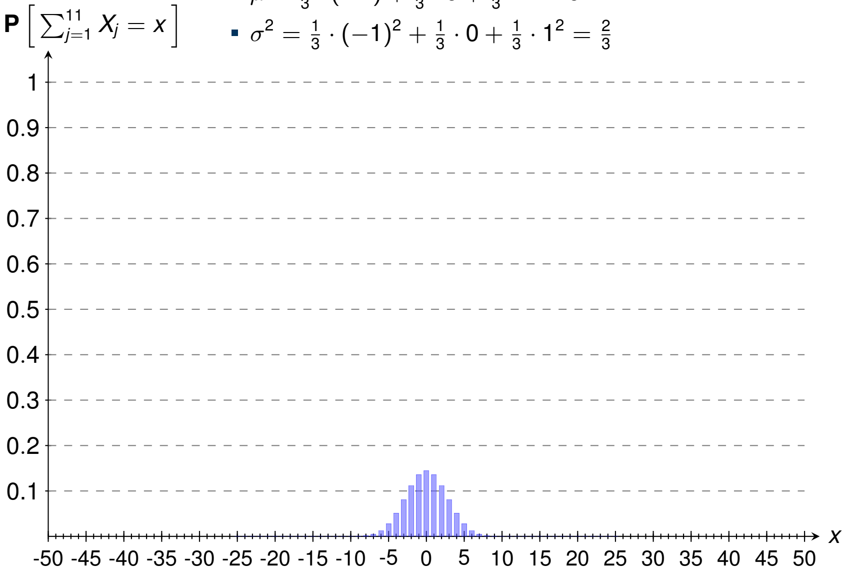

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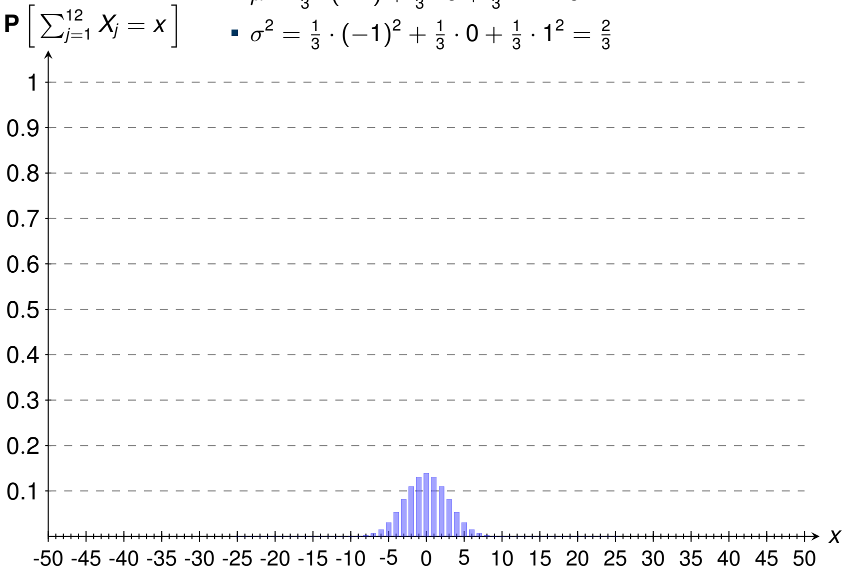

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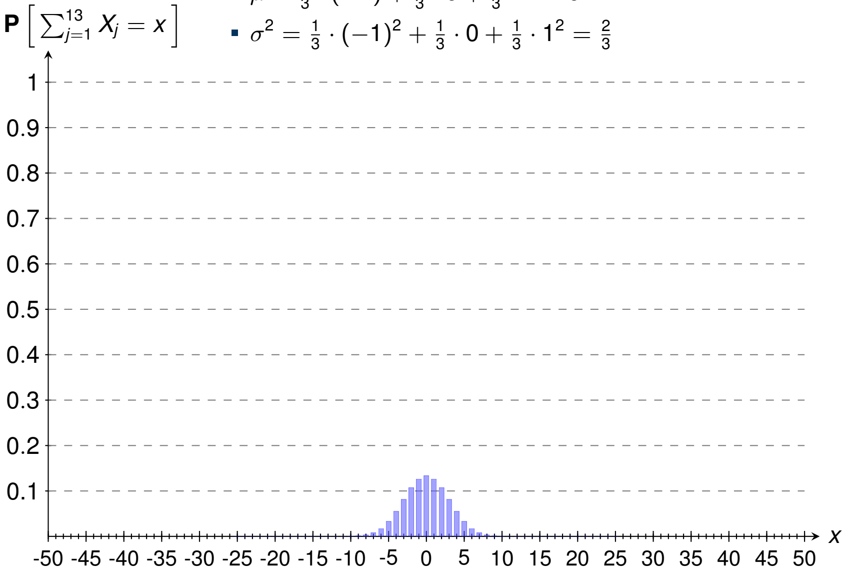

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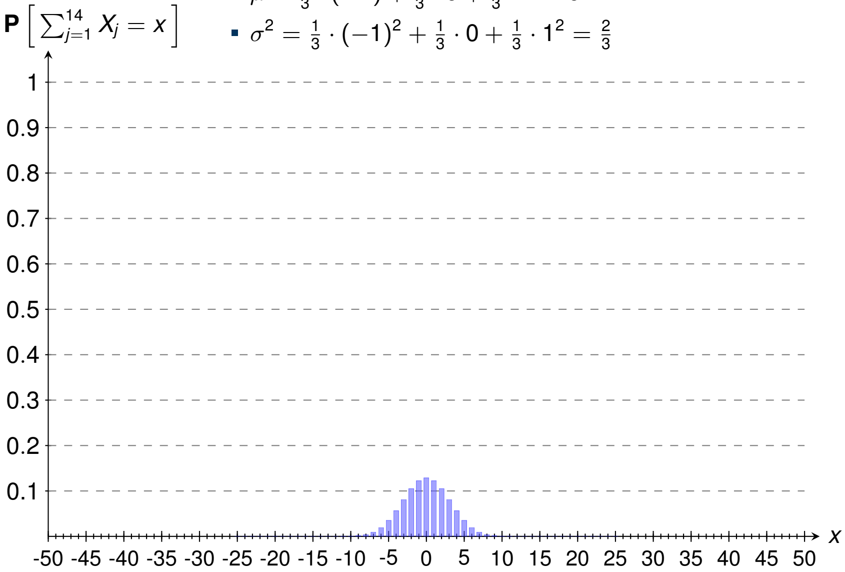

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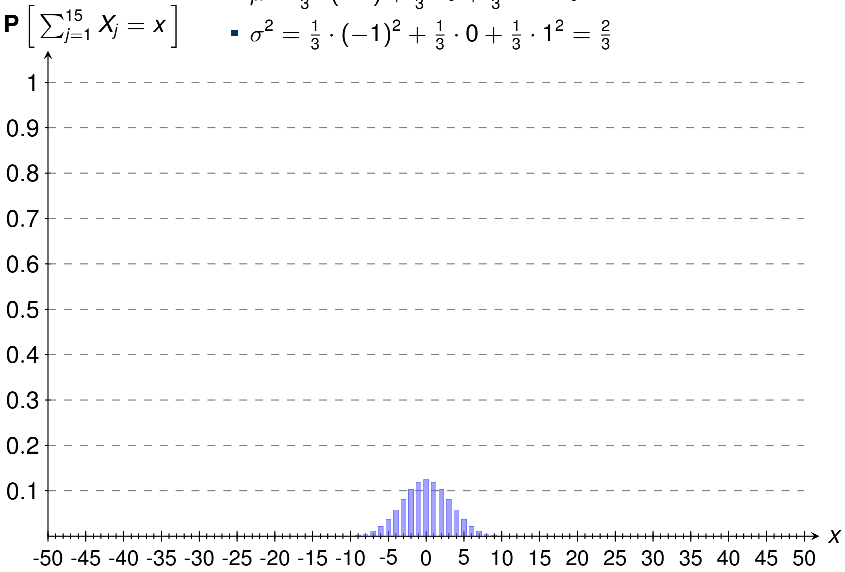

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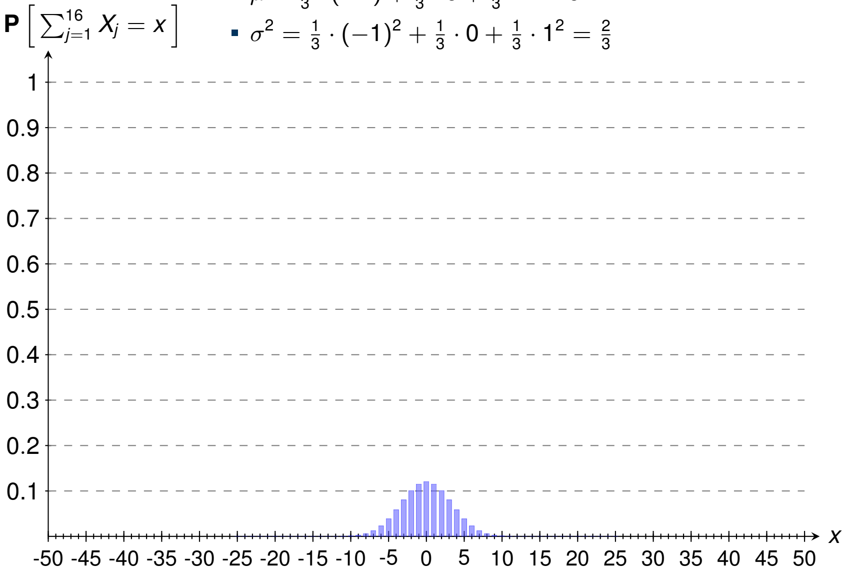

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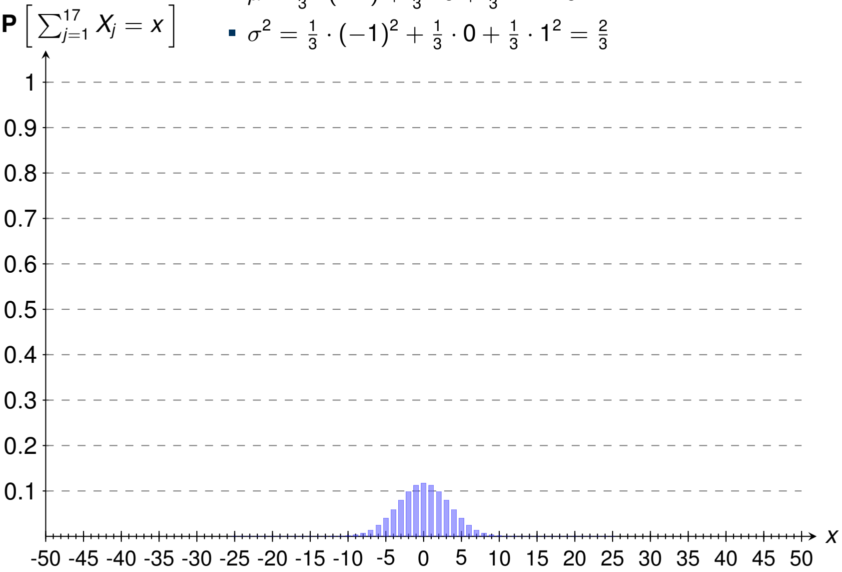

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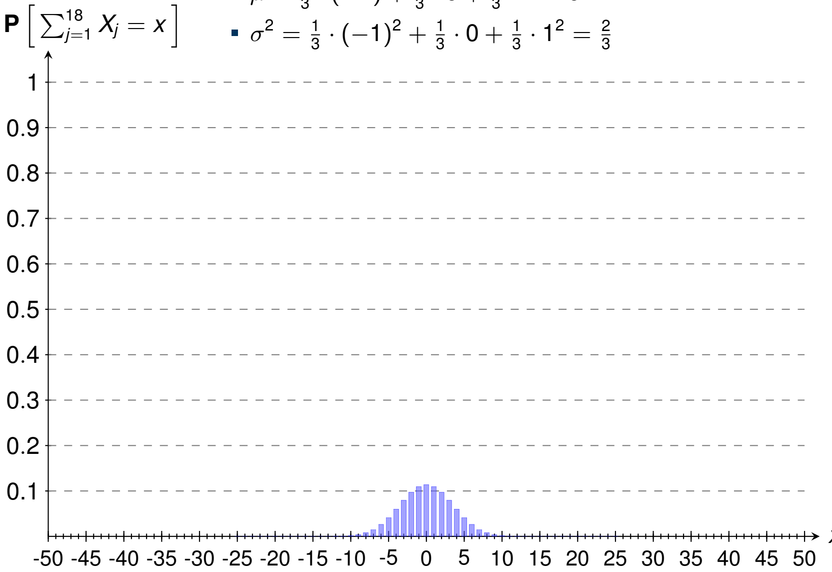

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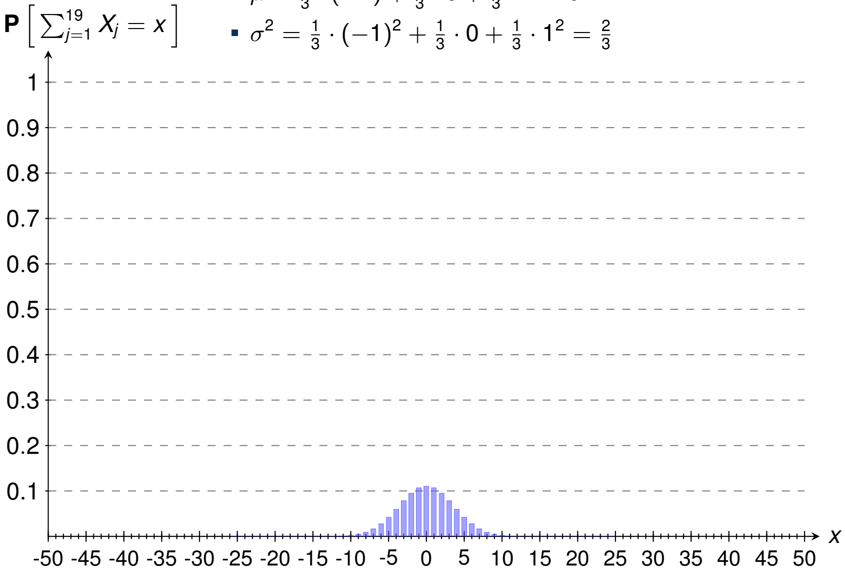

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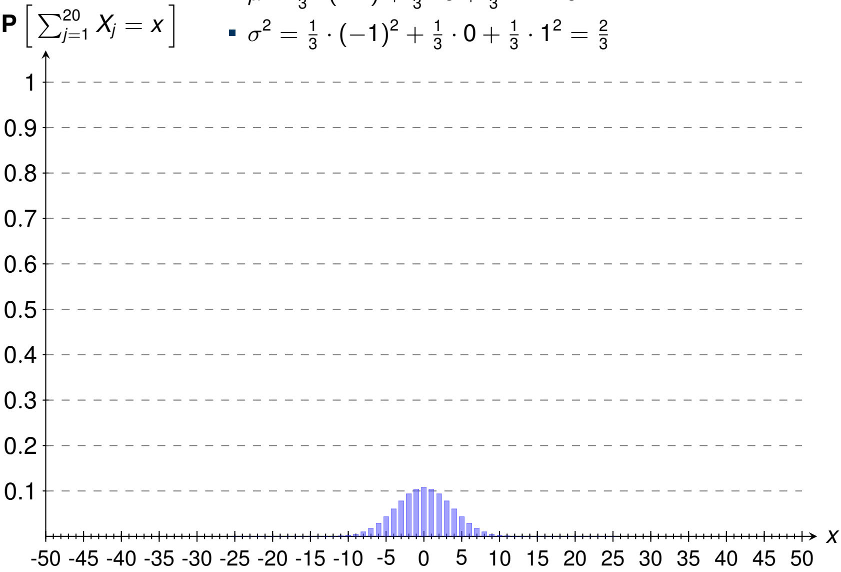

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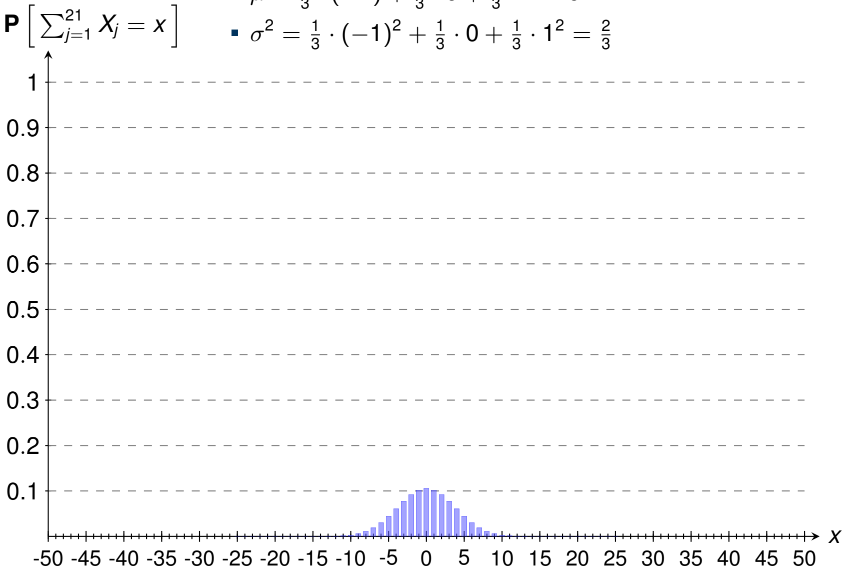

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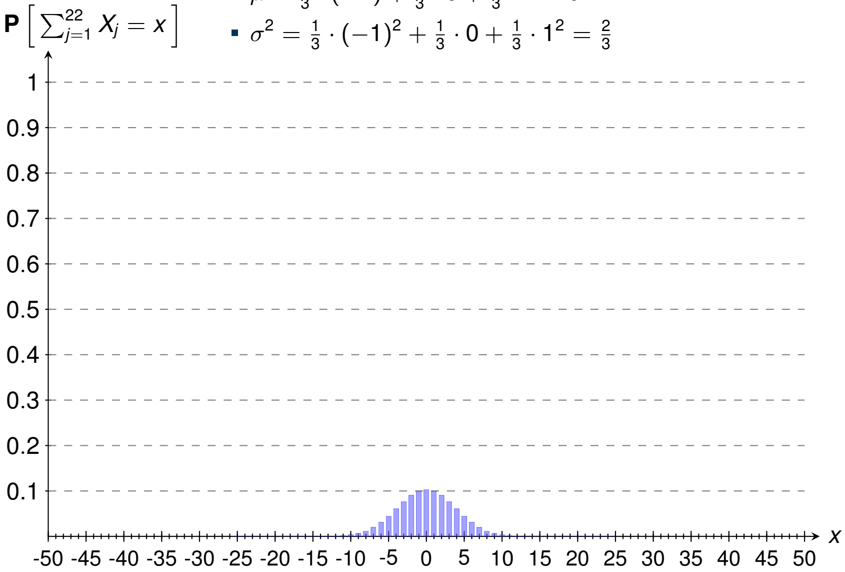

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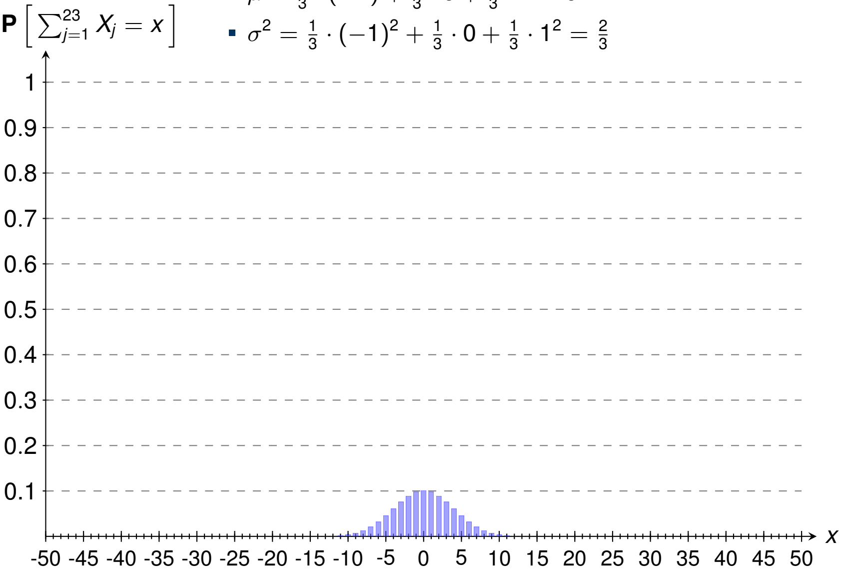

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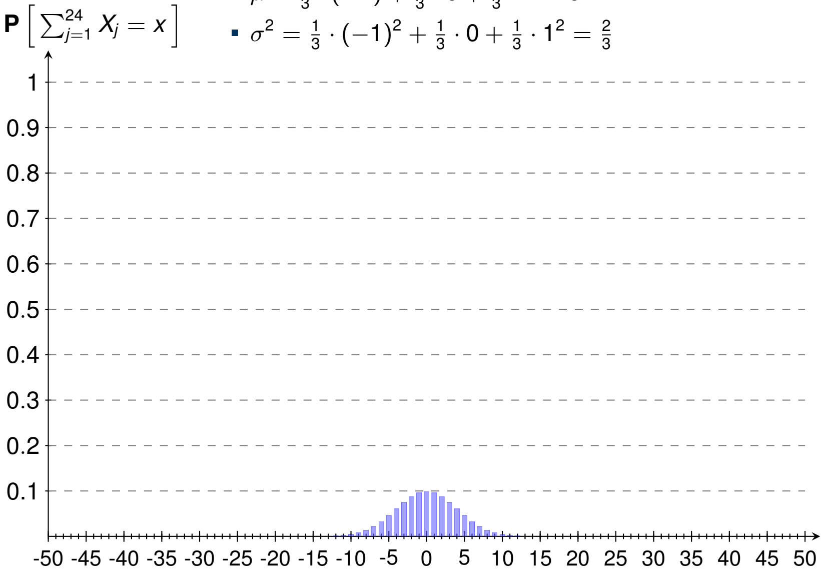

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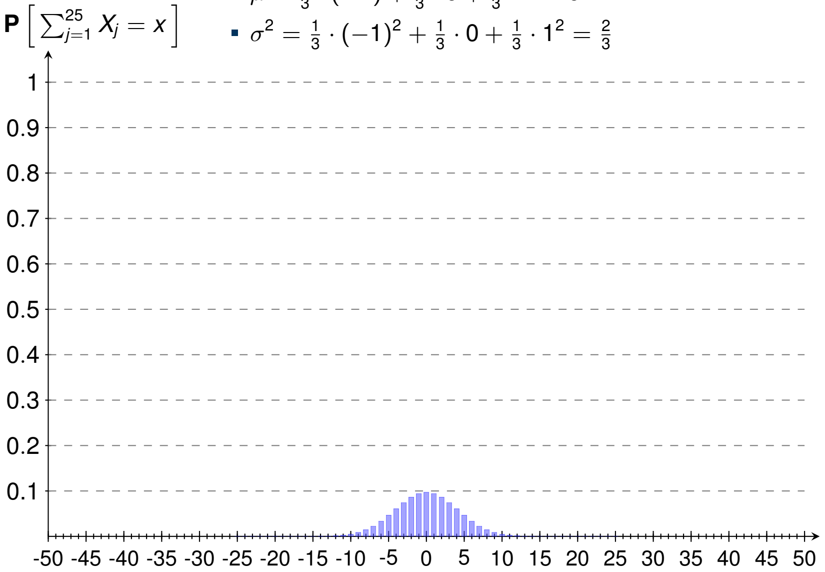

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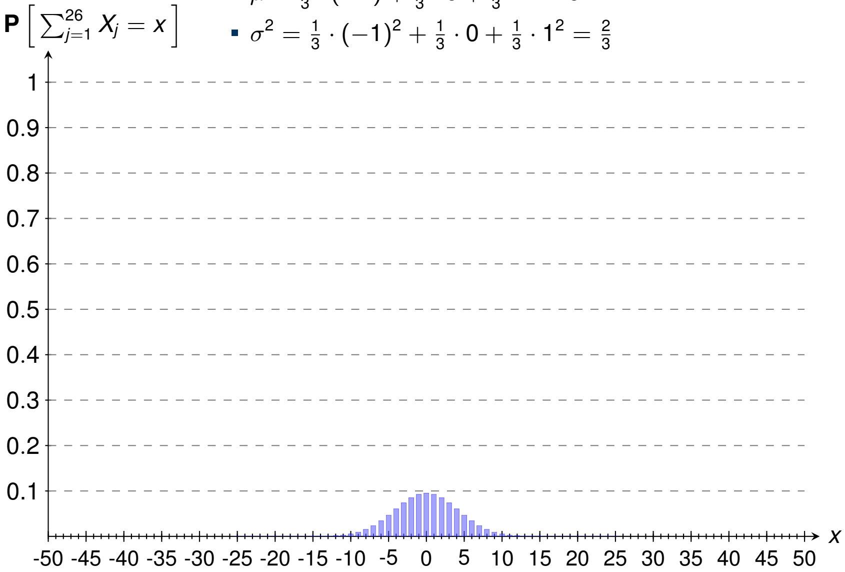

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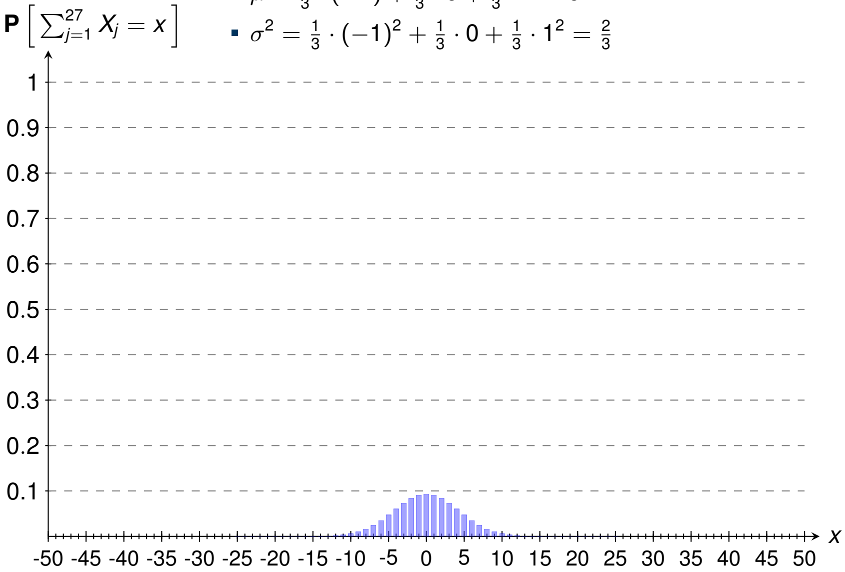

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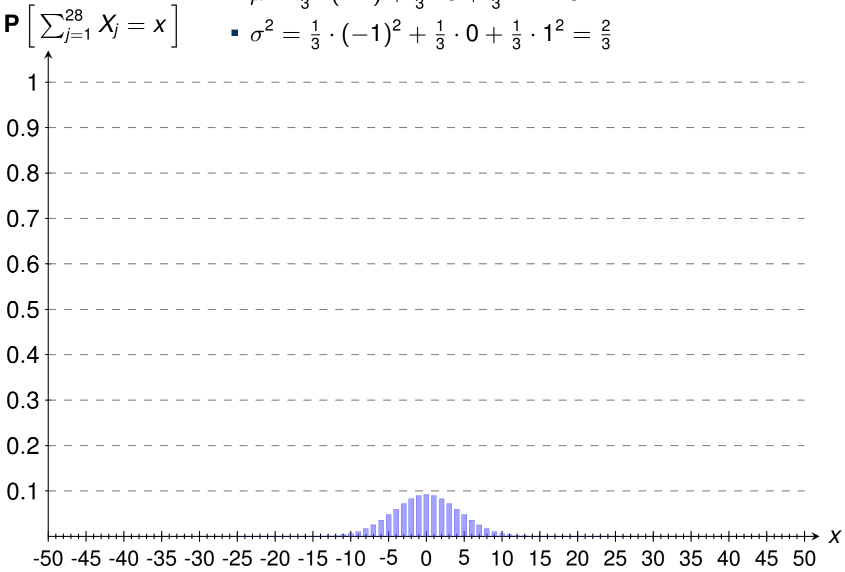

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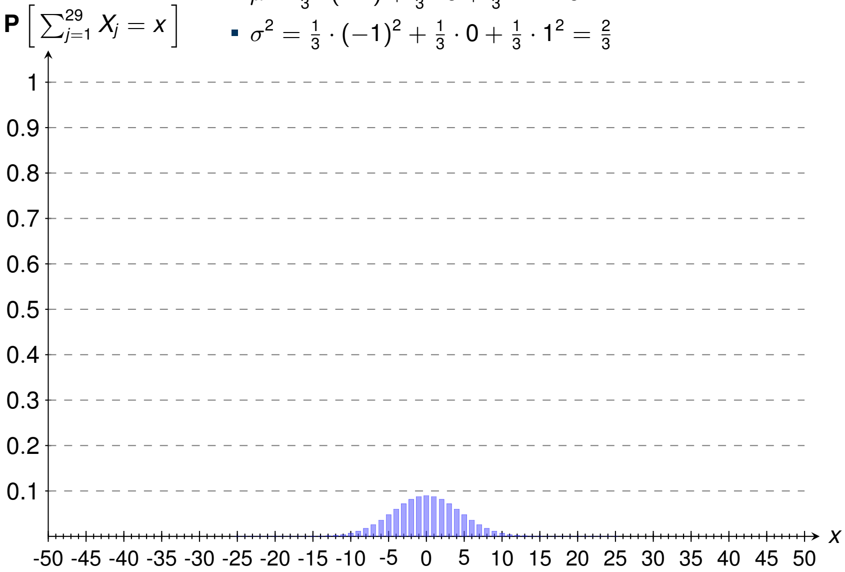

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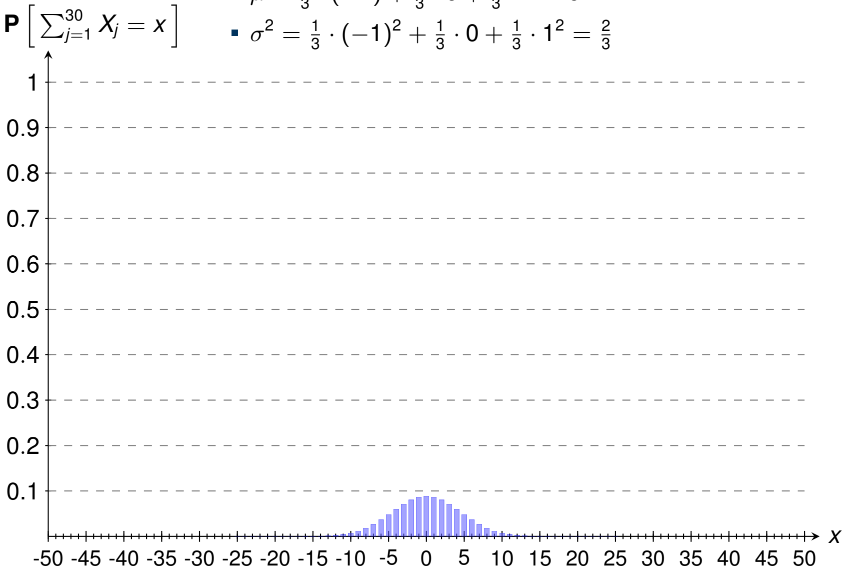

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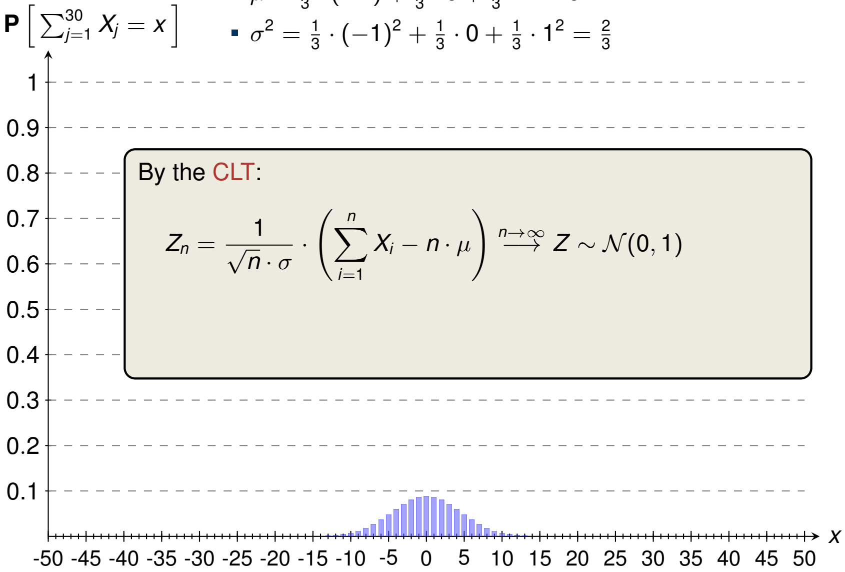

By the CLT:

$$Z_n = \frac{1}{\sqrt{n} \cdot \sigma} \cdot \left(\sum_{i=1}^{n} X_i - n \cdot \mu \right) \xrightarrow{n \to \infty} Z \sim \mathcal{N}(0, 1)$$

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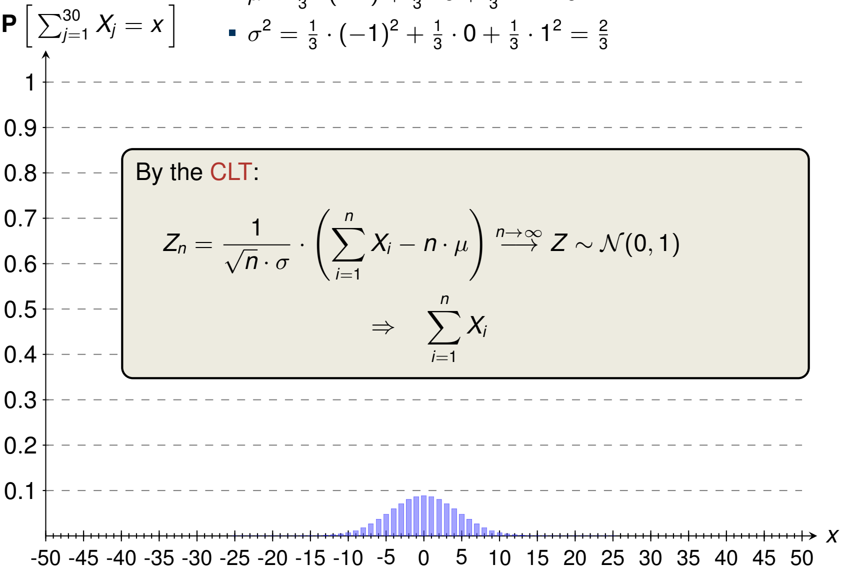

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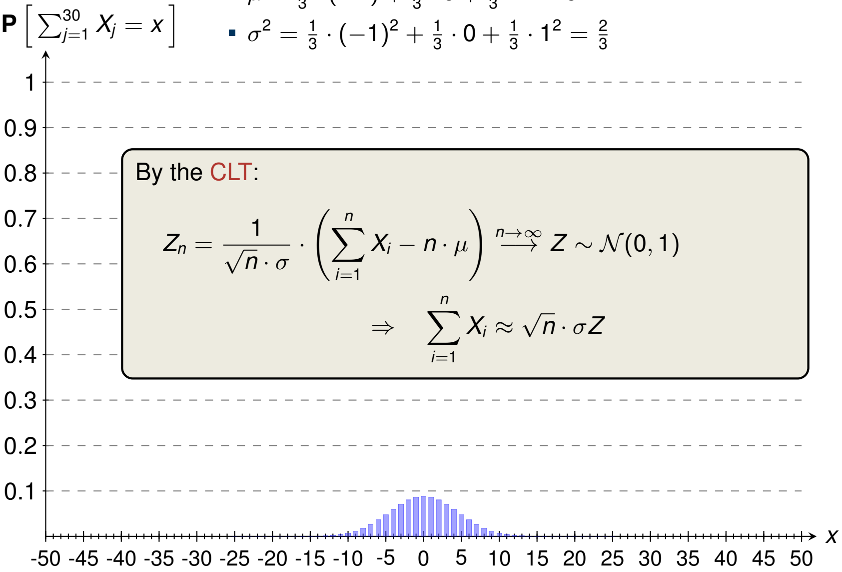

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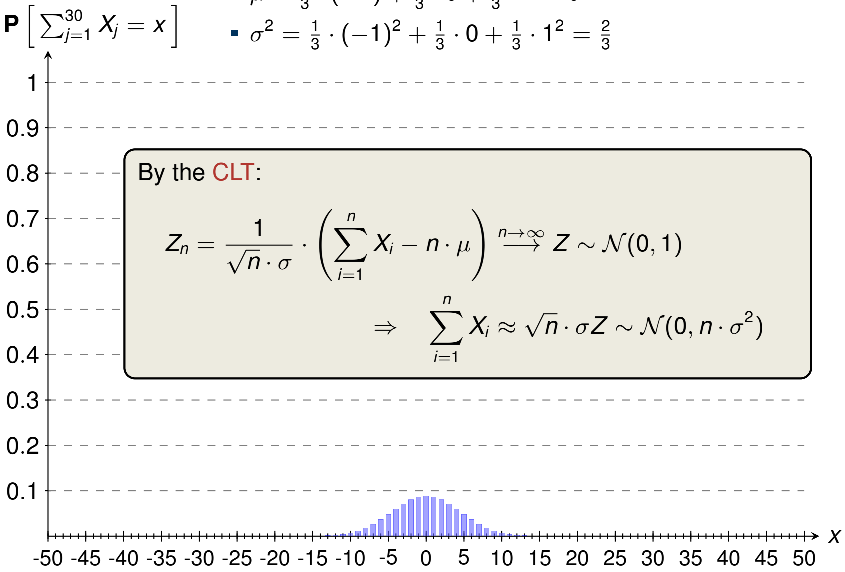

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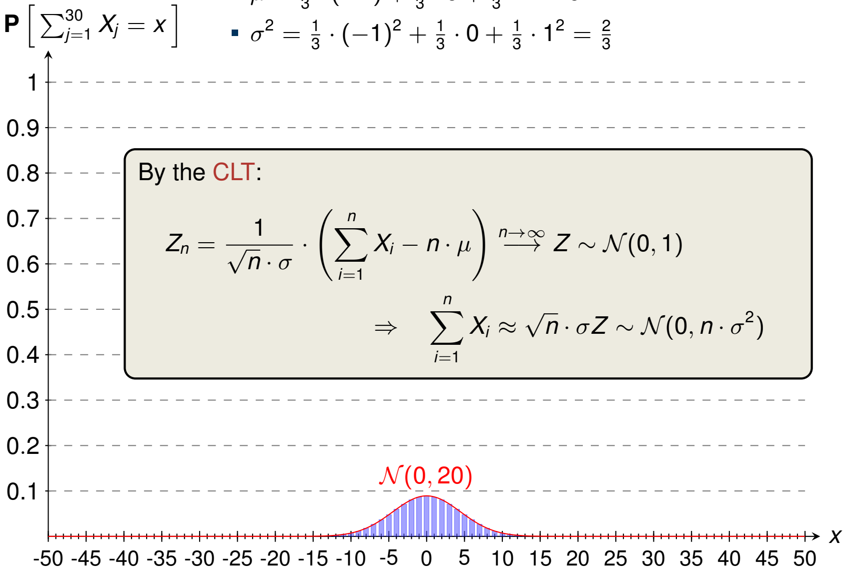

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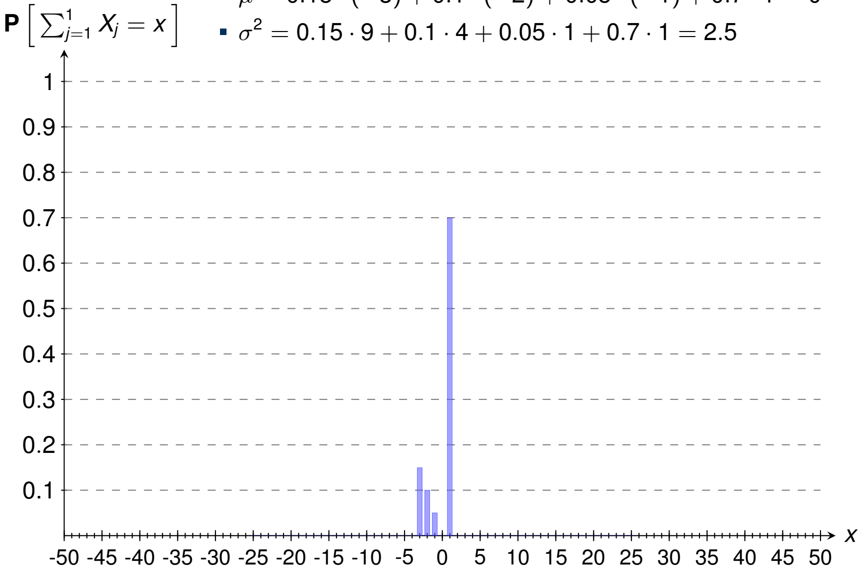

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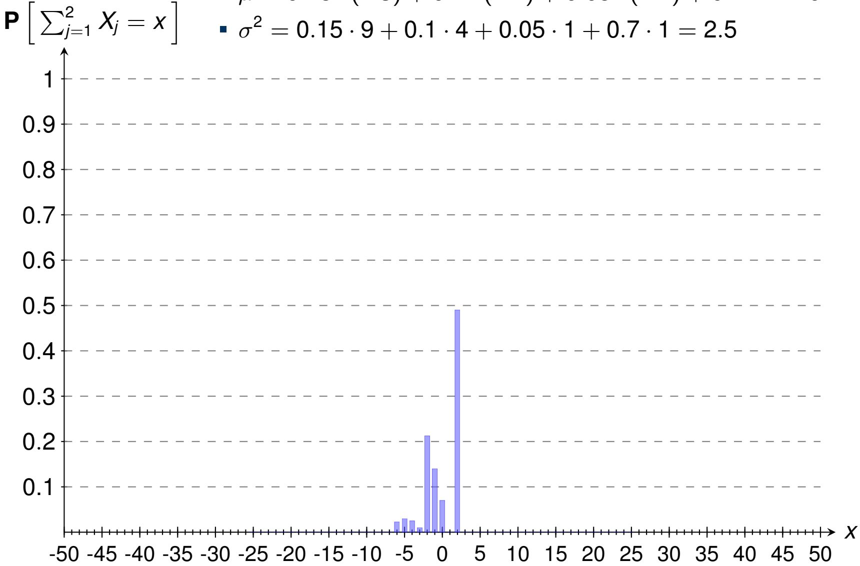

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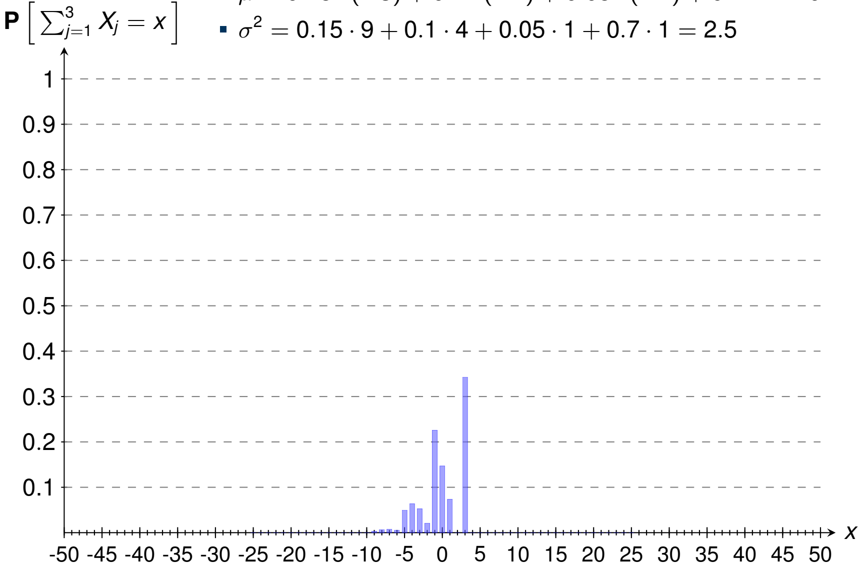

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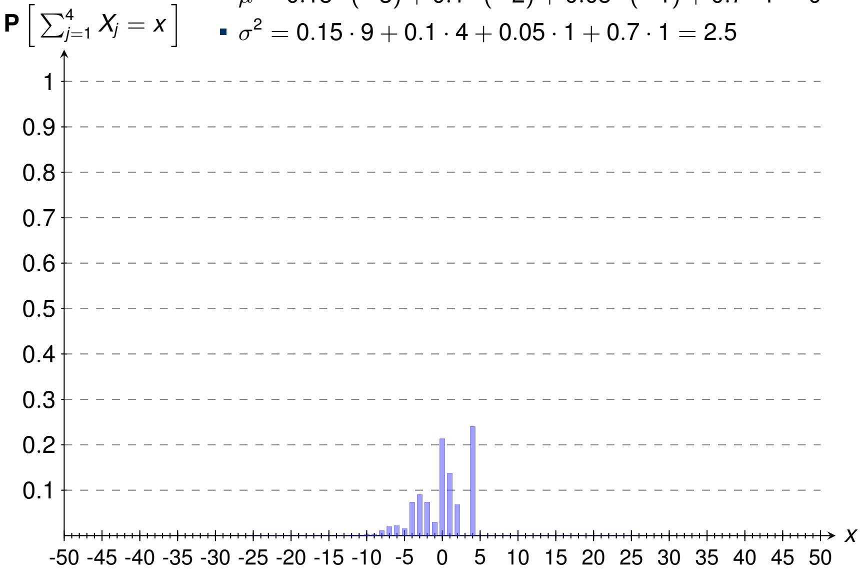

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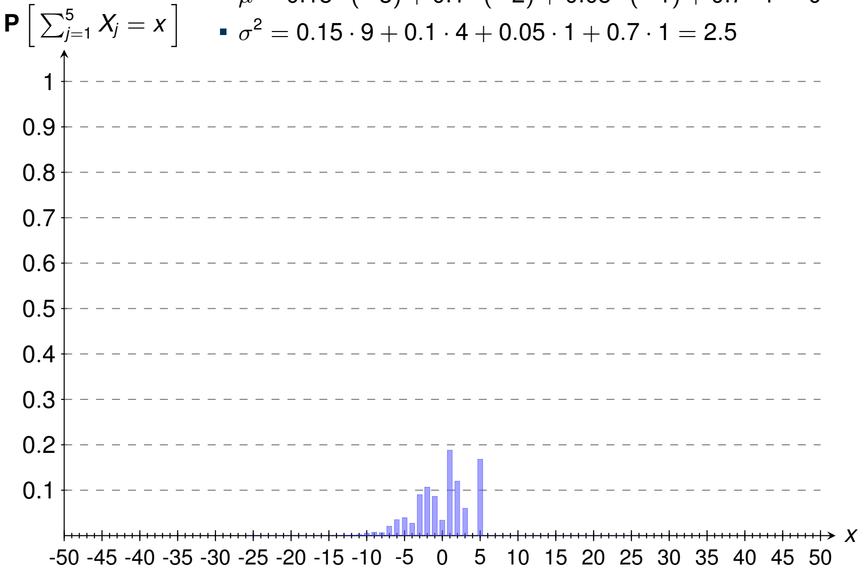

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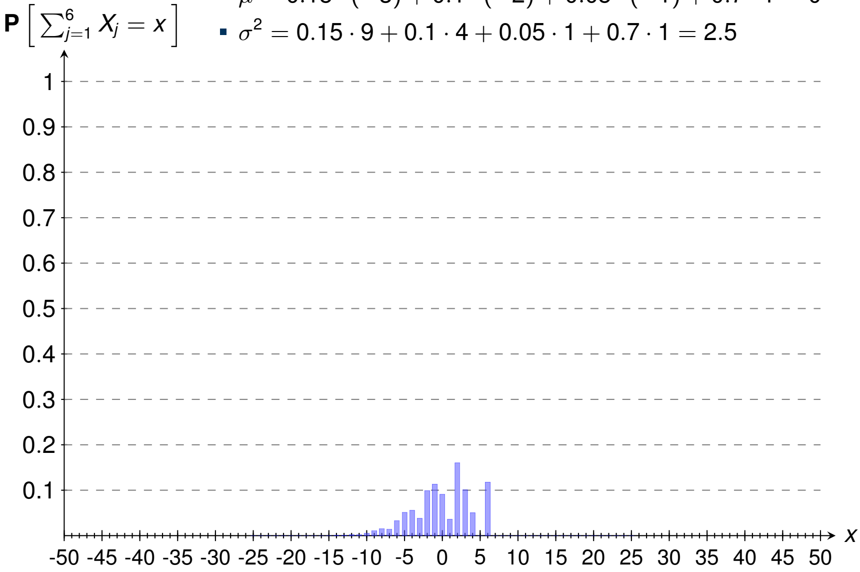

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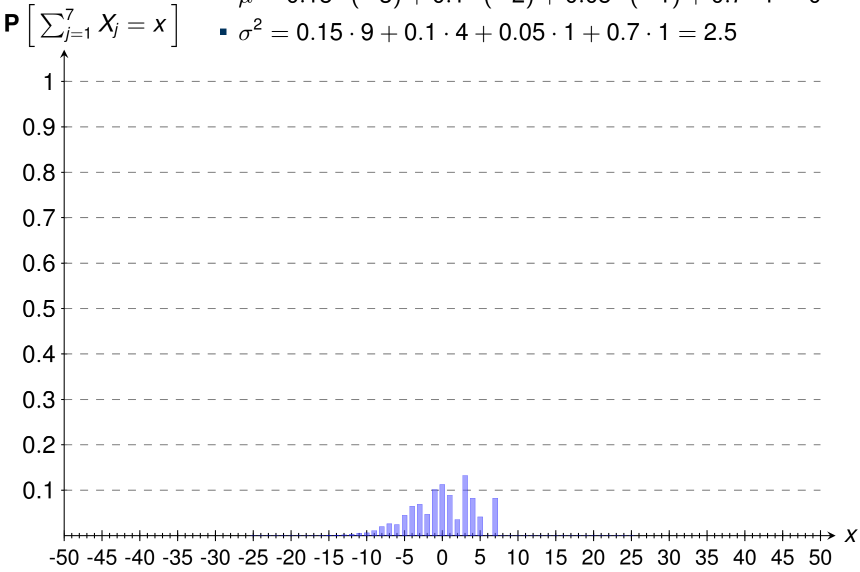

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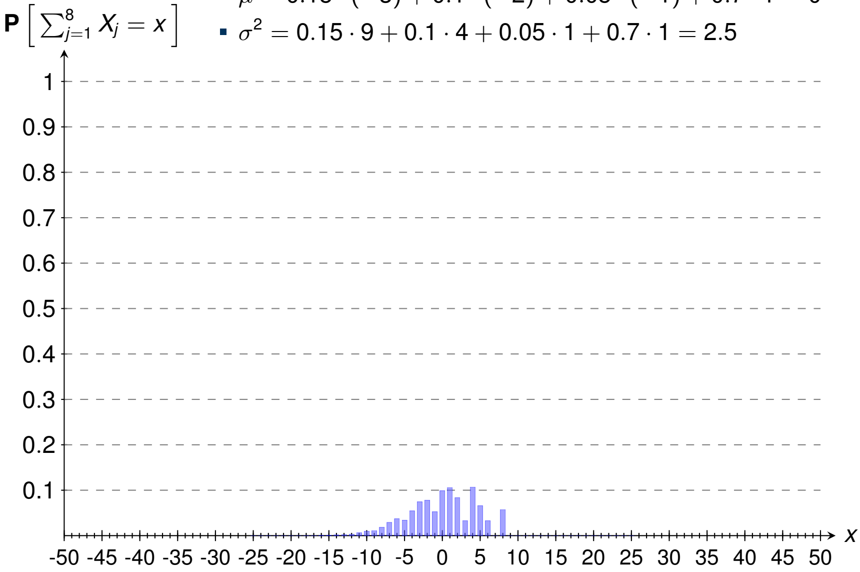

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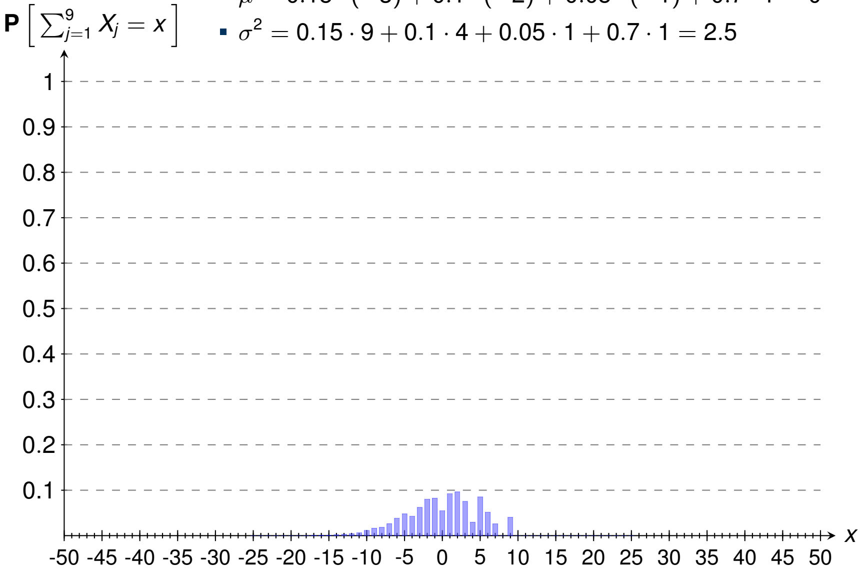

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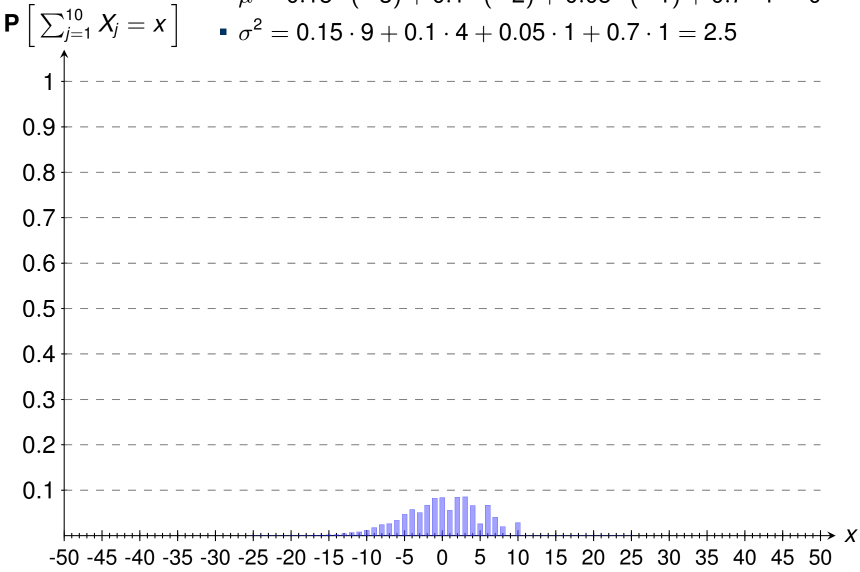

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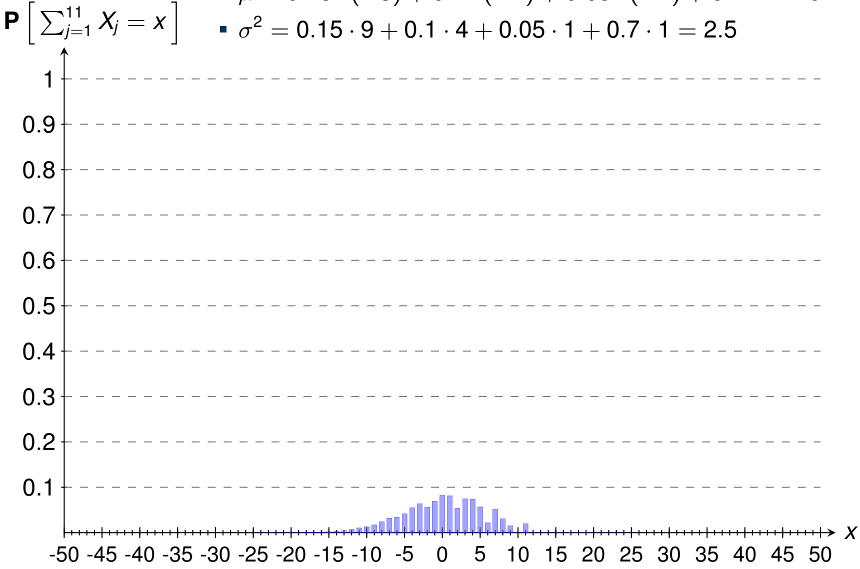

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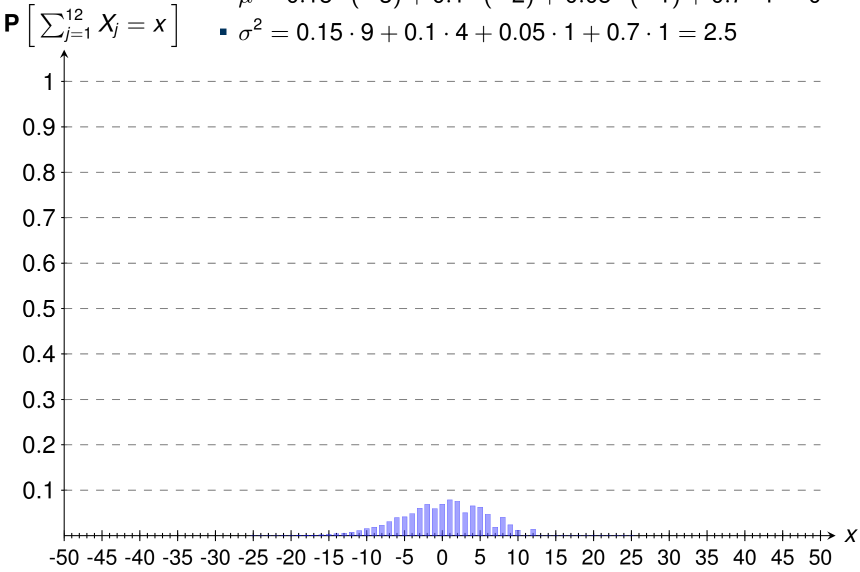

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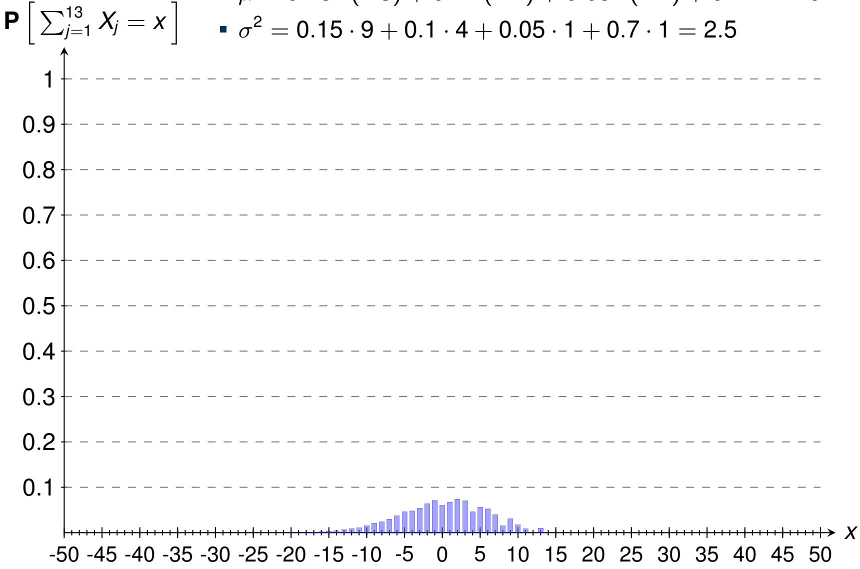

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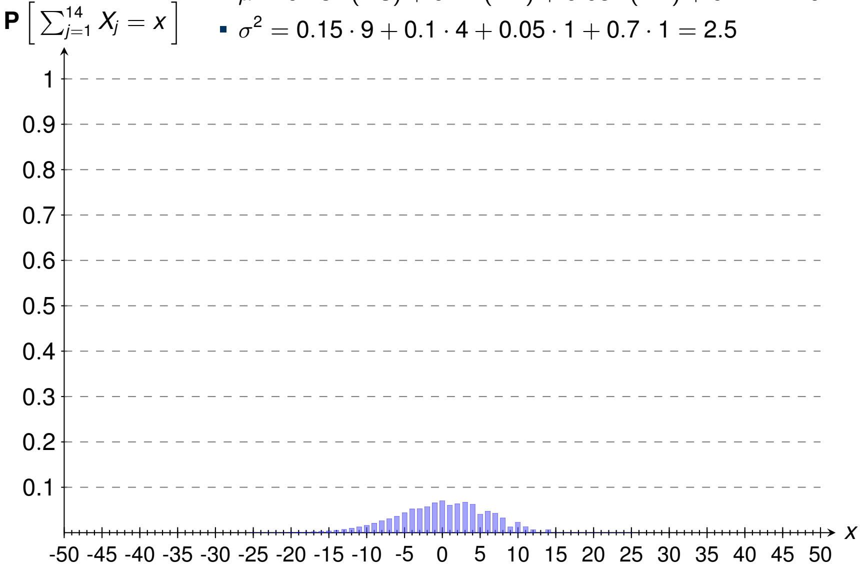

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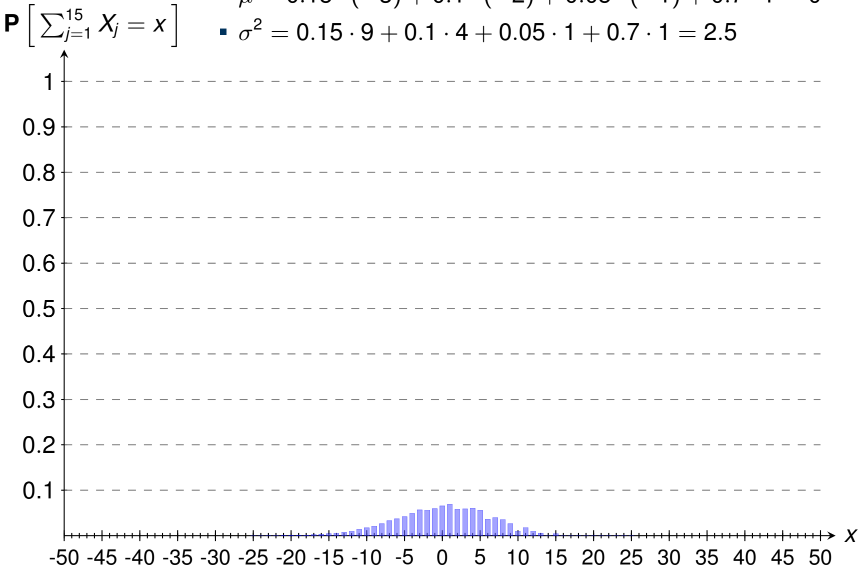

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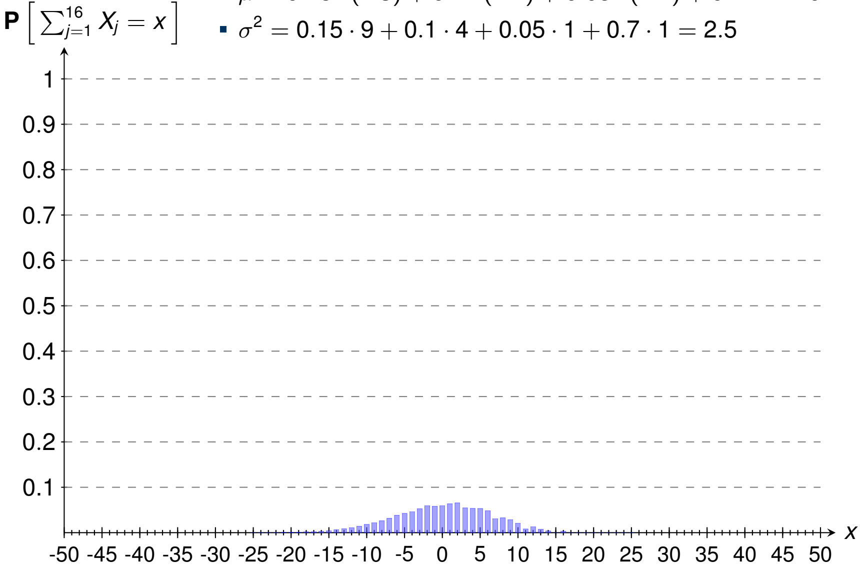

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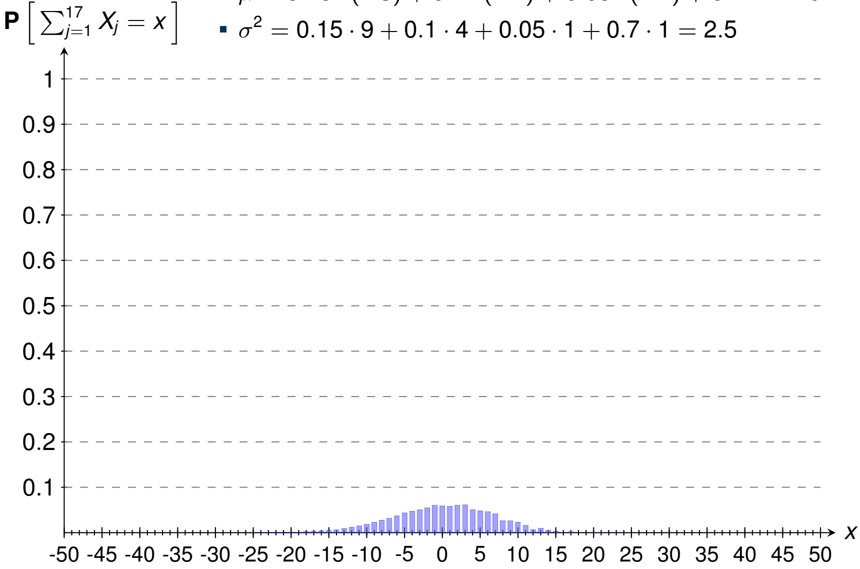

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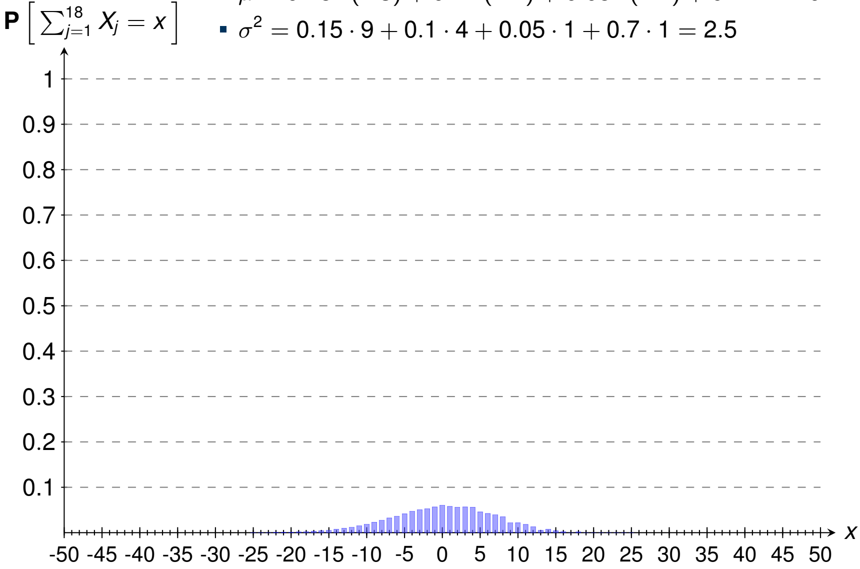

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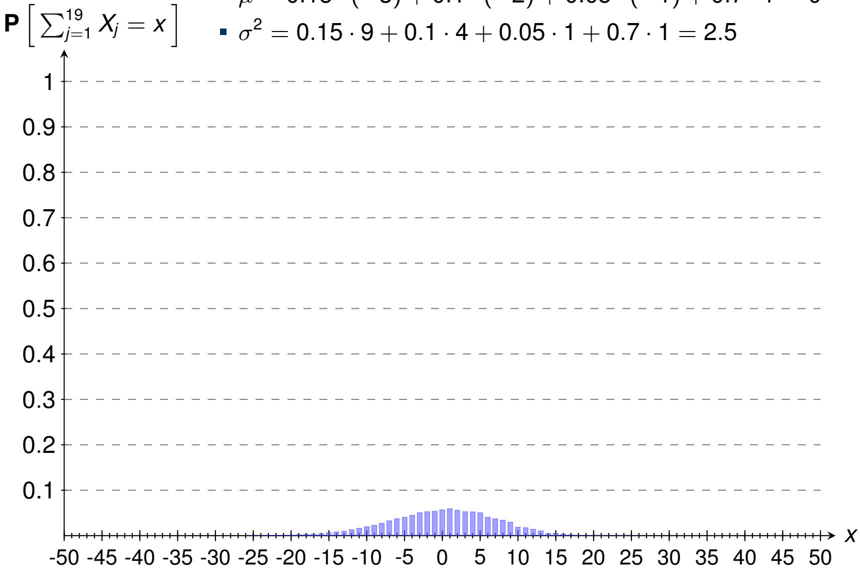

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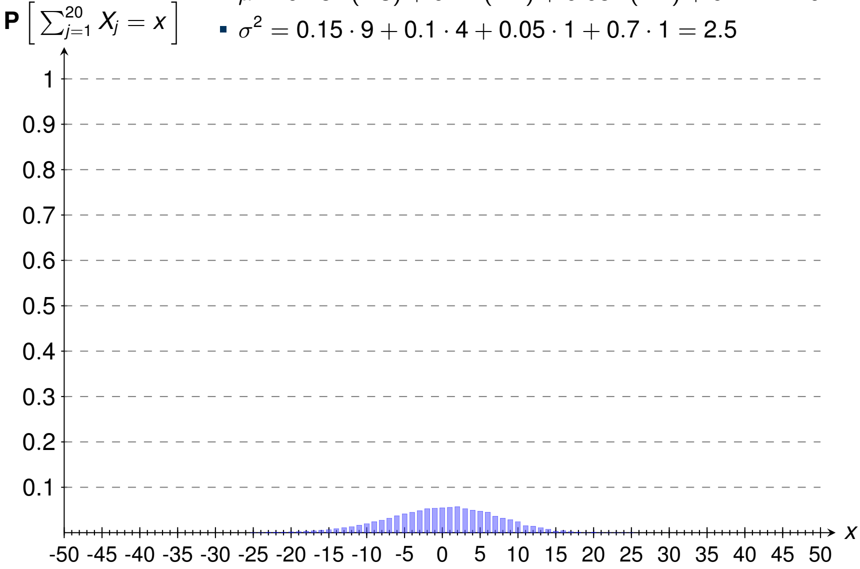

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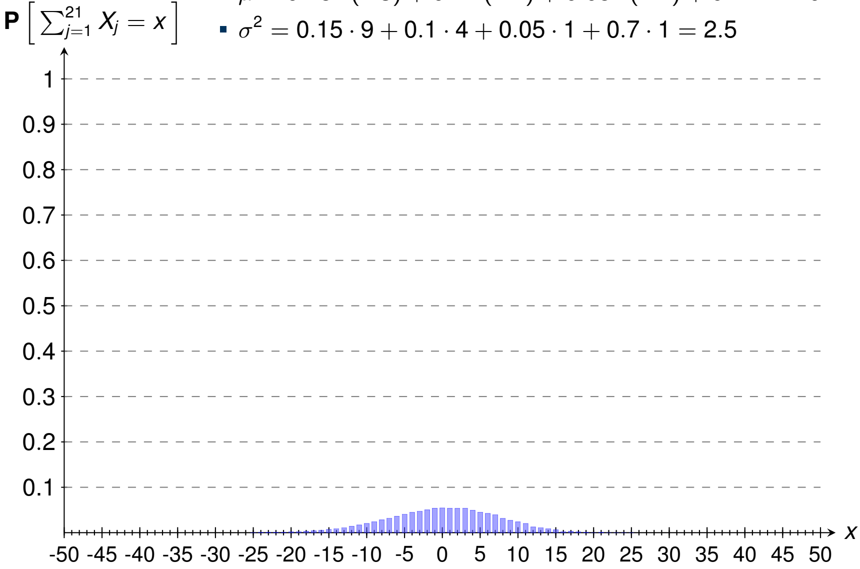

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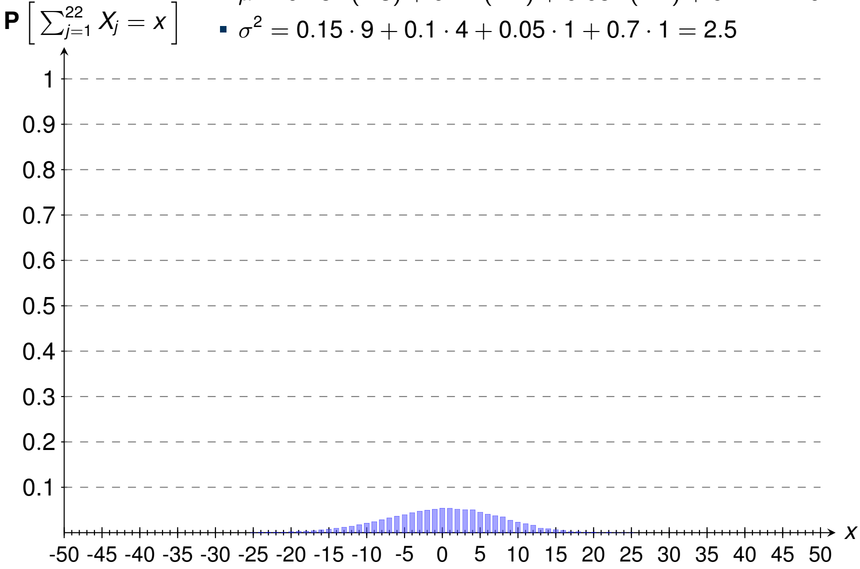

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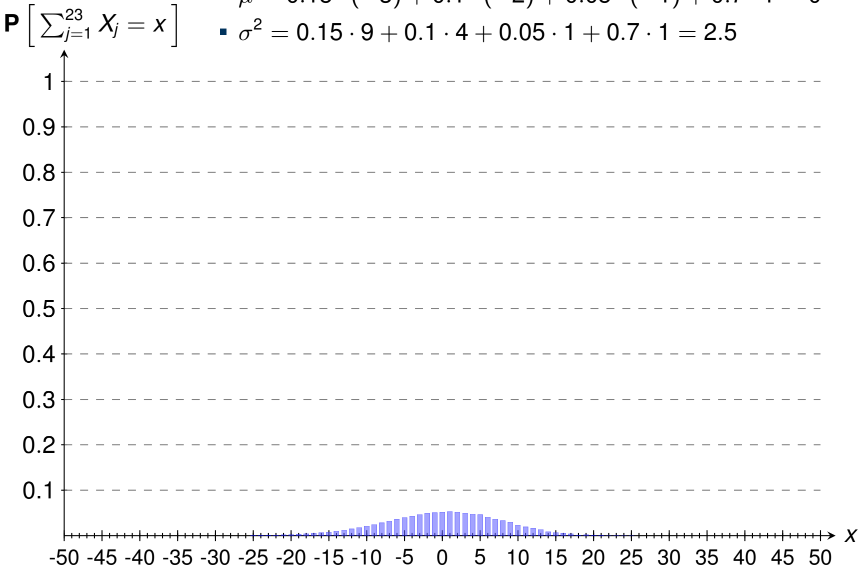

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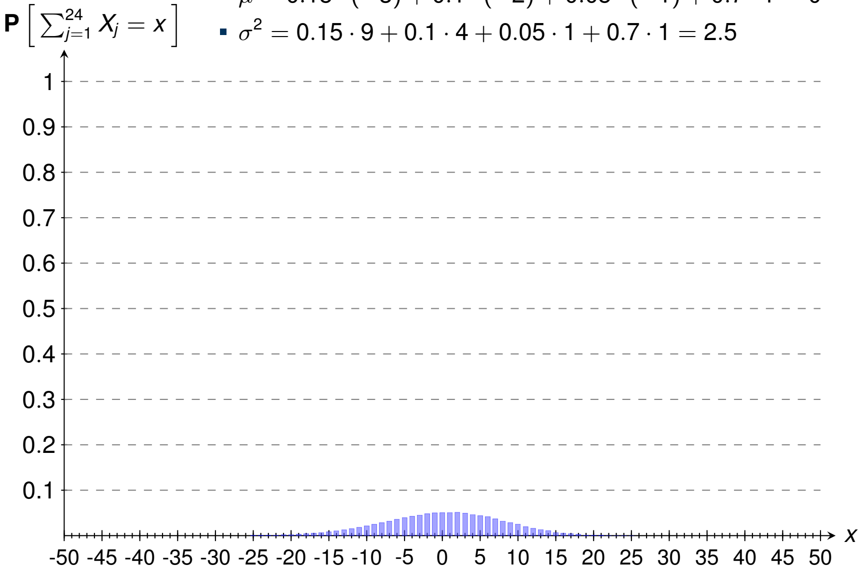

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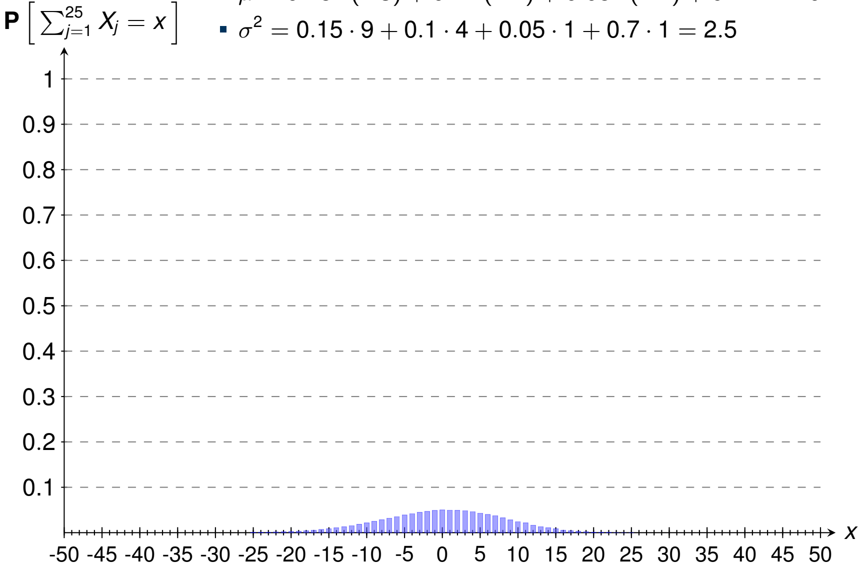

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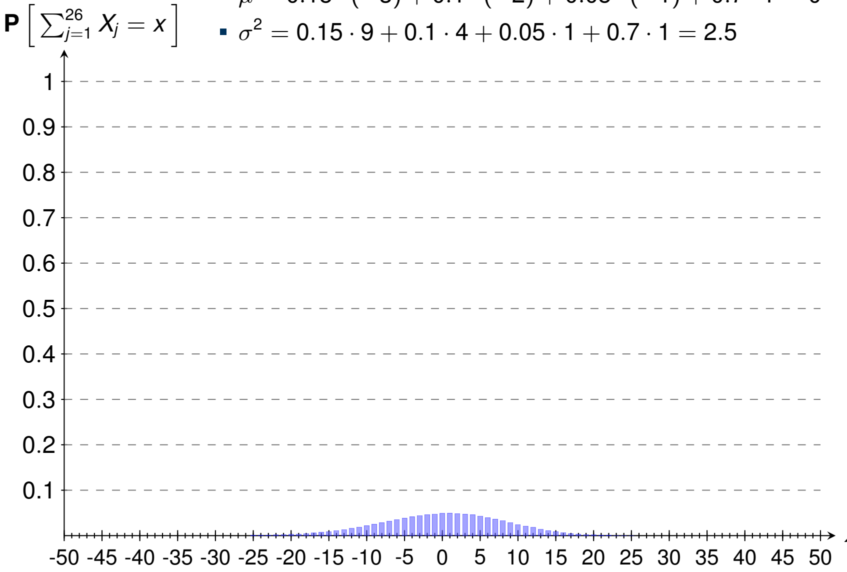

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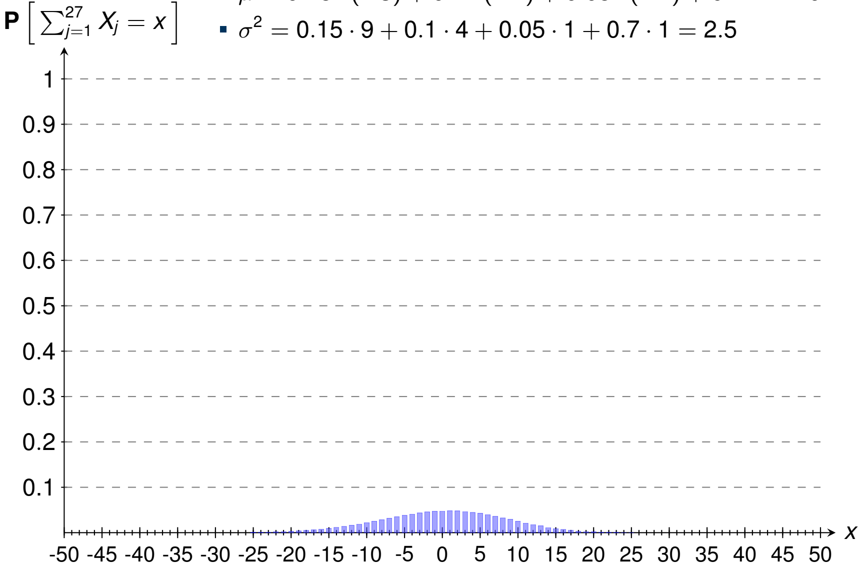

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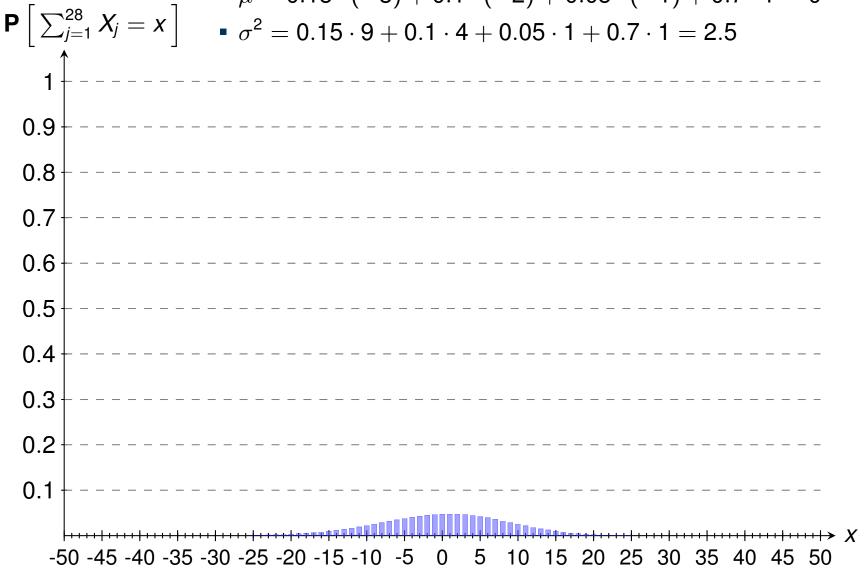

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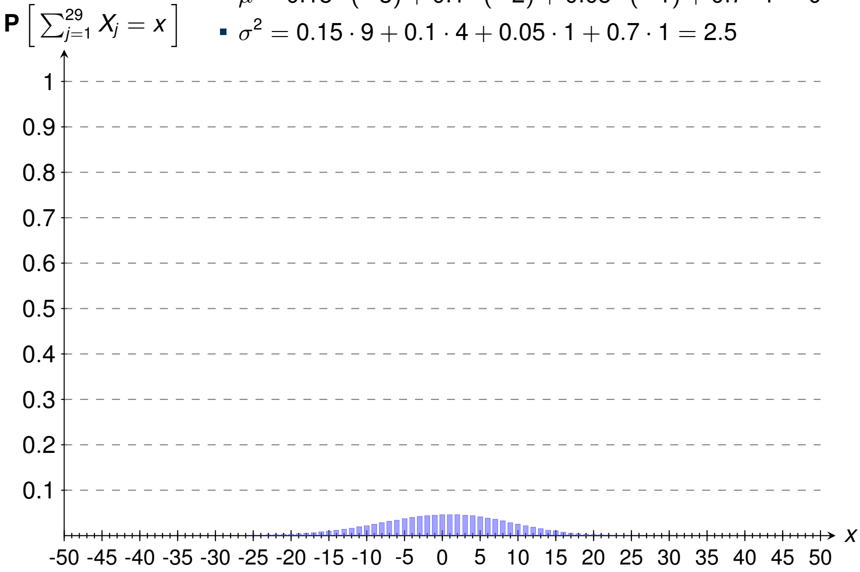

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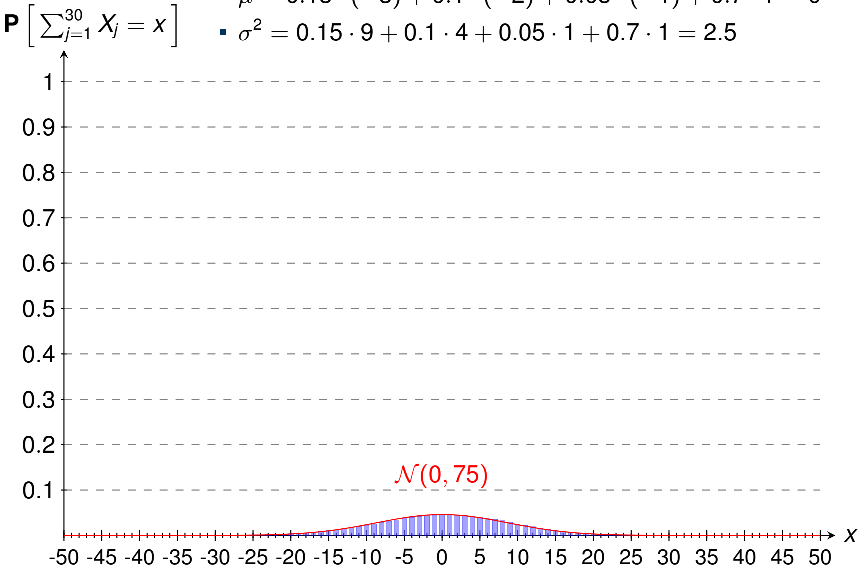

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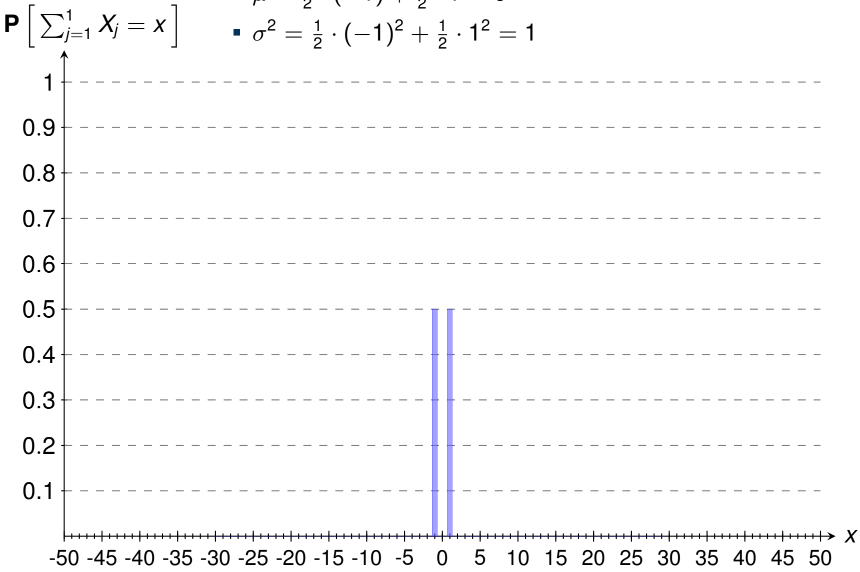

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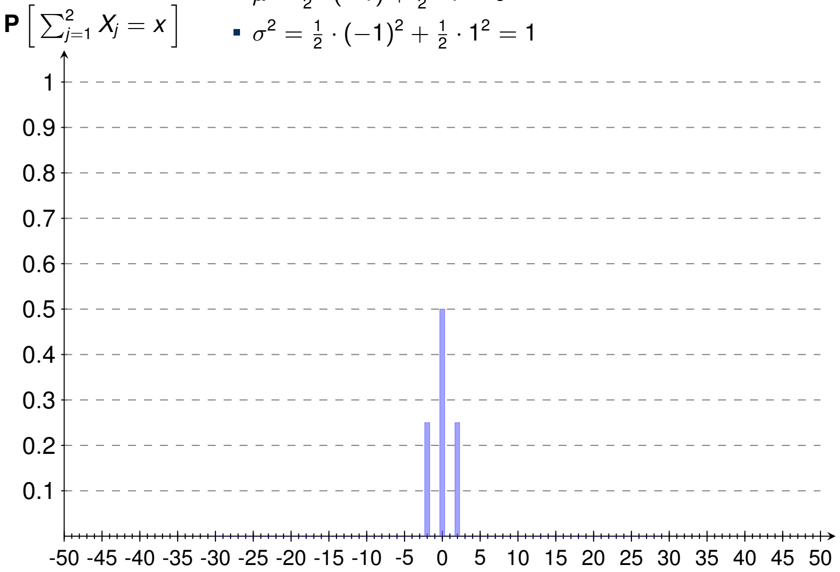

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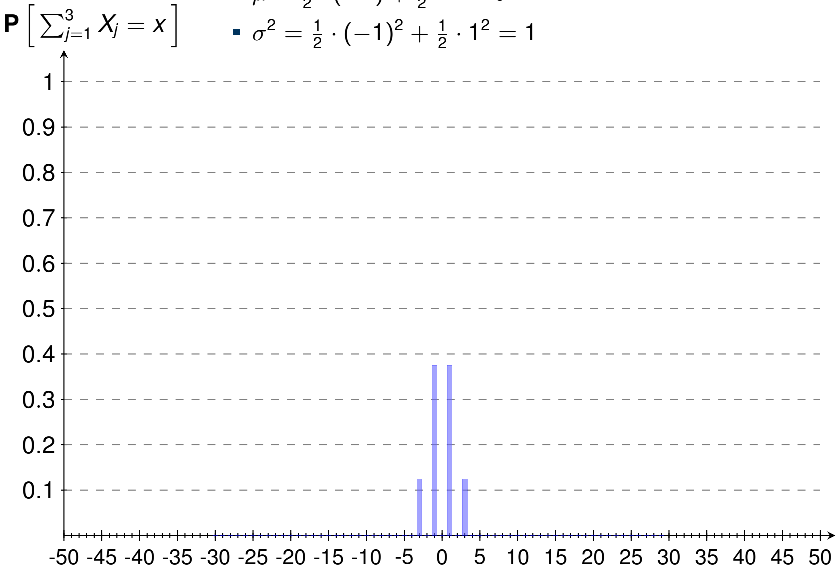

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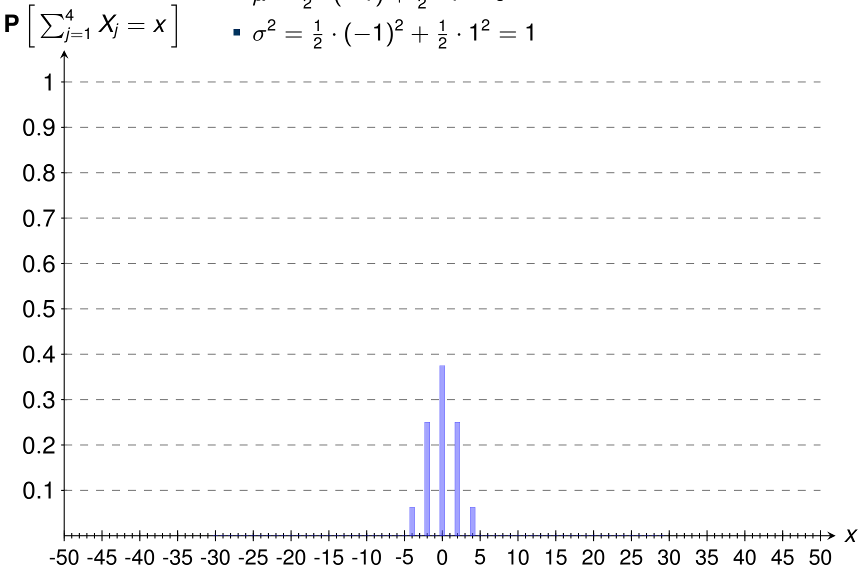

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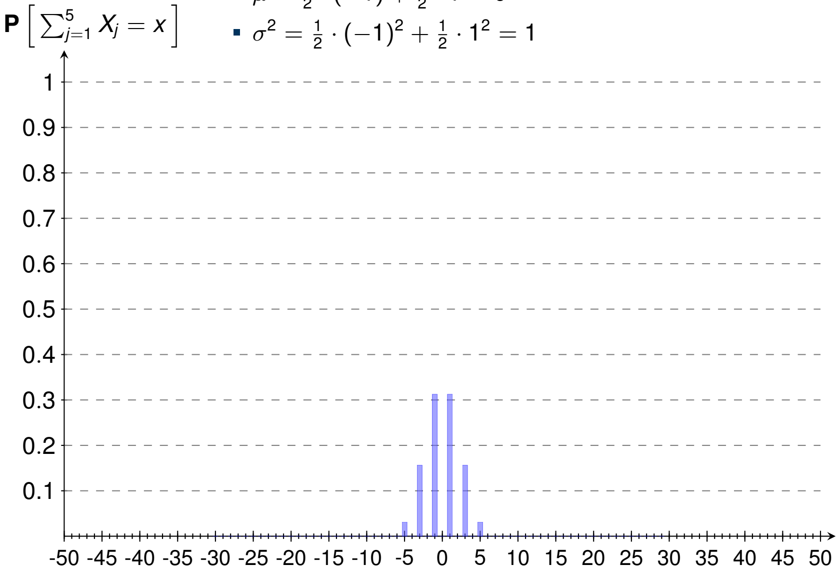

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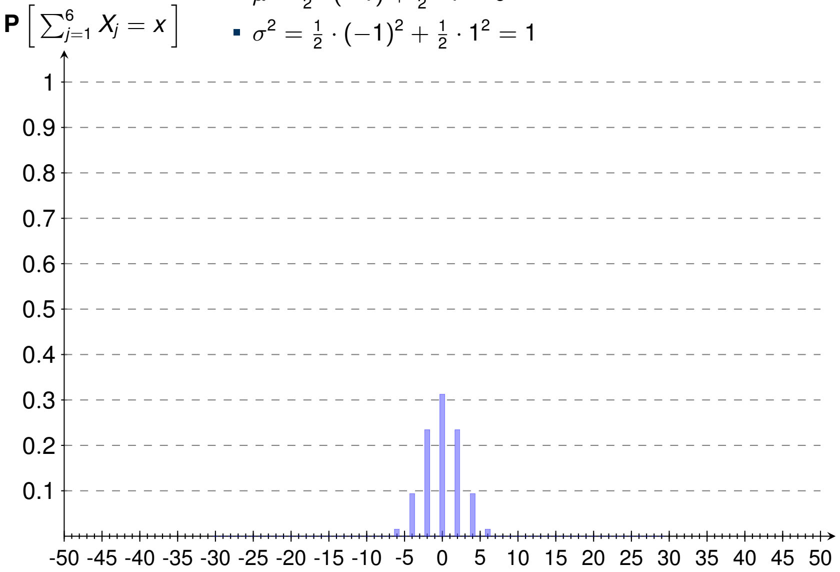

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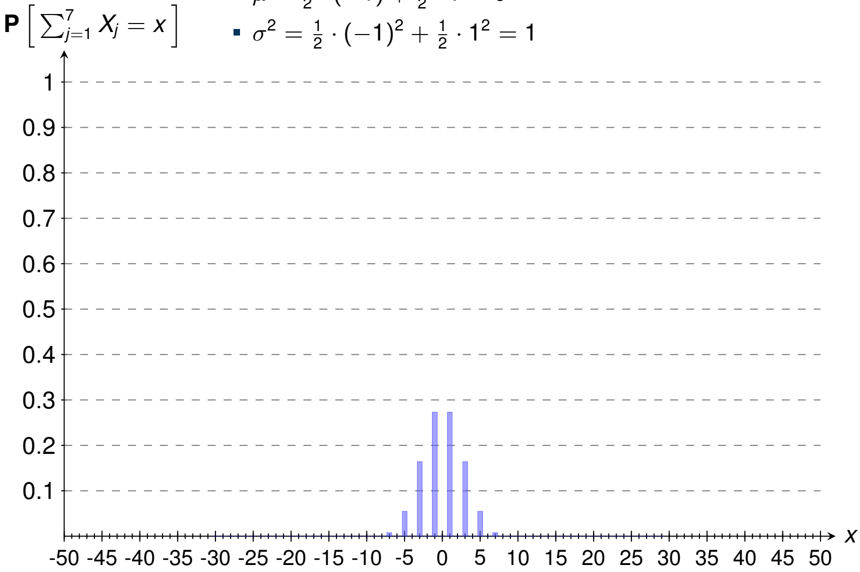

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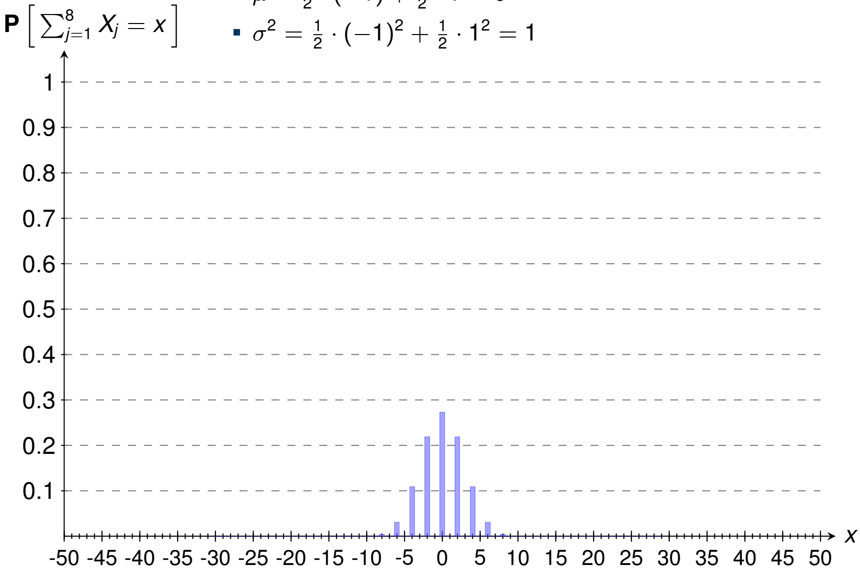

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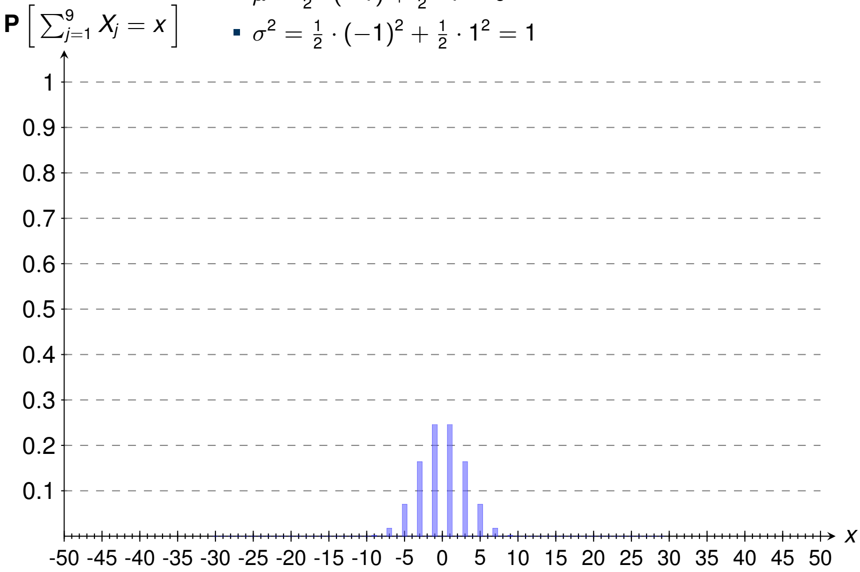

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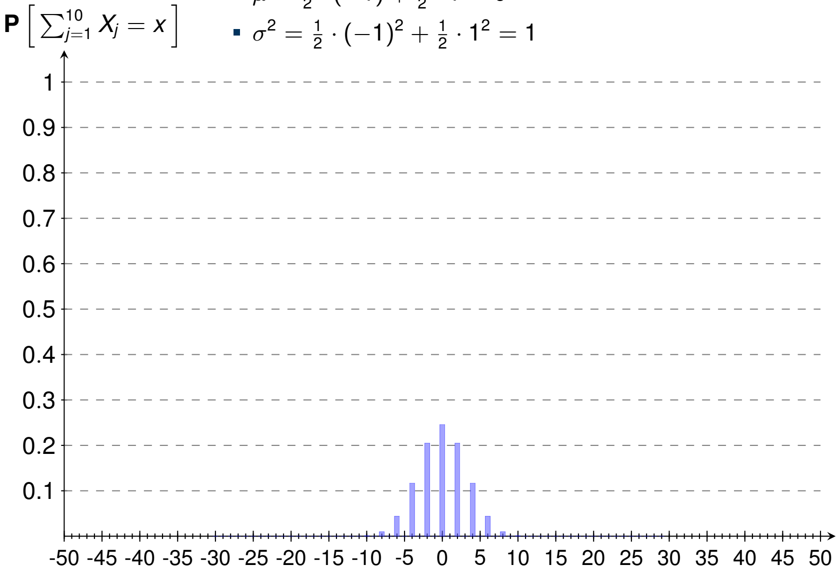

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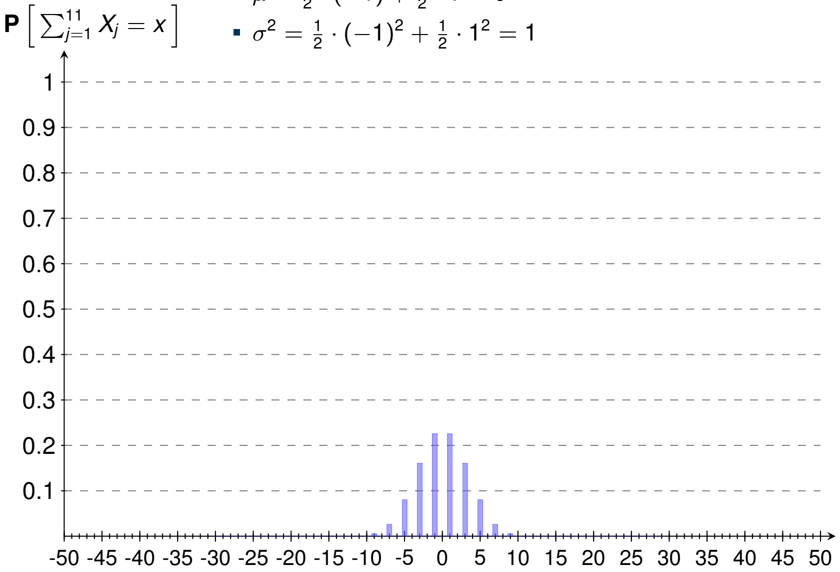

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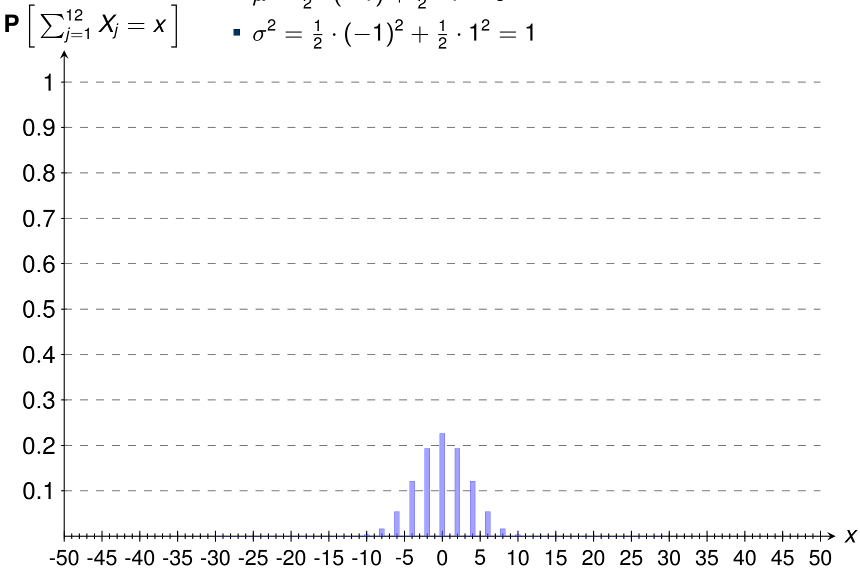

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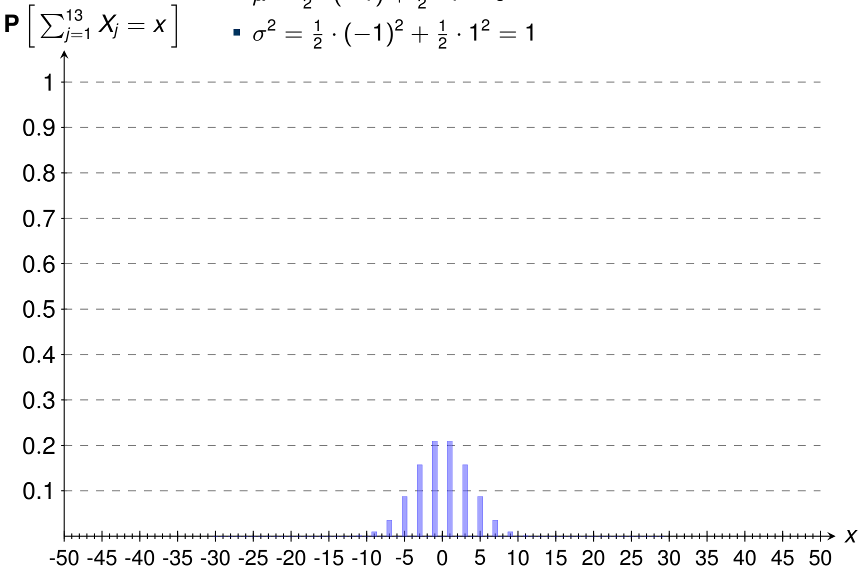

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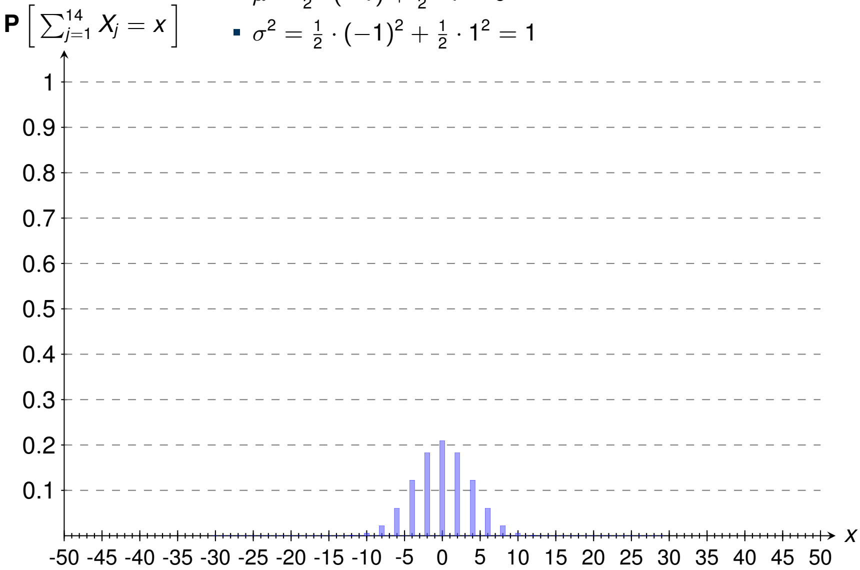

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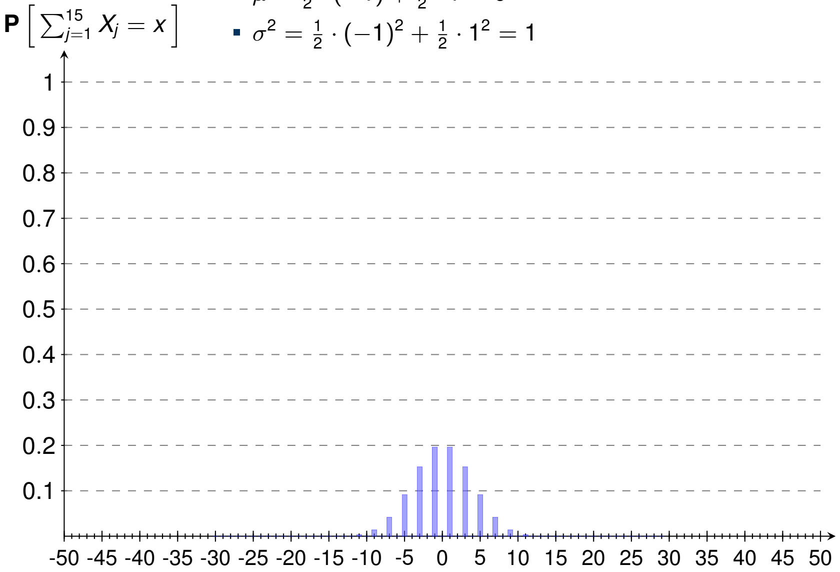

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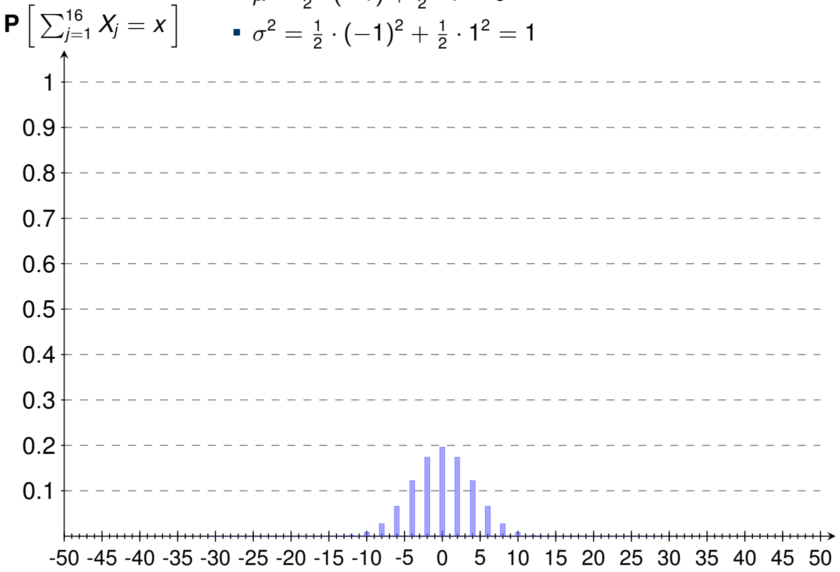

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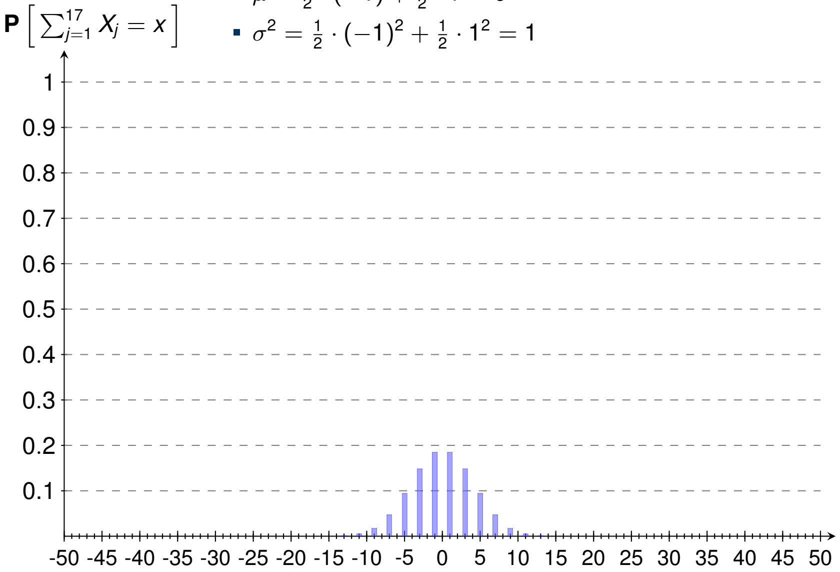

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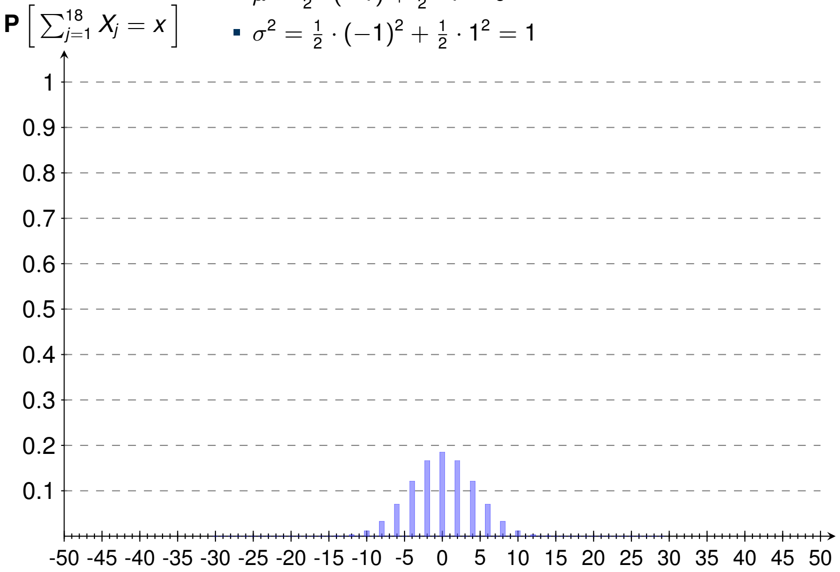

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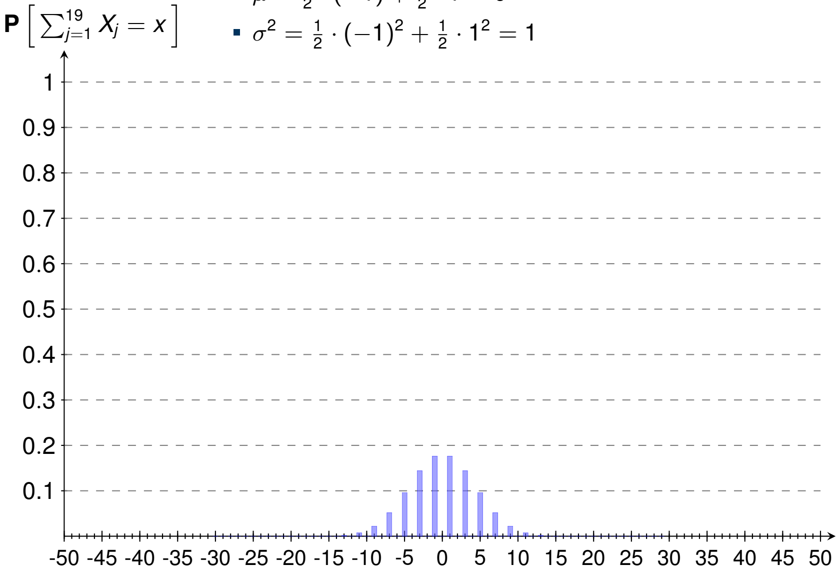

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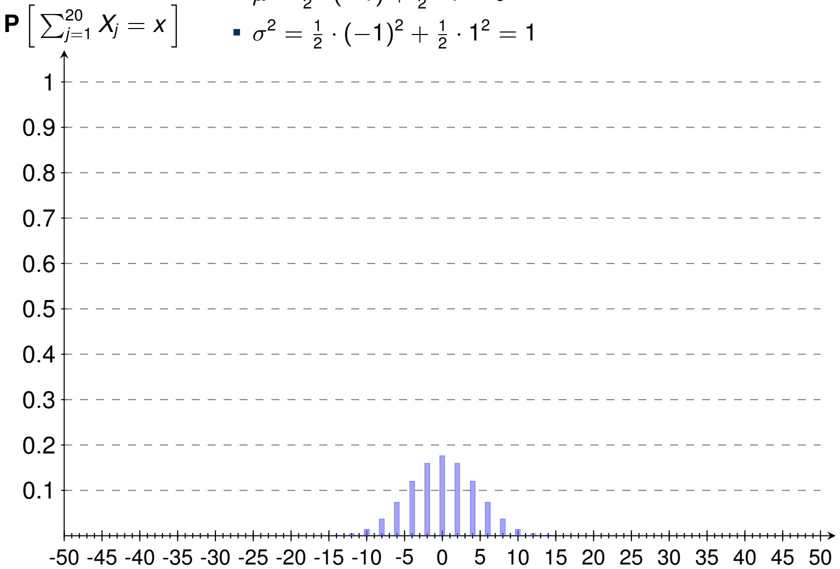

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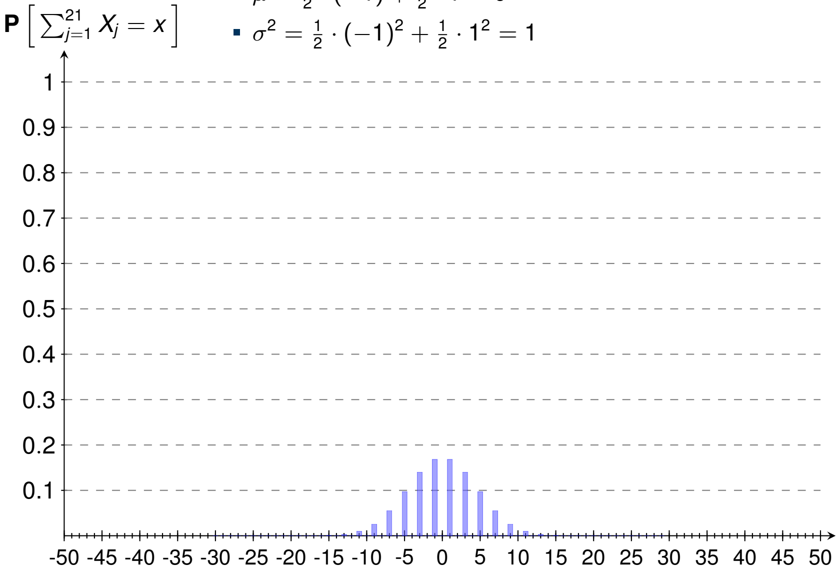

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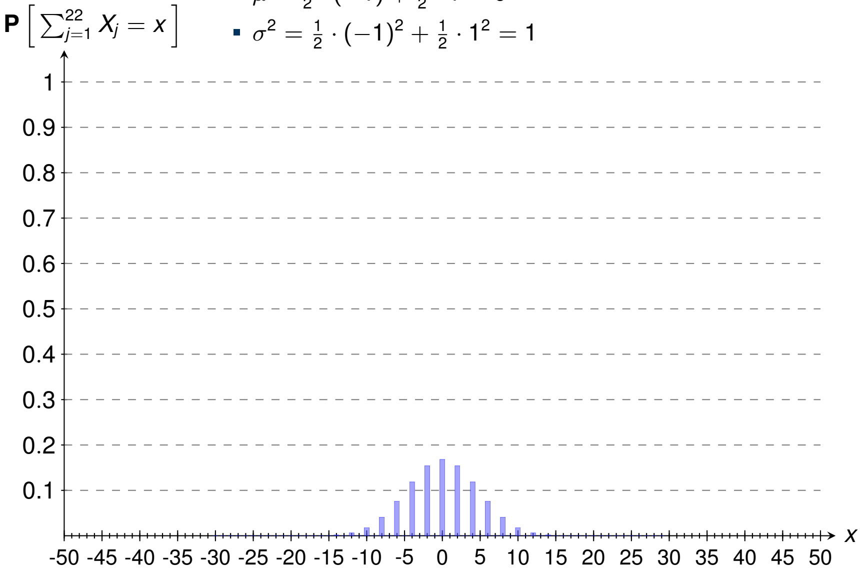

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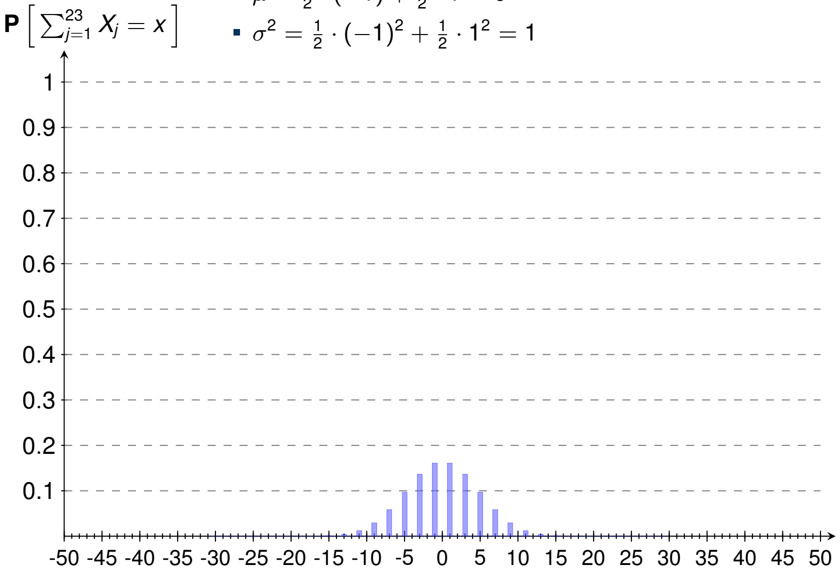

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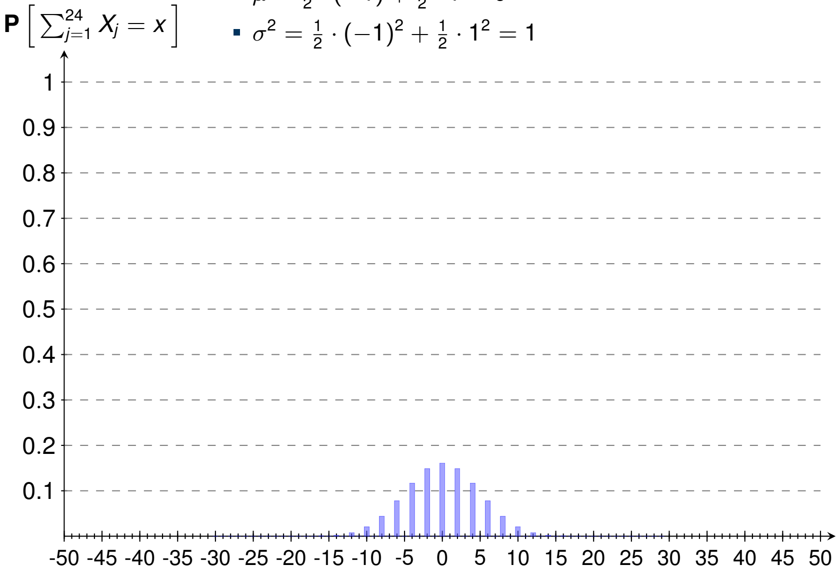

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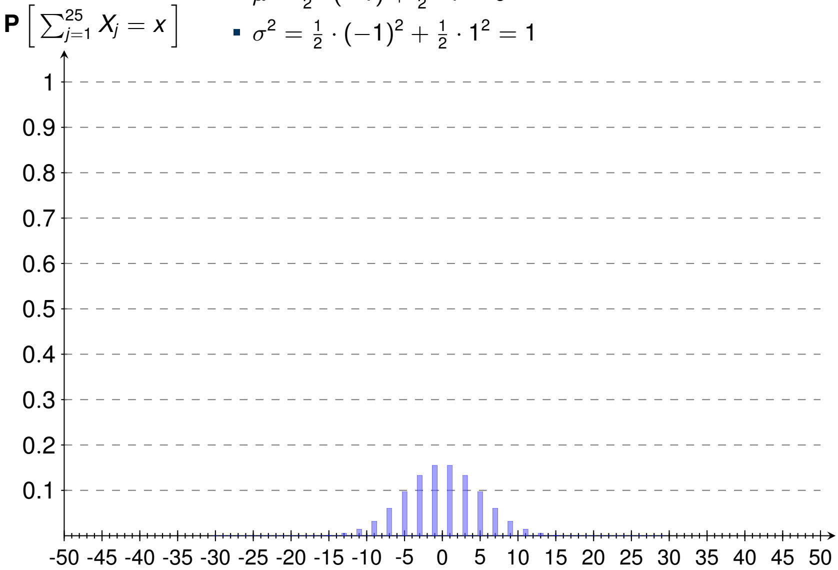

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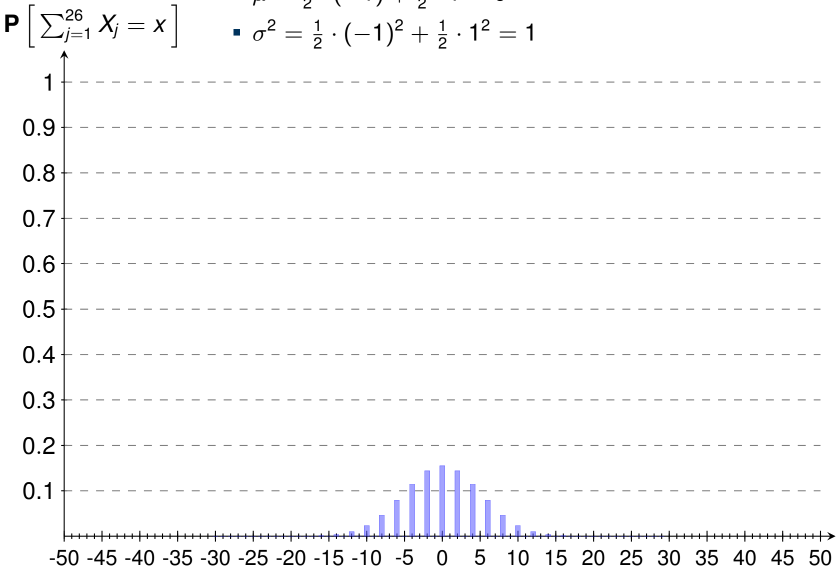

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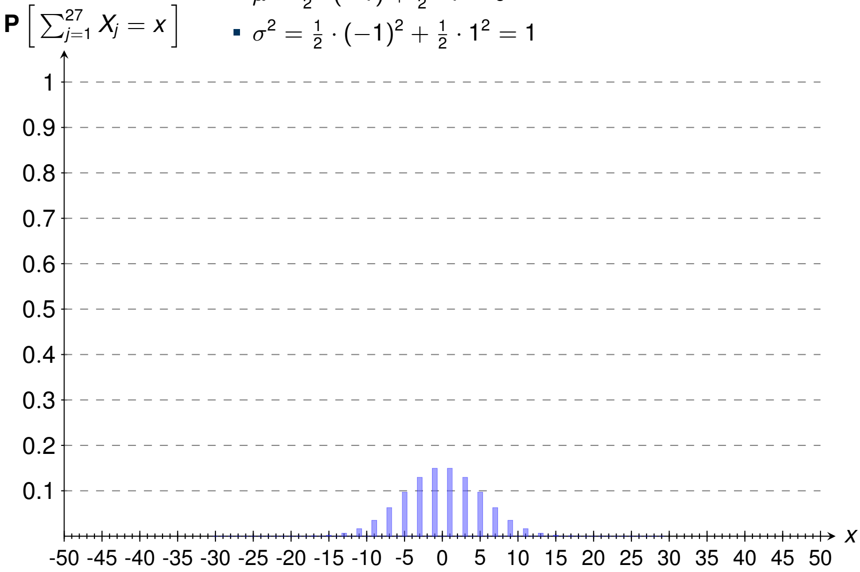

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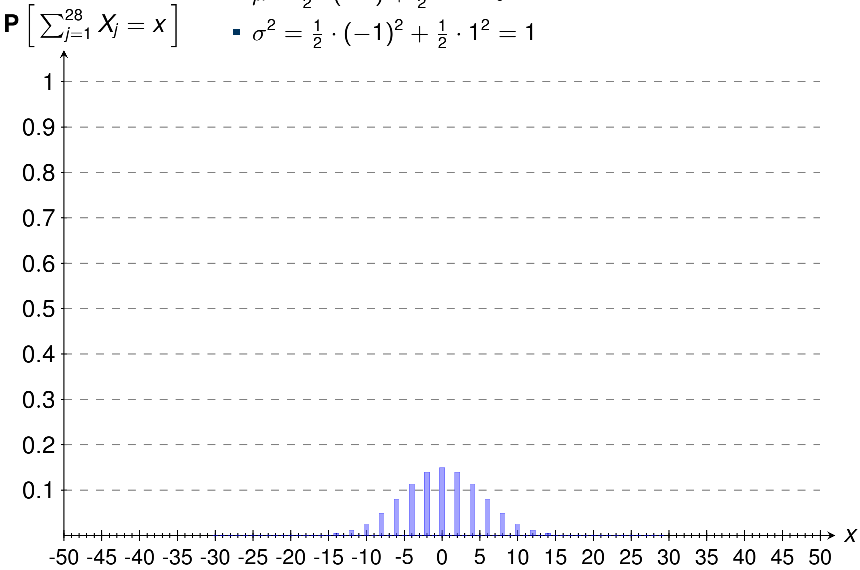

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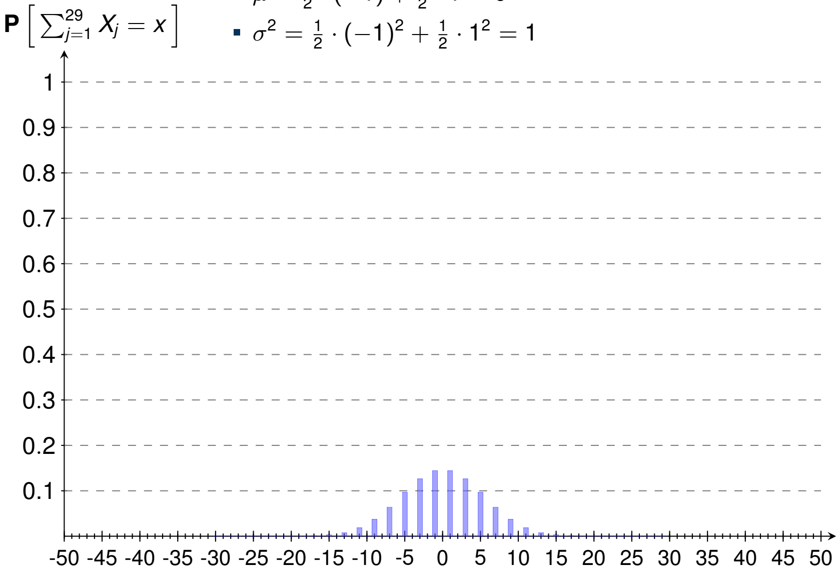

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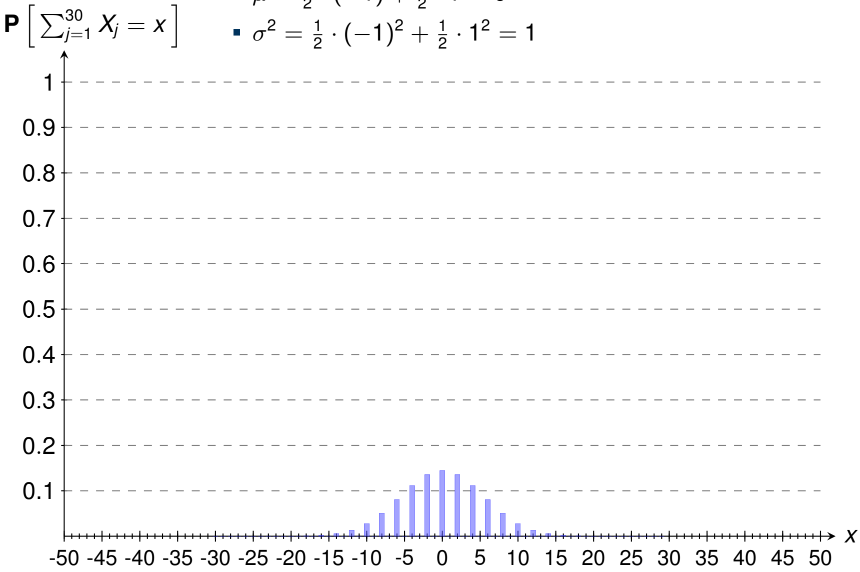

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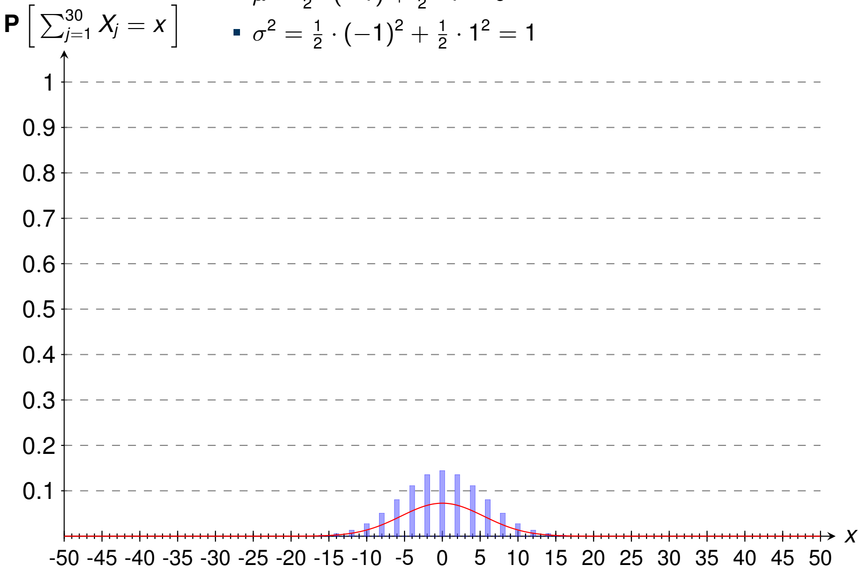

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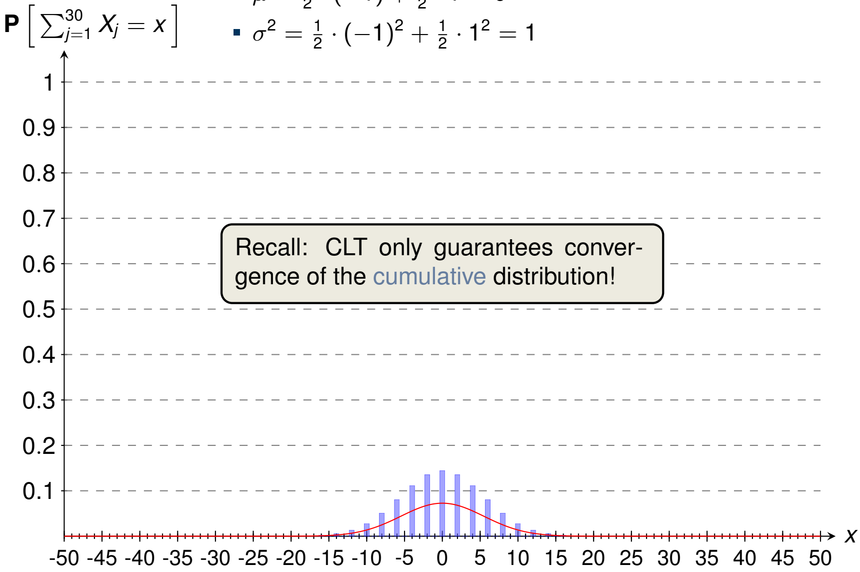

Recall: CLT only guarantees convergence of the cumulative distribution!

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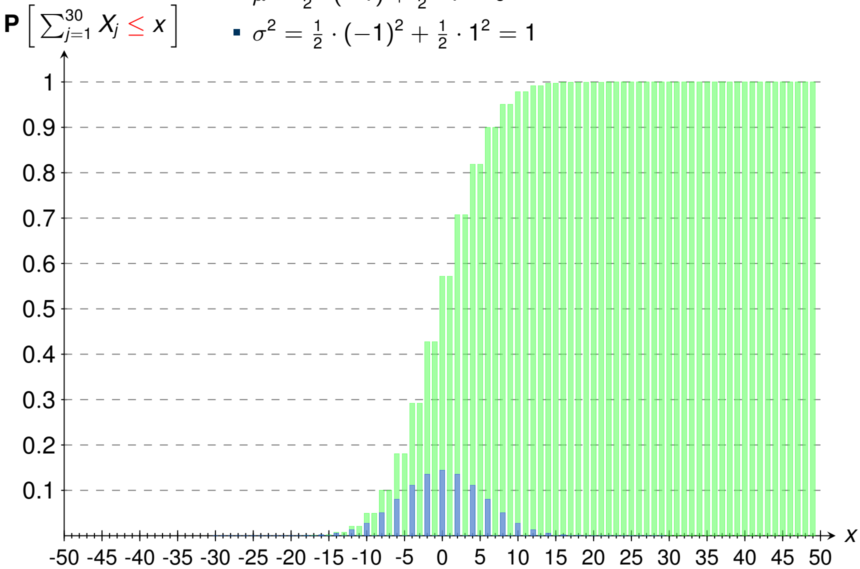

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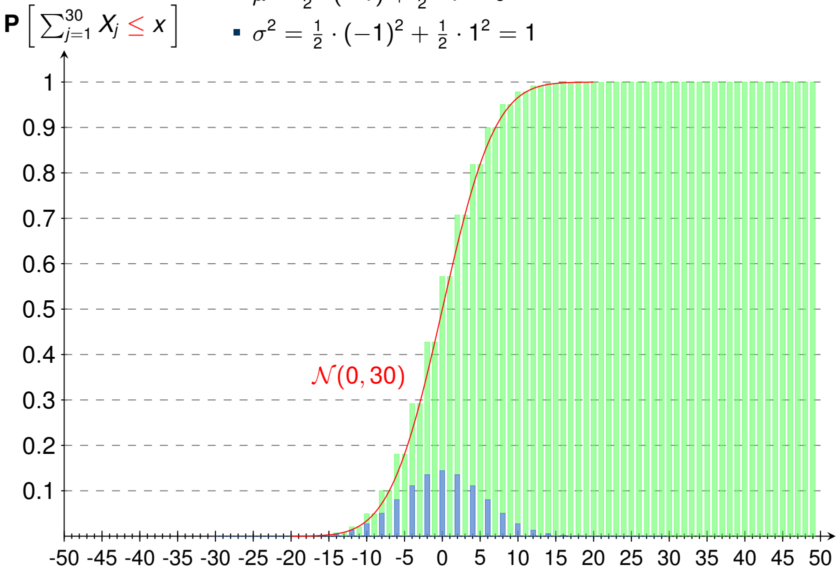

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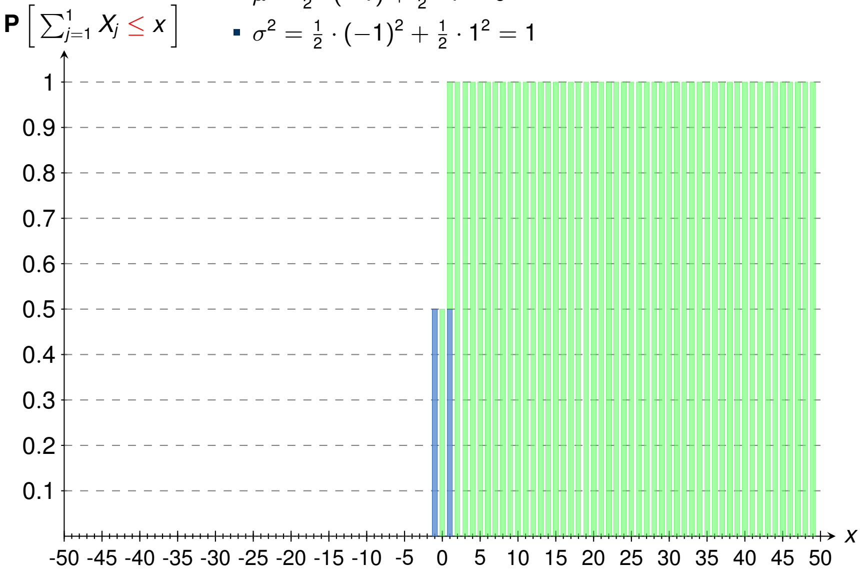

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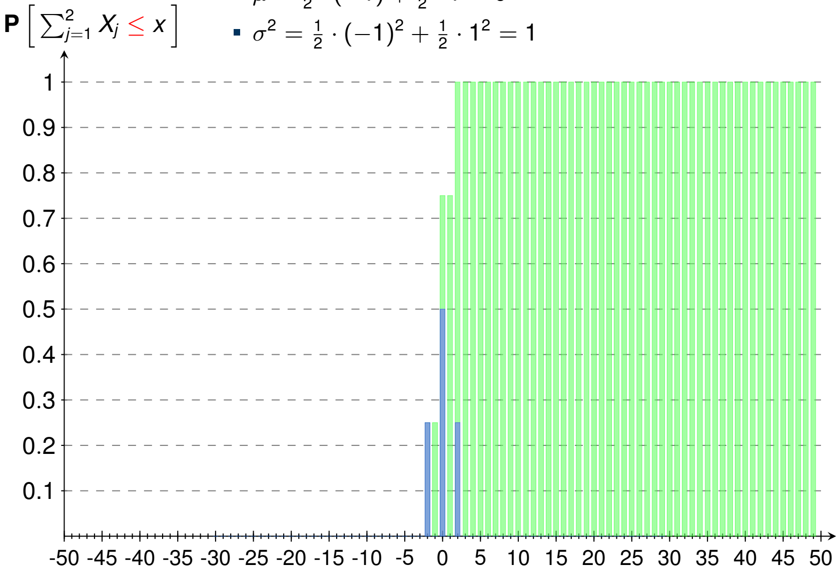

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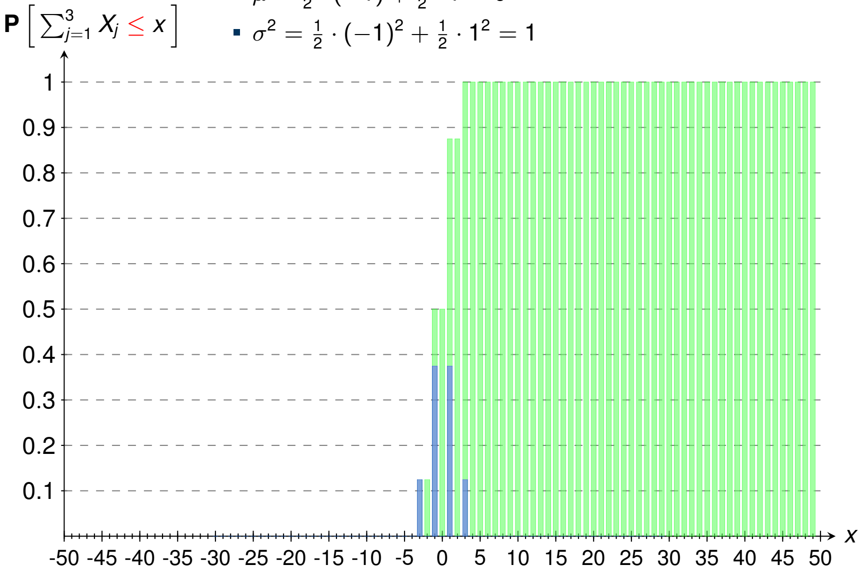

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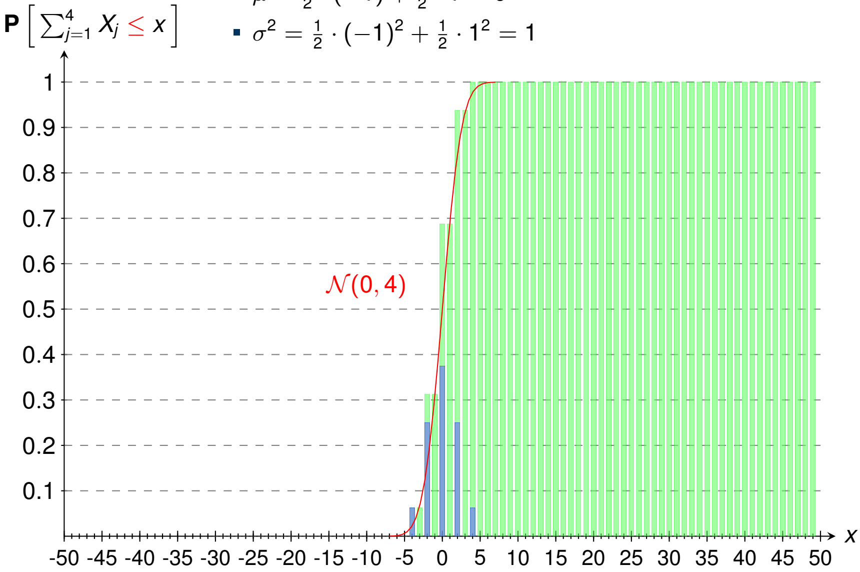

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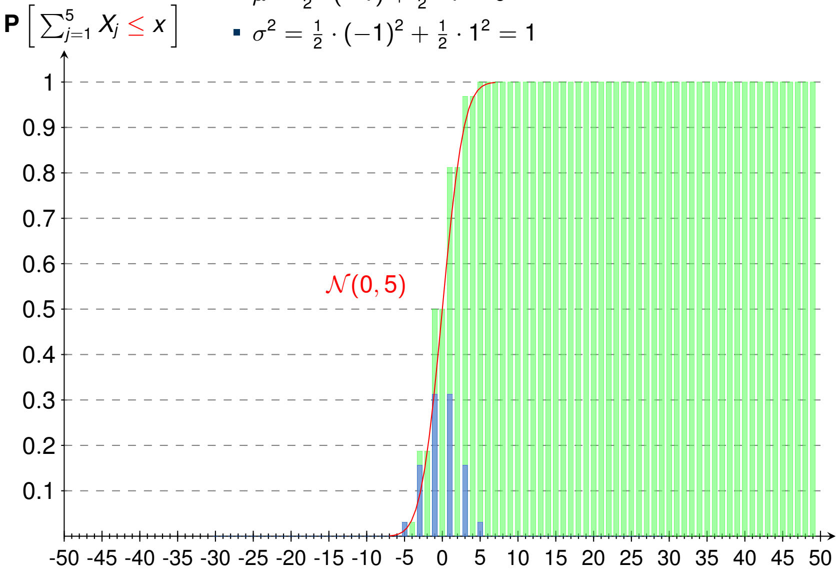

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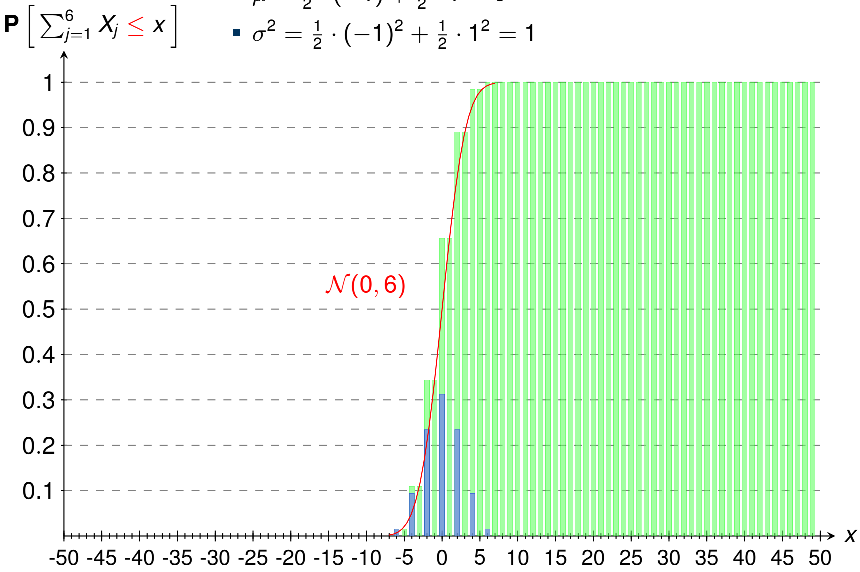

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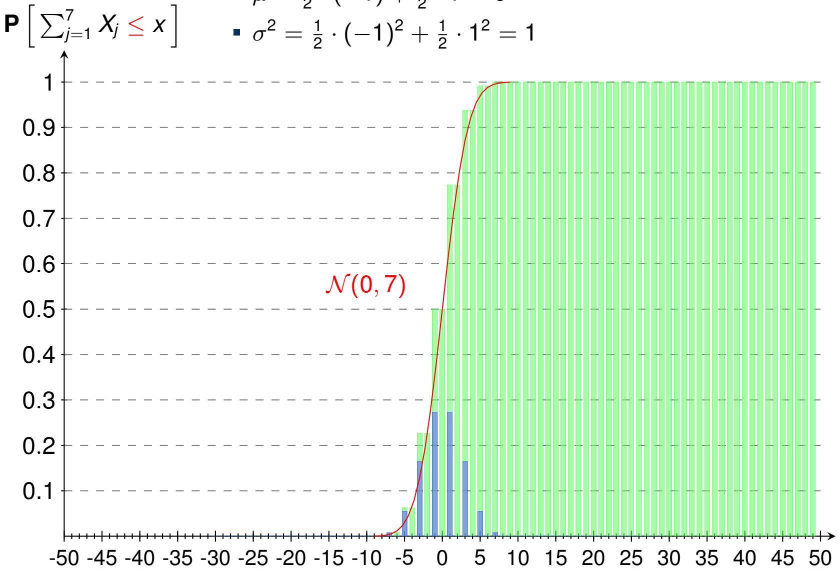

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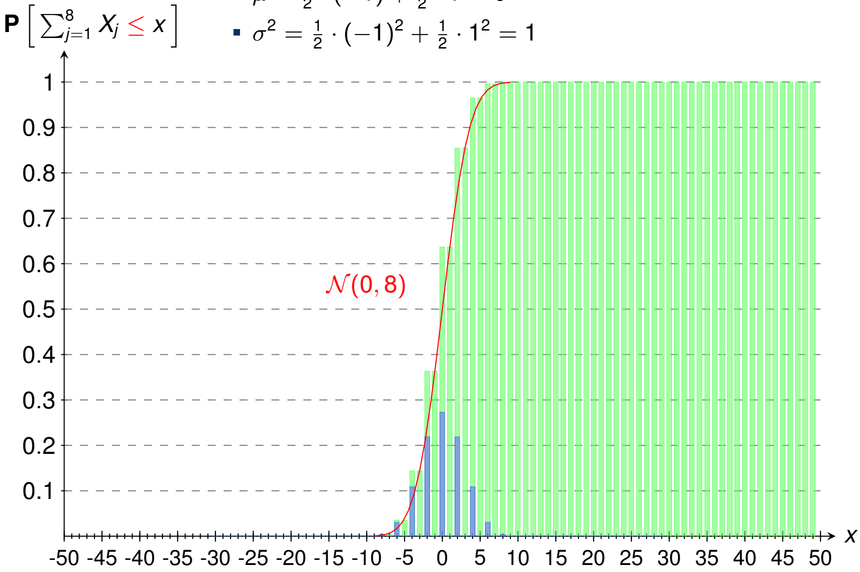

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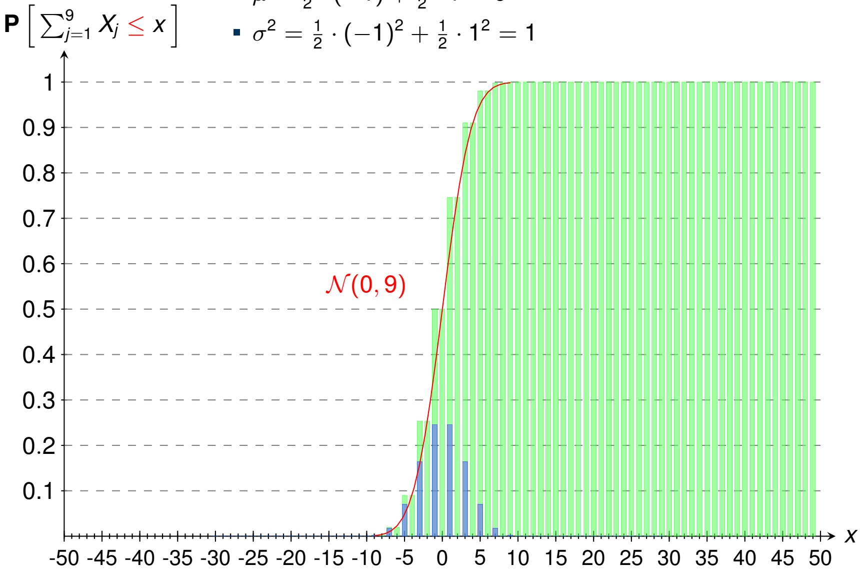

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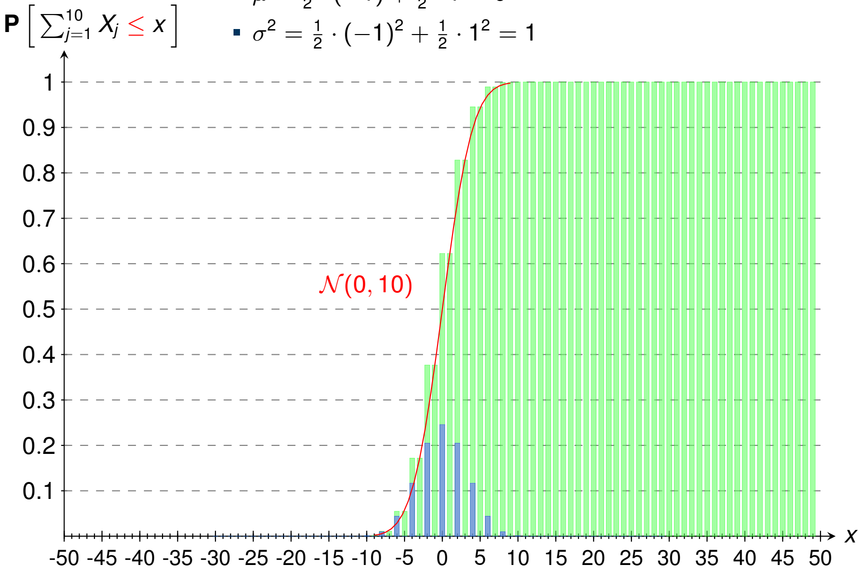

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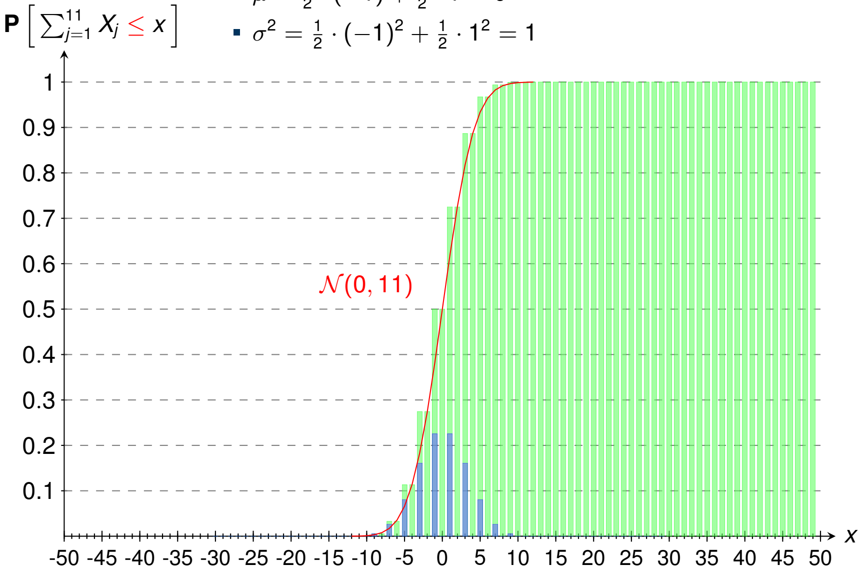

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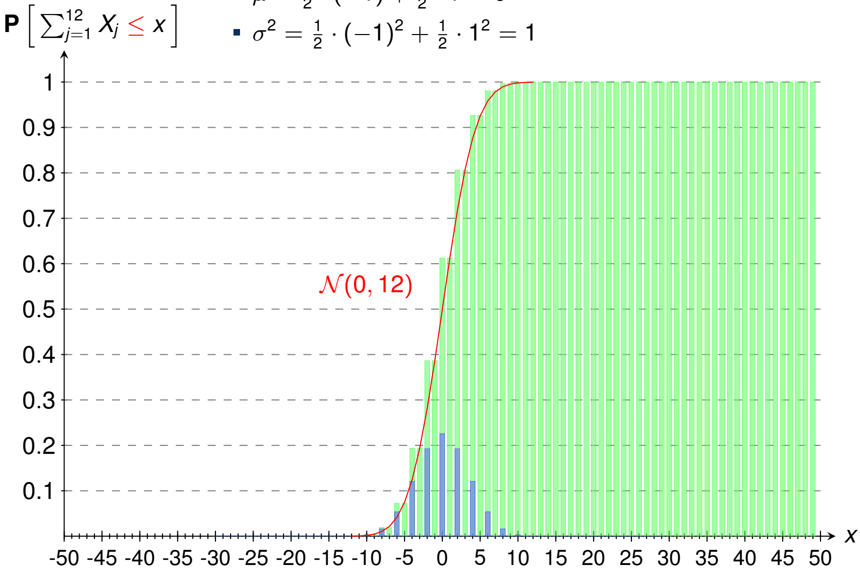

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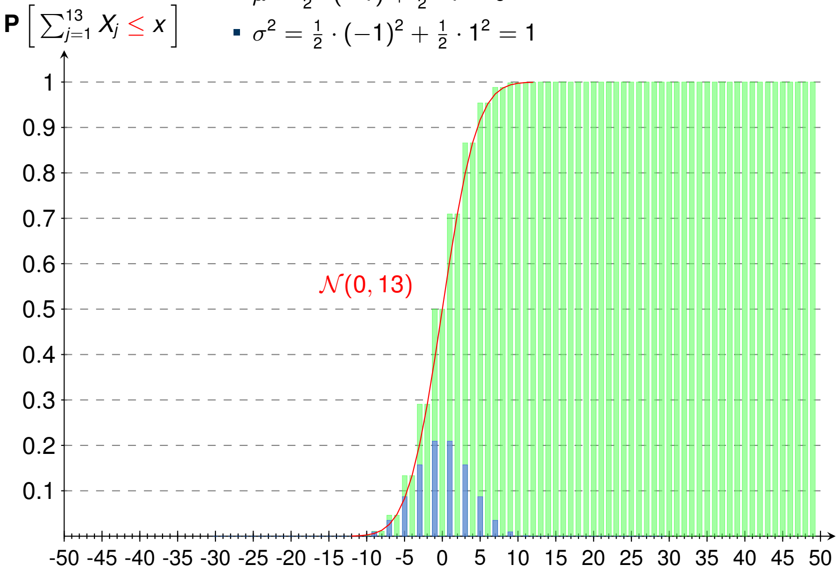

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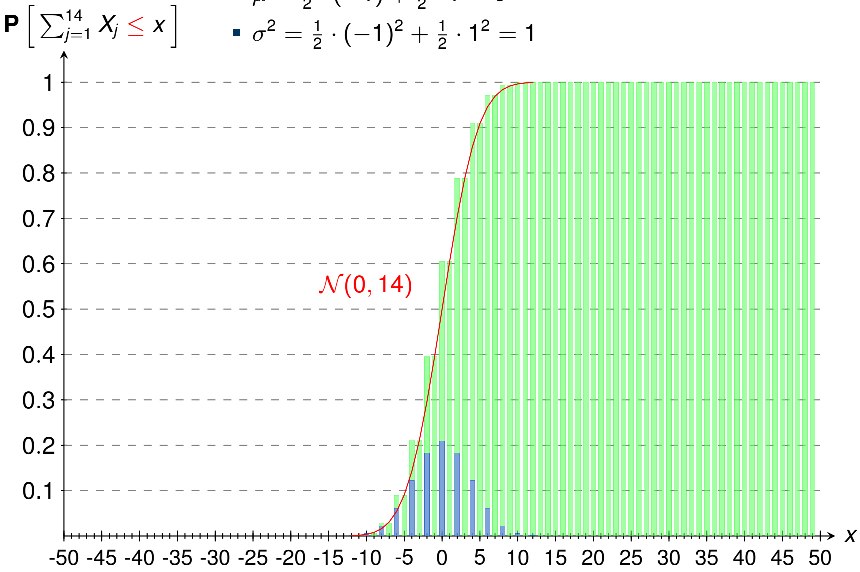

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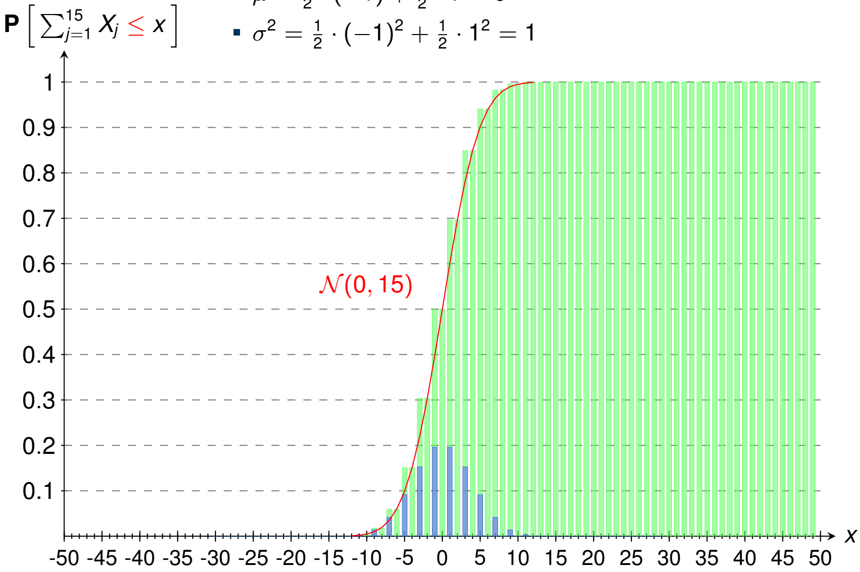

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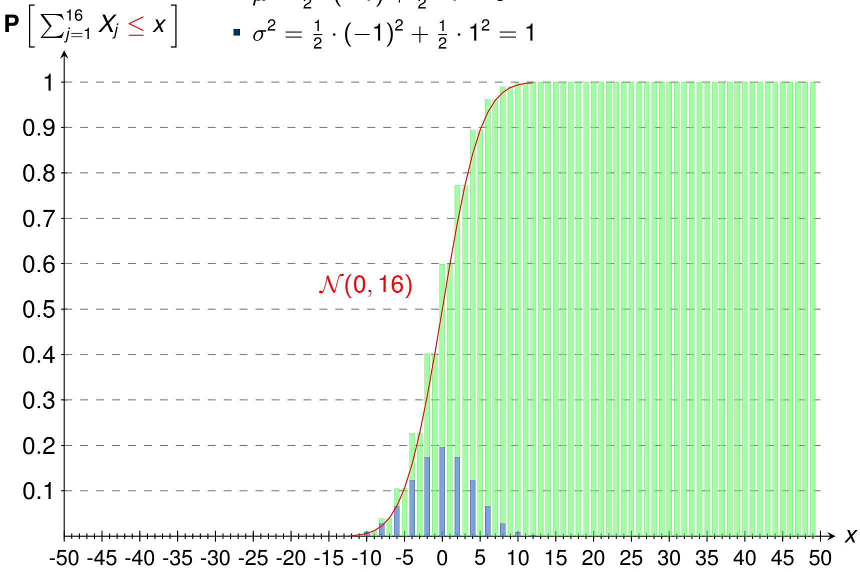

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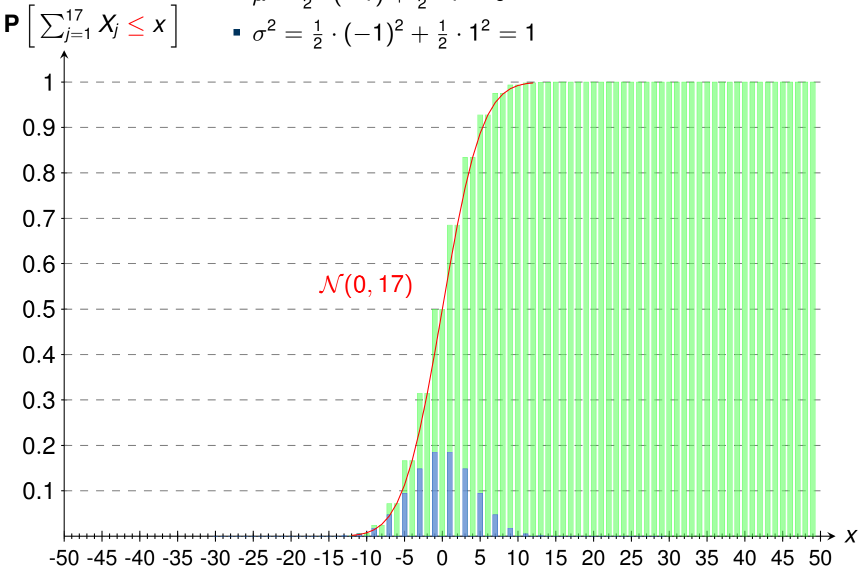

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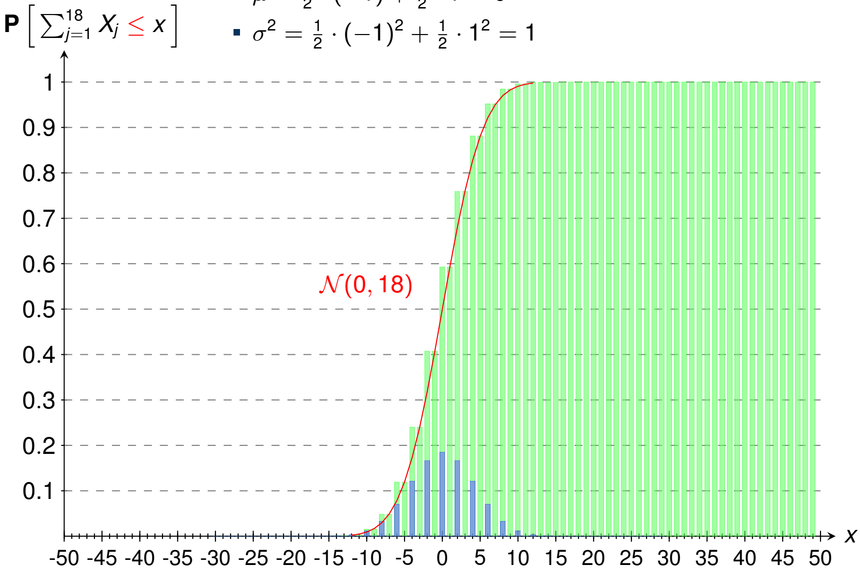

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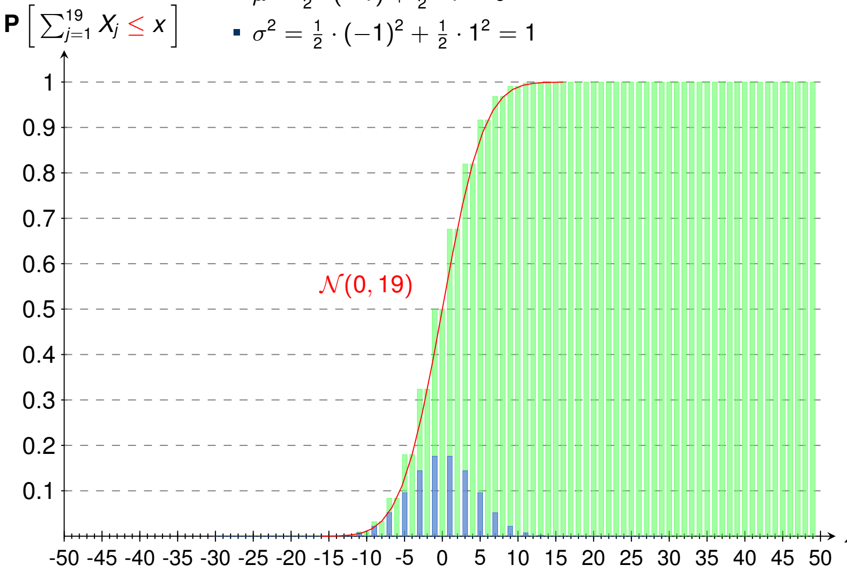

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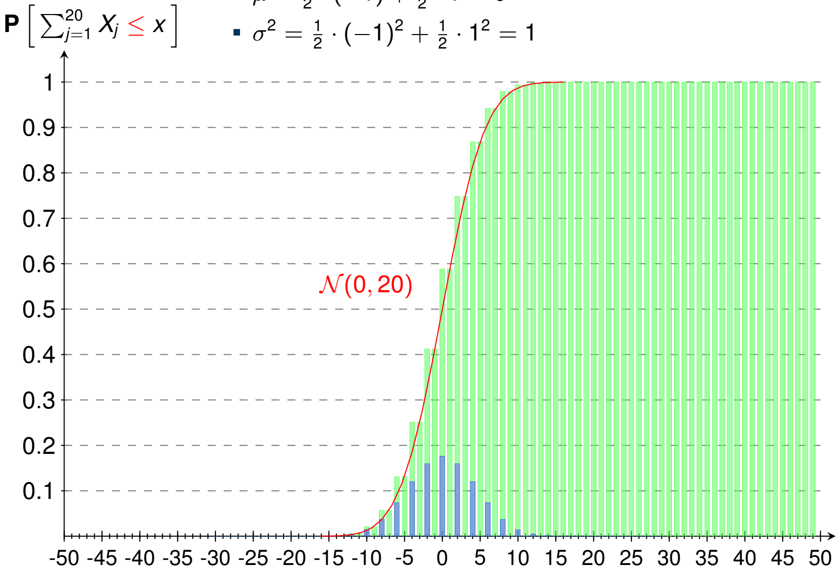

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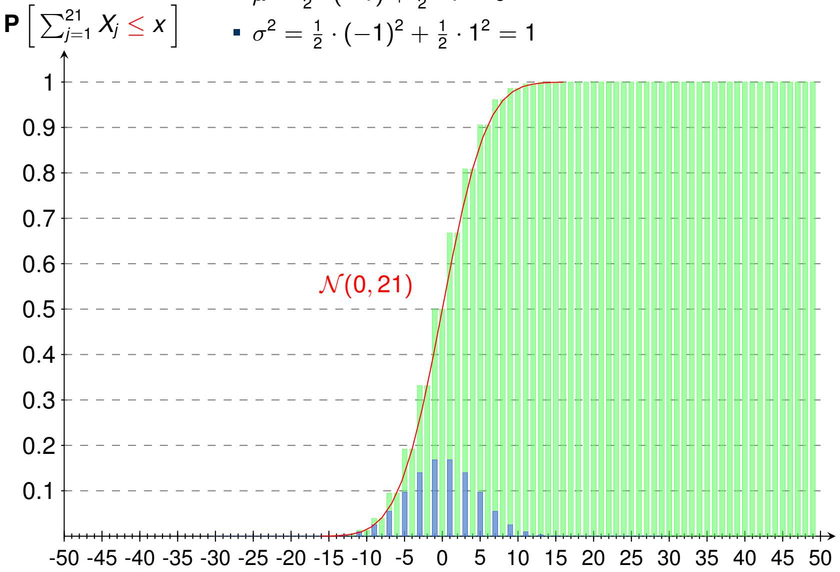

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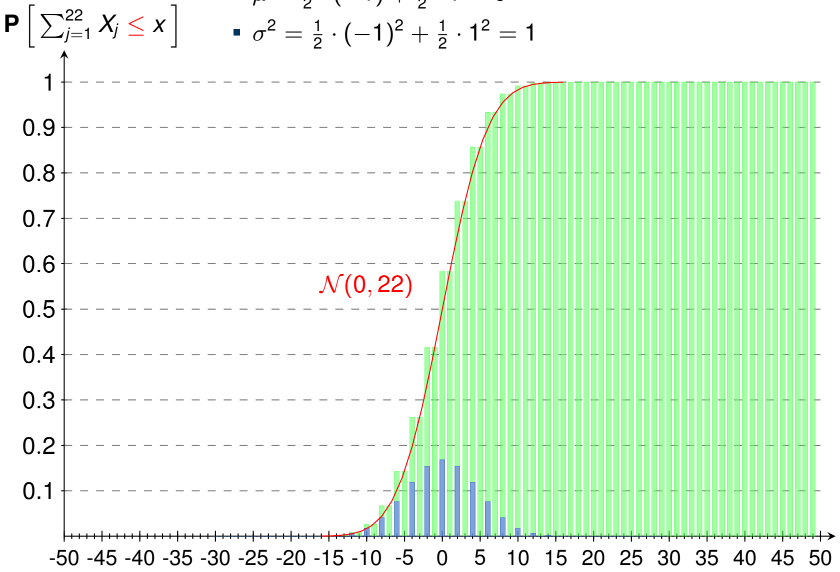

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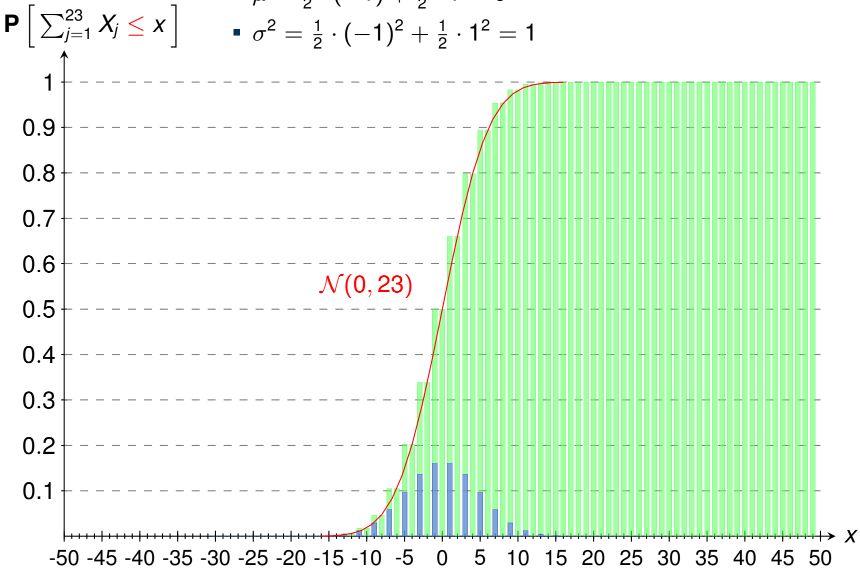

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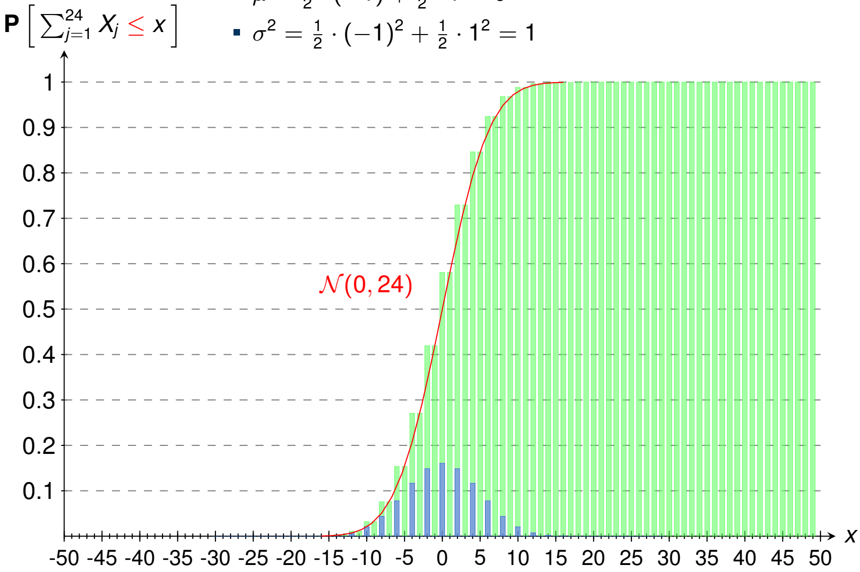

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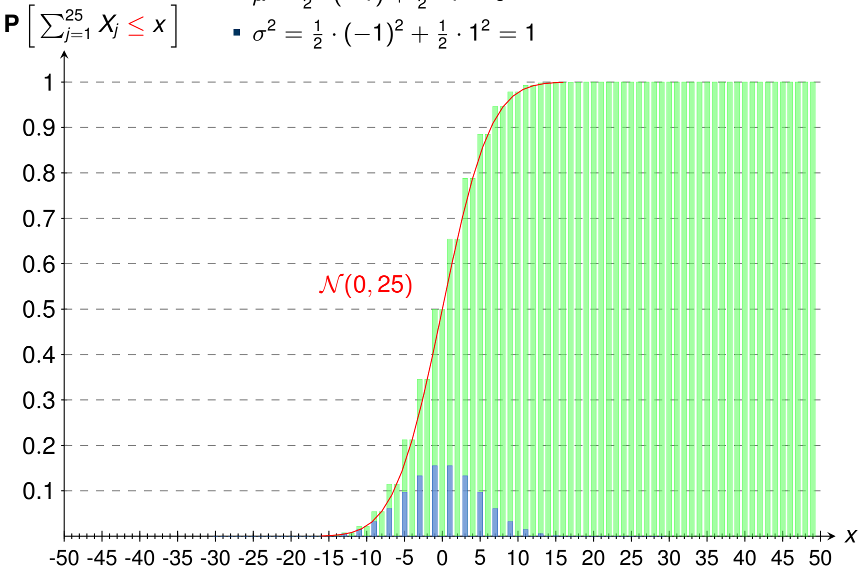

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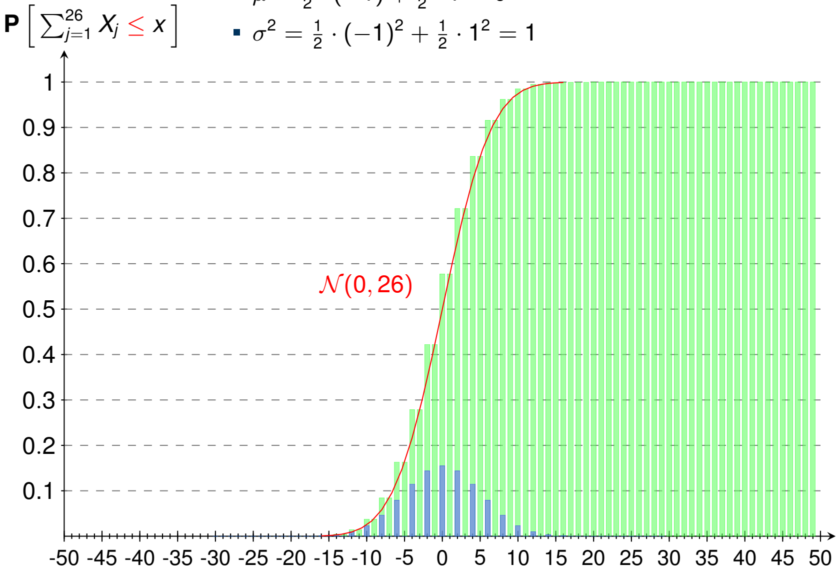

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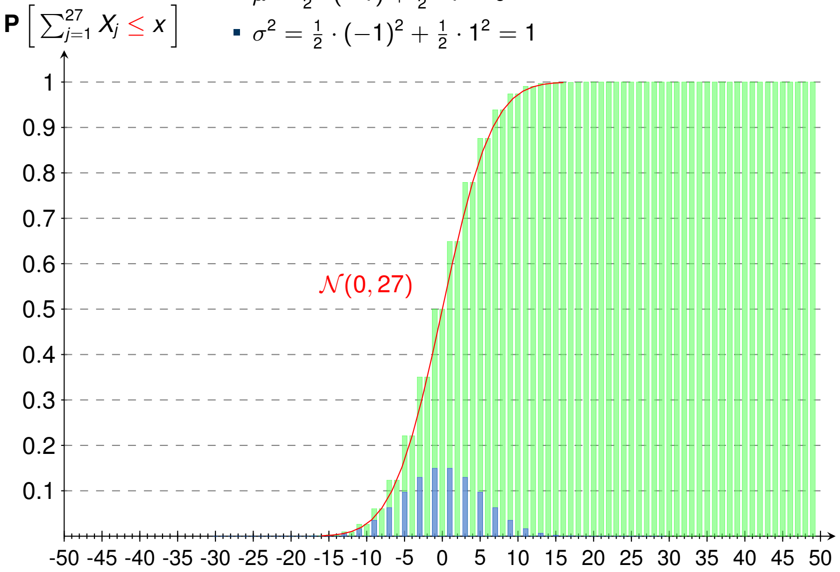

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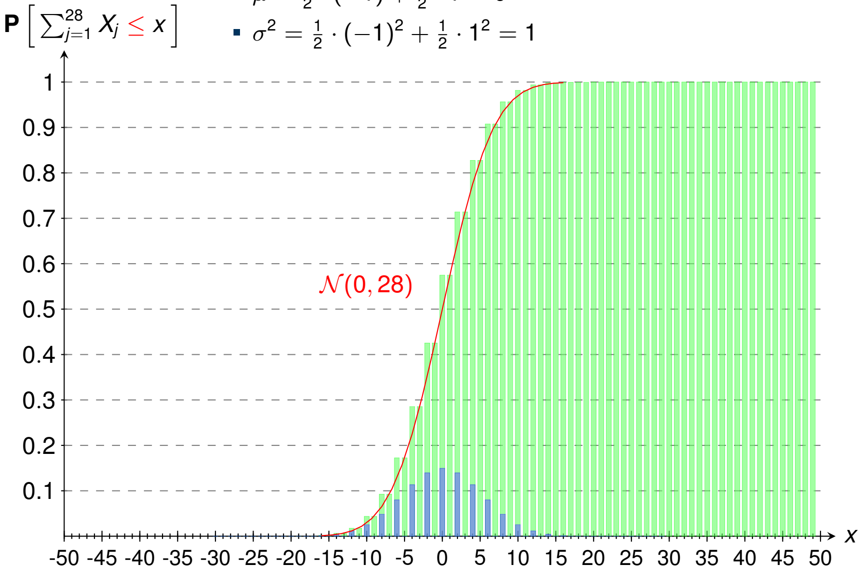

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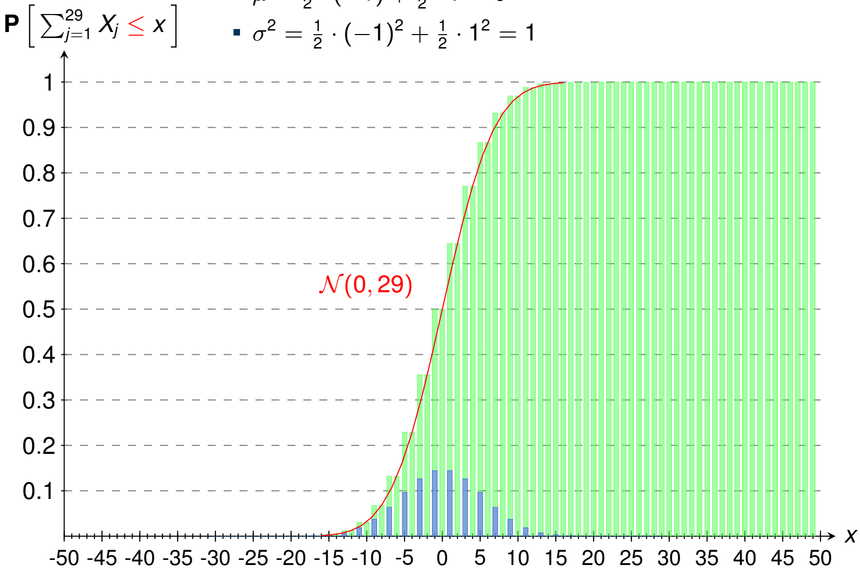

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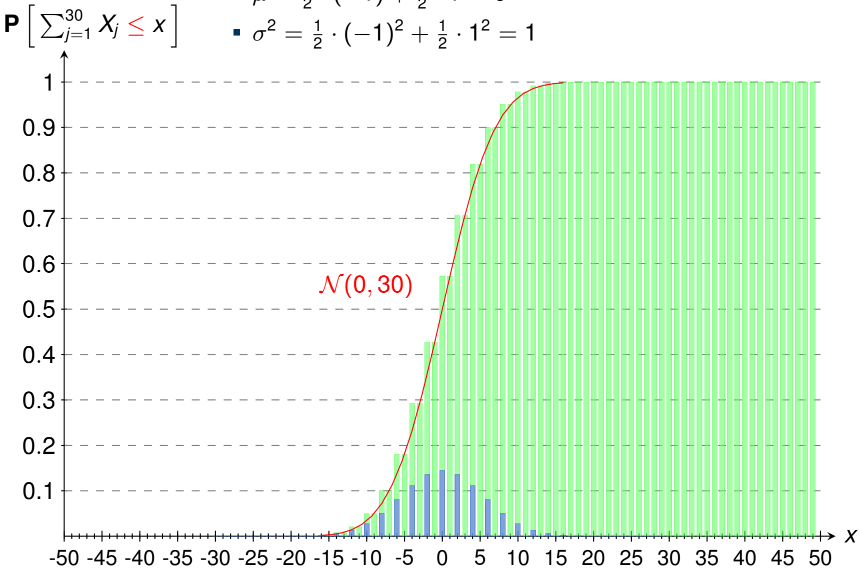

Illustration of CLT with Standardising (1/2)

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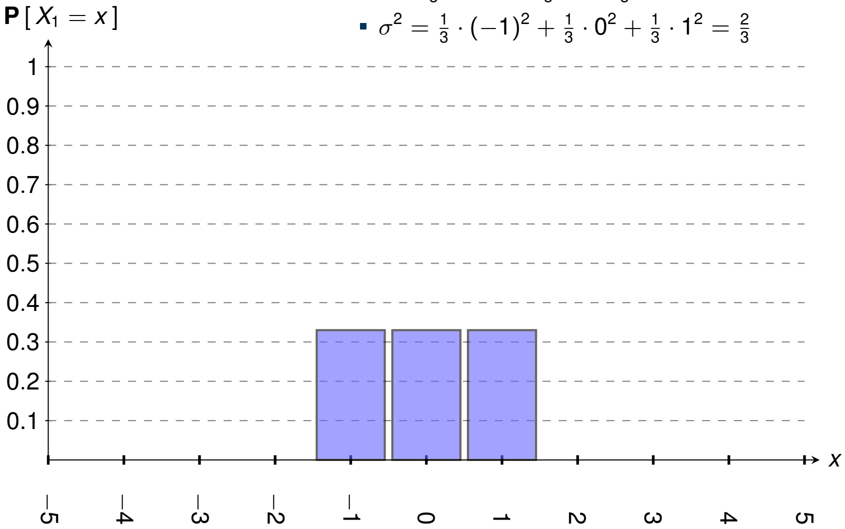

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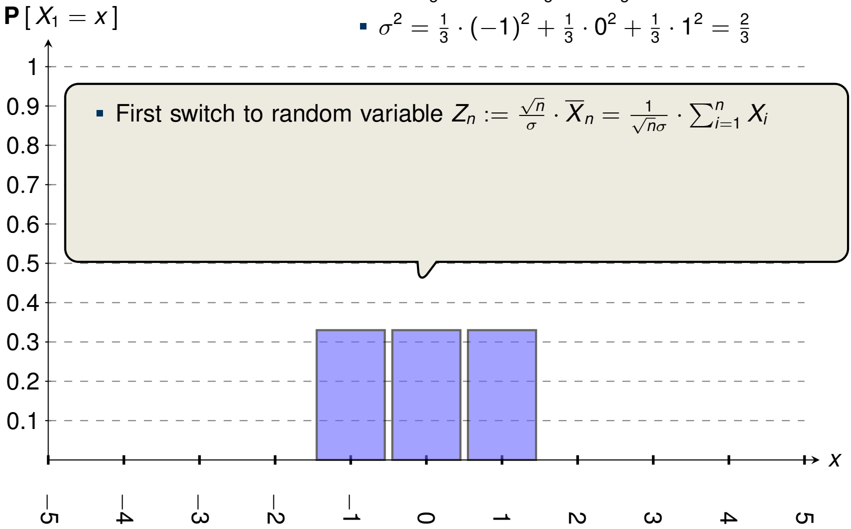

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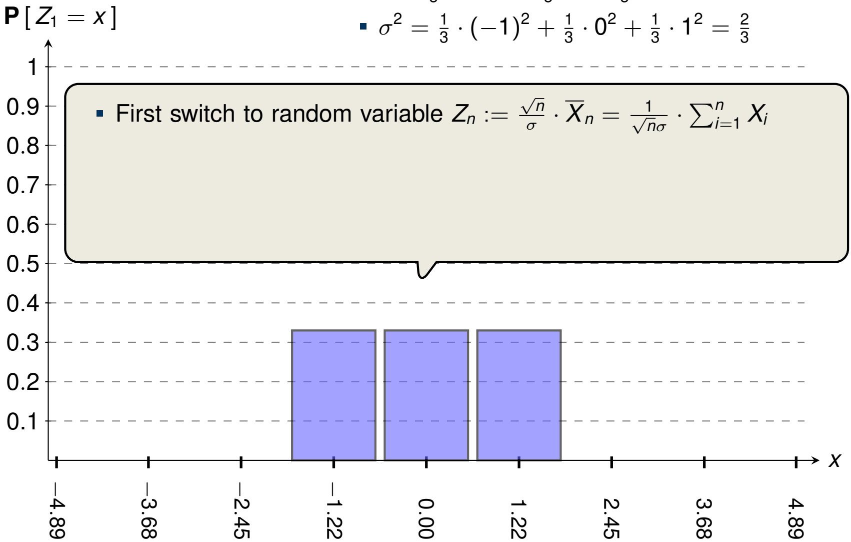

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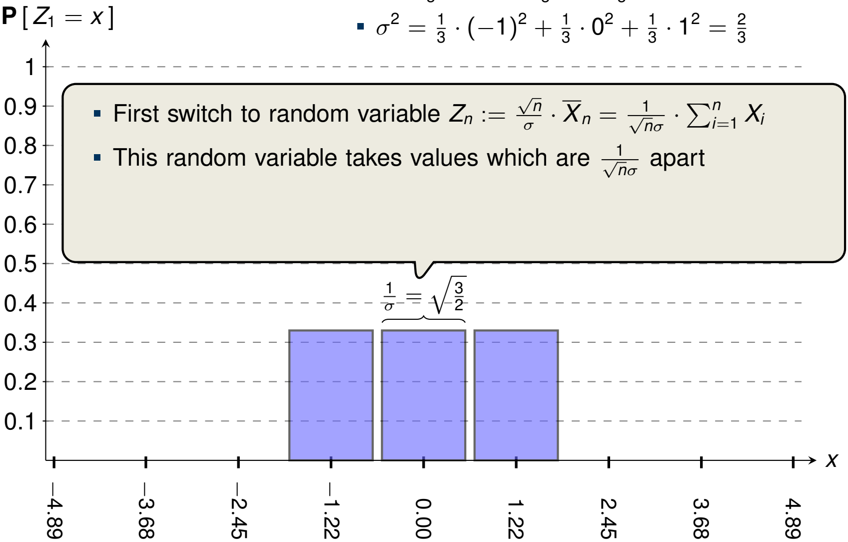

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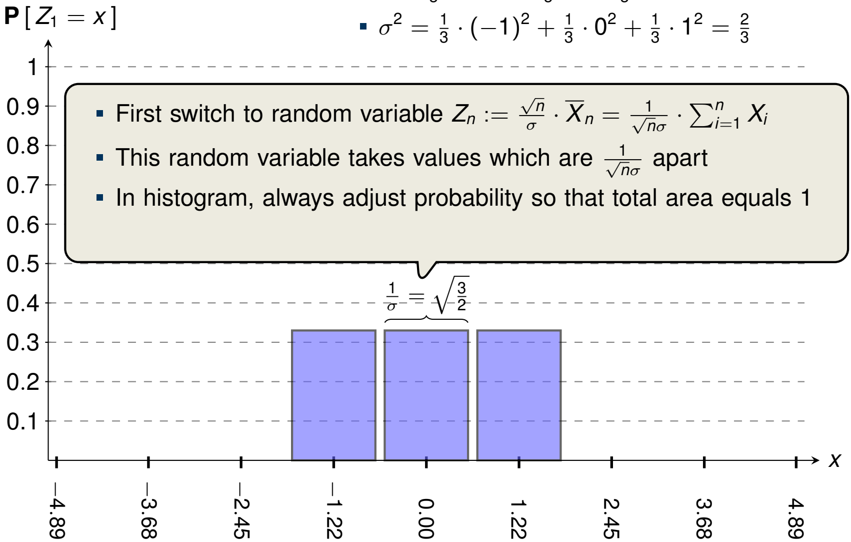

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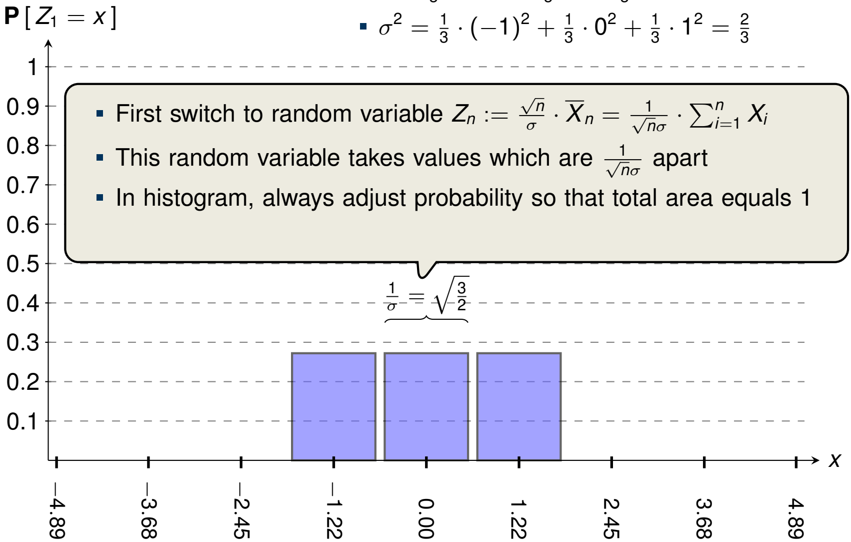

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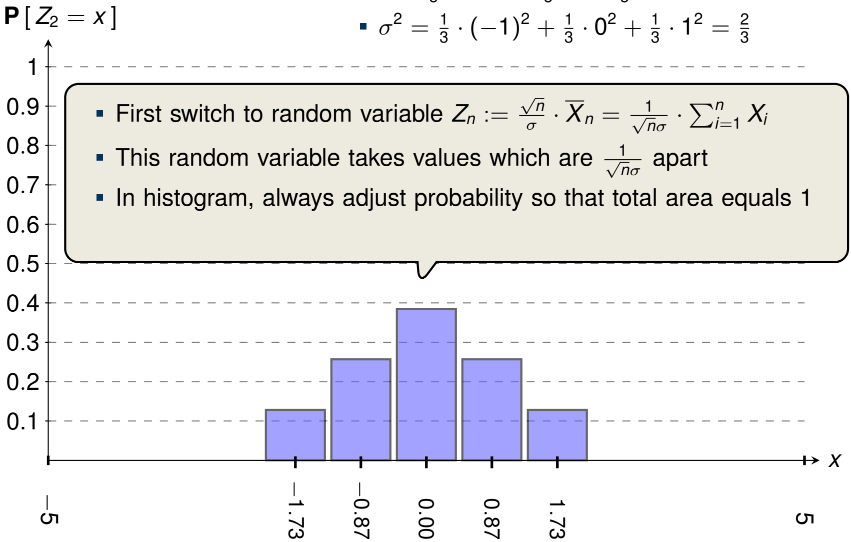

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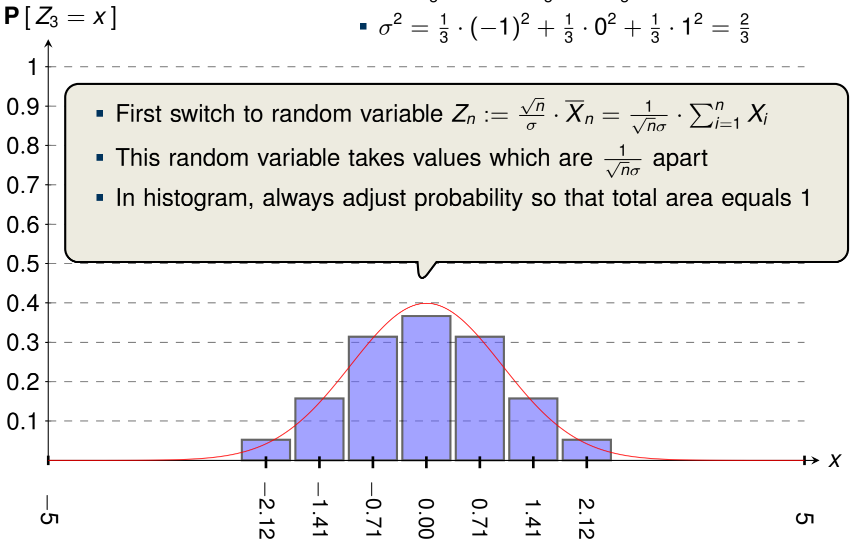

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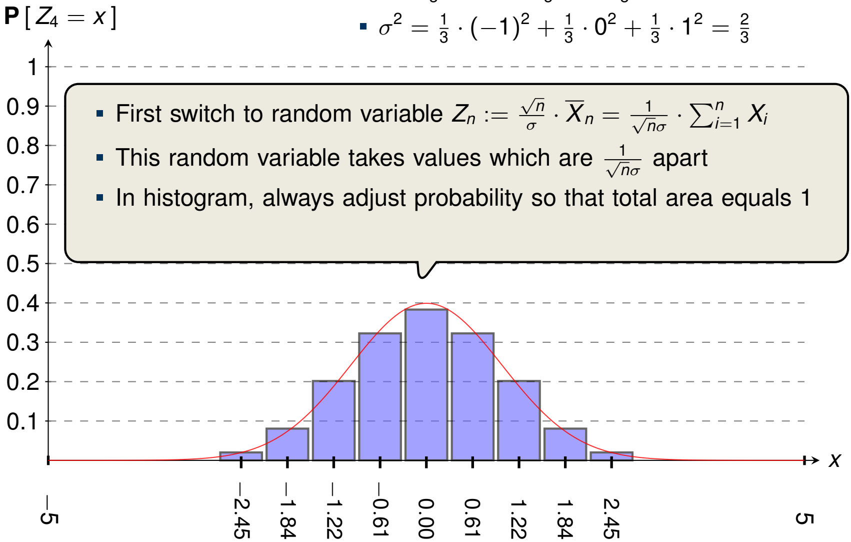

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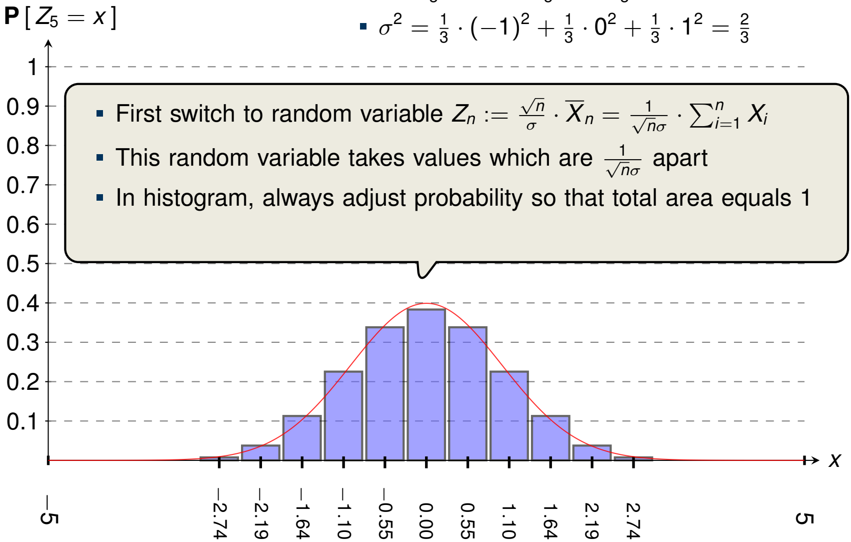

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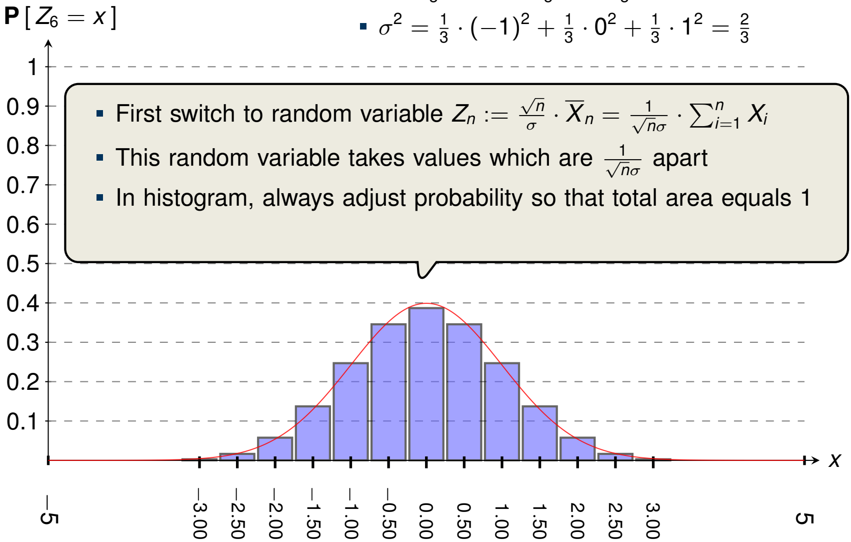

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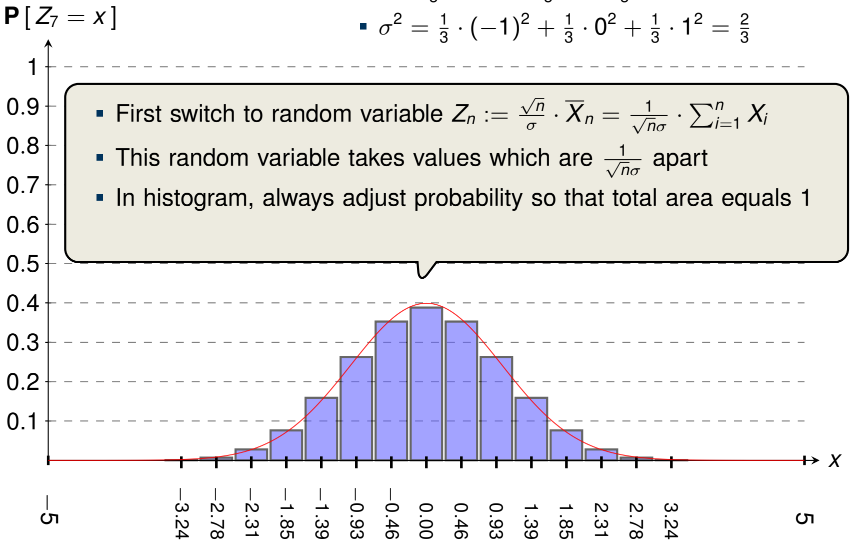

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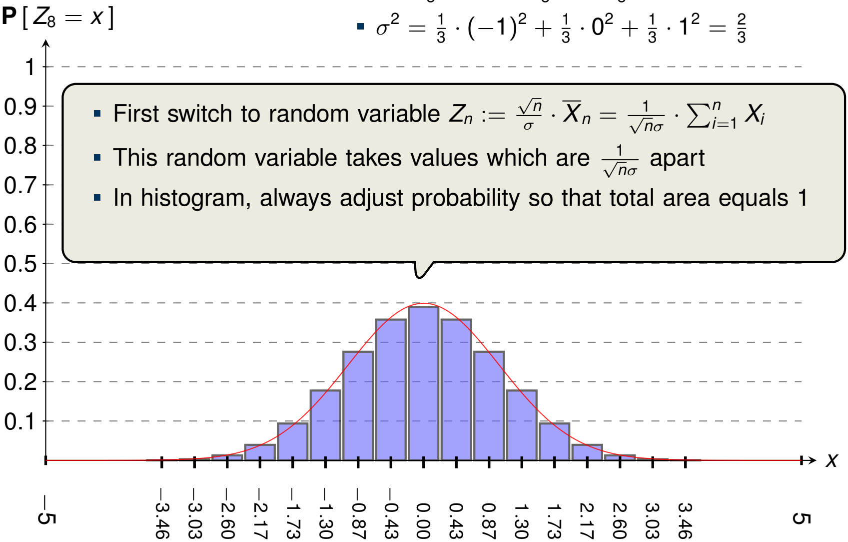

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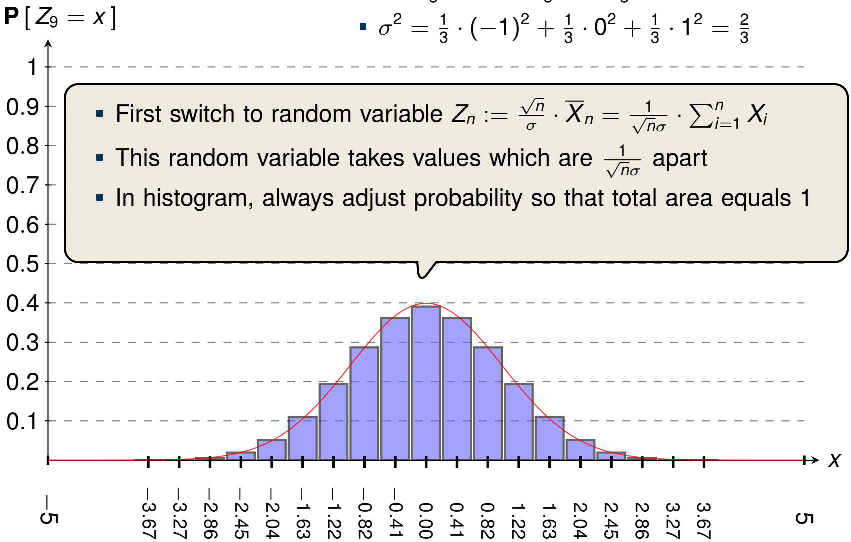

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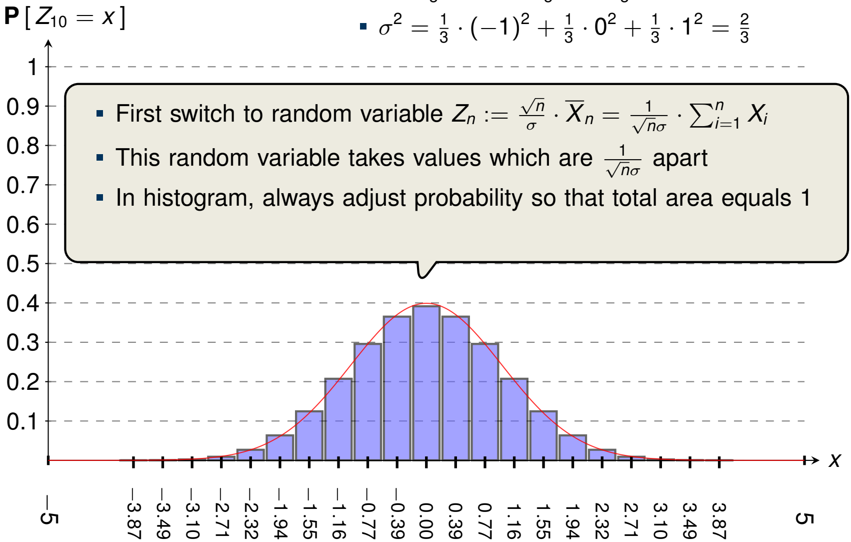

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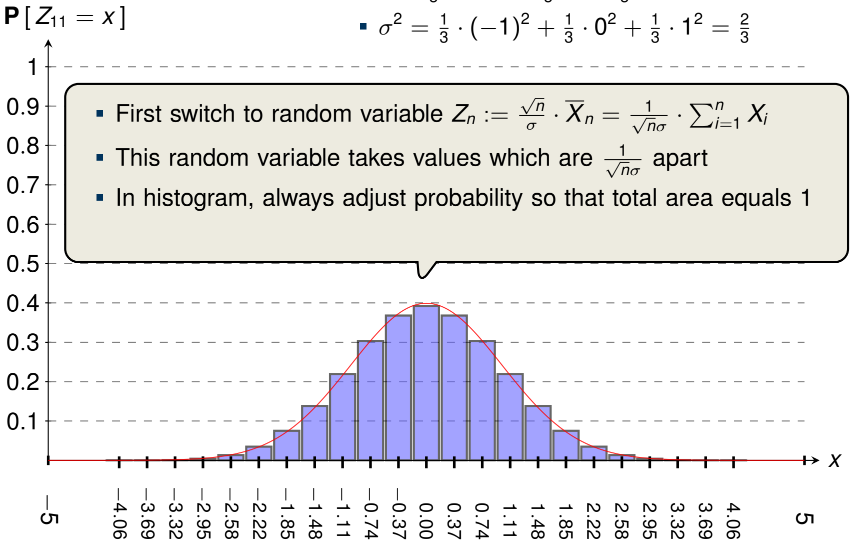

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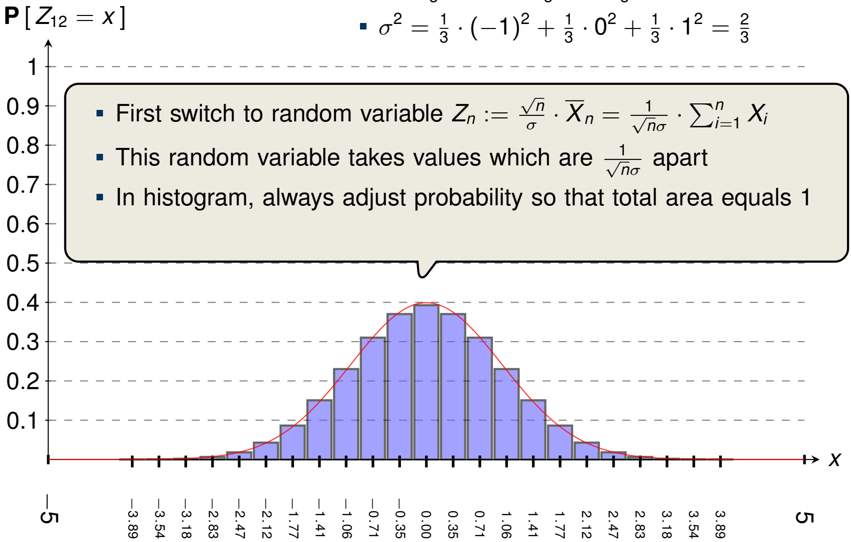

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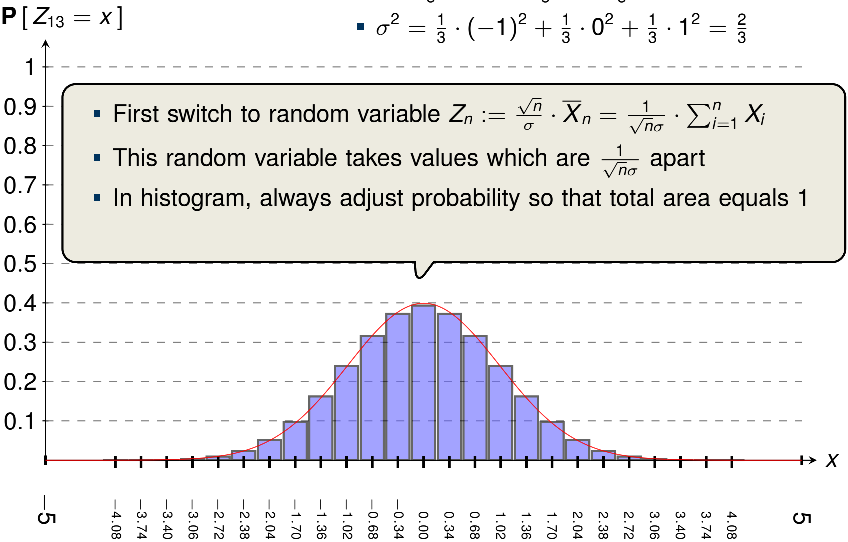

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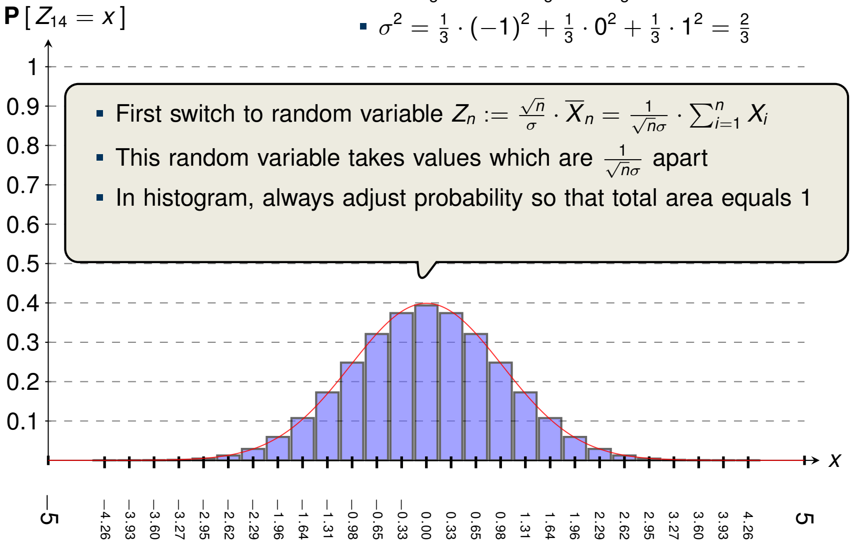

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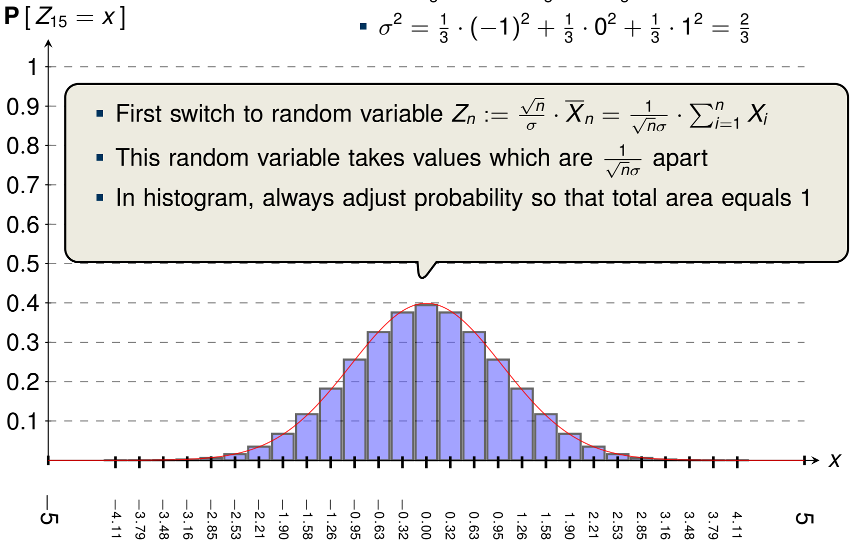

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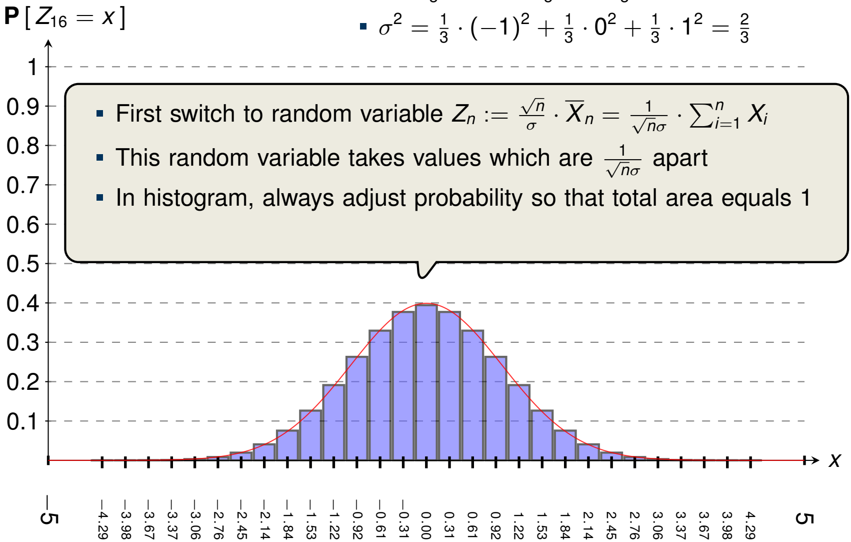

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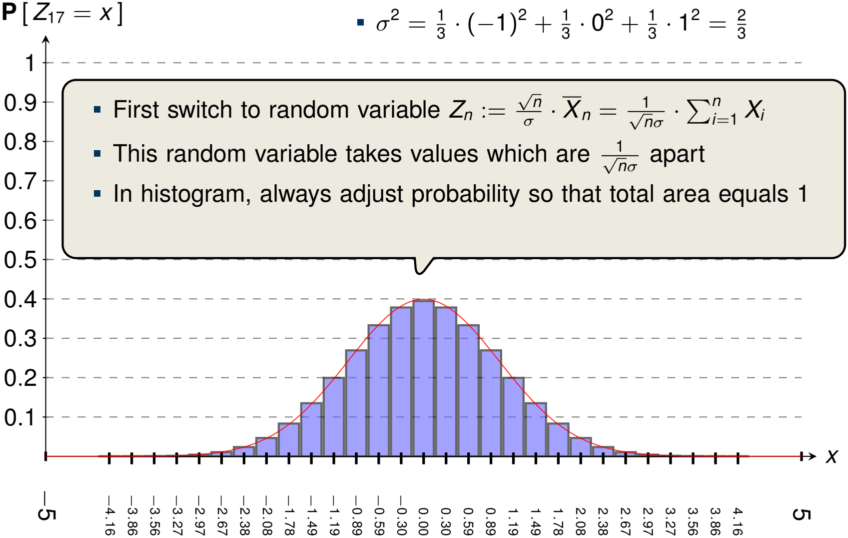

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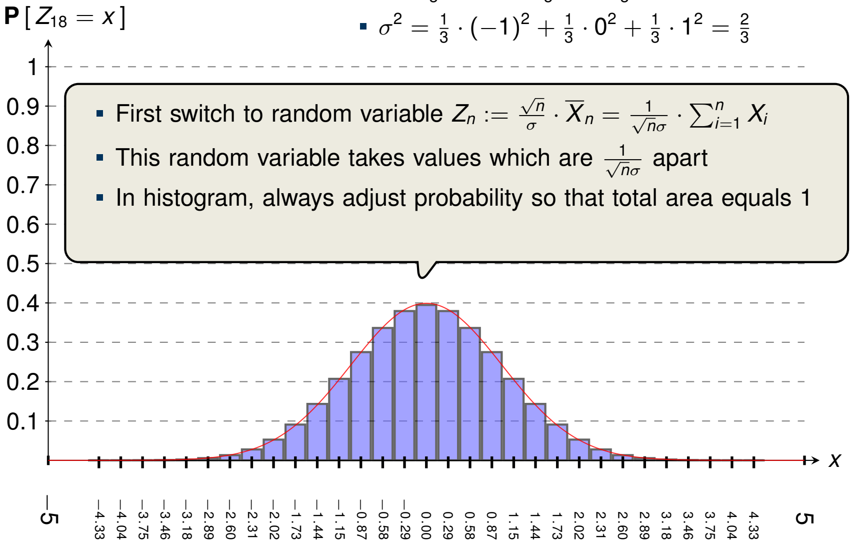

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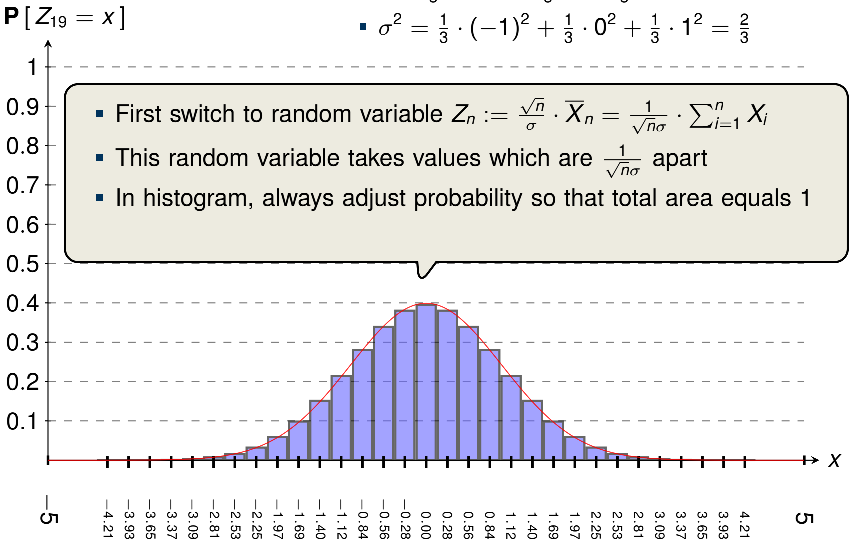

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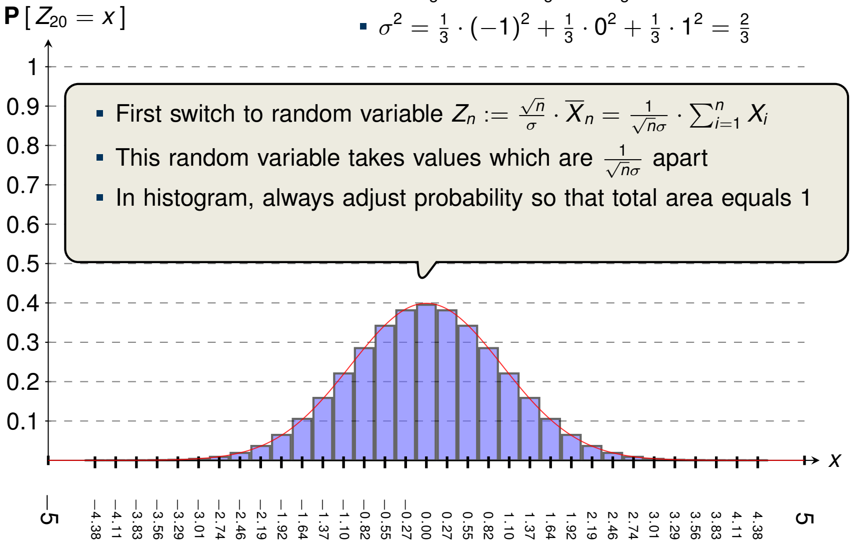

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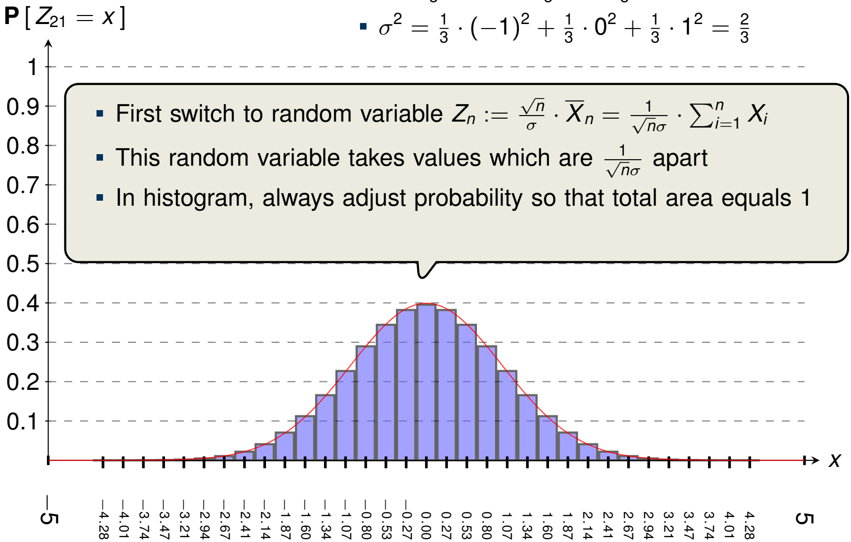

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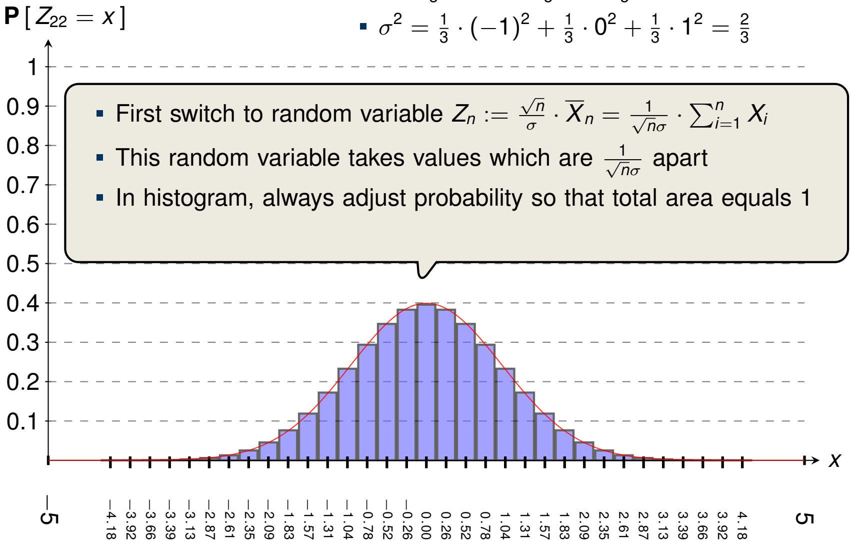
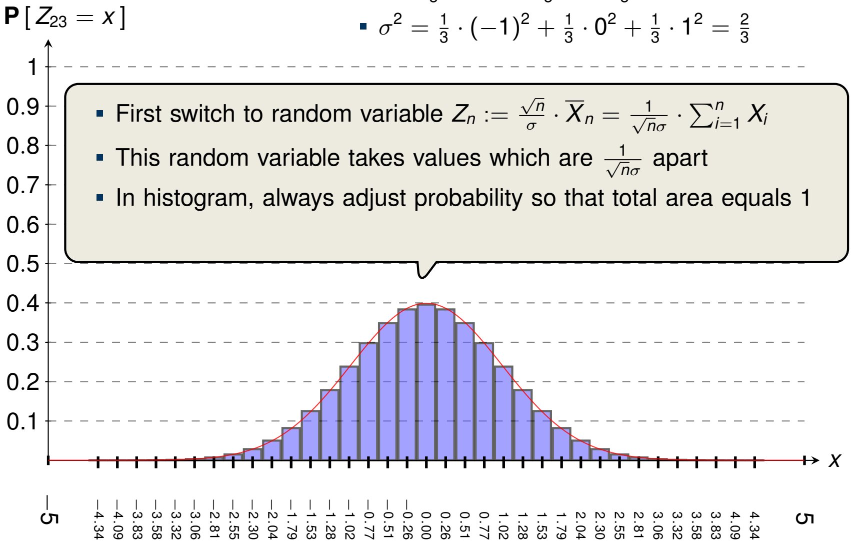


- First switch to random variable $Z_n := \frac{\sqrt{n}}{\sigma} \cdot \overline{X}_n = \frac{1}{\sqrt{n}\sigma} \cdot \sum_{i=1}^{n} X_i$
- This random variable takes values which are $\frac{1}{\sqrt{n}\sigma}$ apart
- In histogram, always adjust probability so that total area equals 1

Illustration of CLT with Standardising (1/2)

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0^2 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$



- First switch to random variable $Z_n := \frac{\sqrt{n}}{\sigma} \cdot \overline{X}_n = \frac{1}{\sqrt{n}\sigma} \cdot \sum_{i=1}^{n} X_i$
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- In histogram, always adjust probability so that total area equals 1

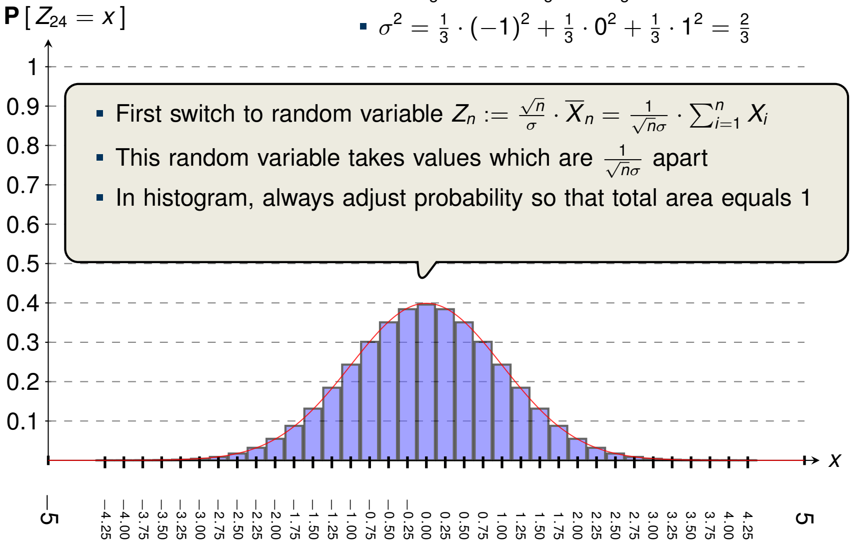

Illustration of CLT with Standardising (1/2)

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0^2 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$

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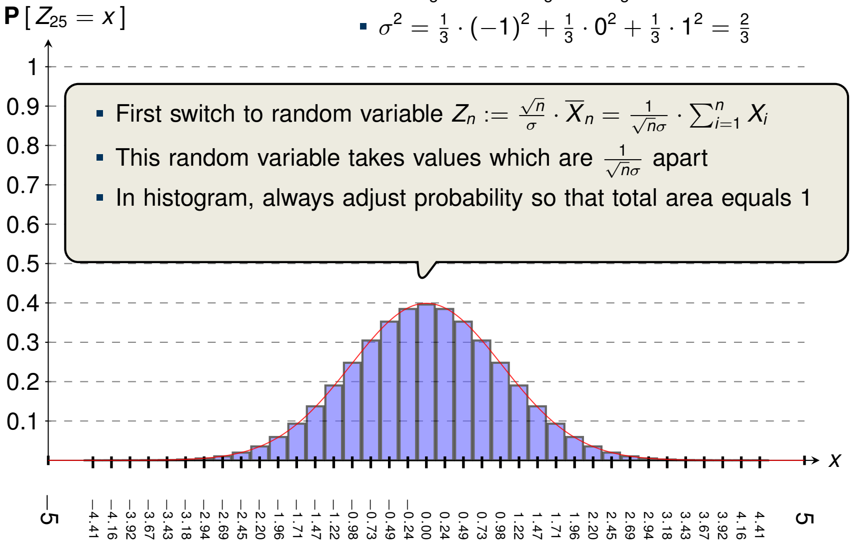

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- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
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- This random variable takes values which are $\frac{1}{\sqrt{n}\sigma}$ apart
- In histogram, always adjust probability so that total area equals 1

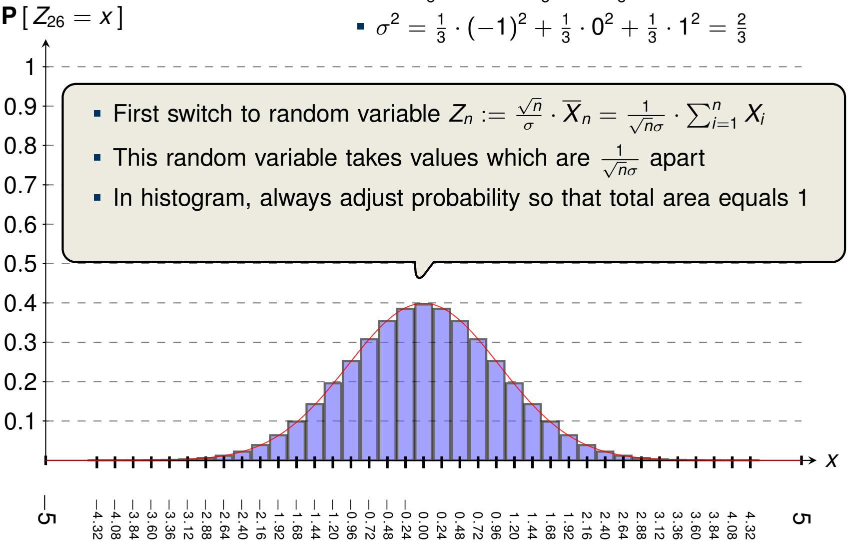
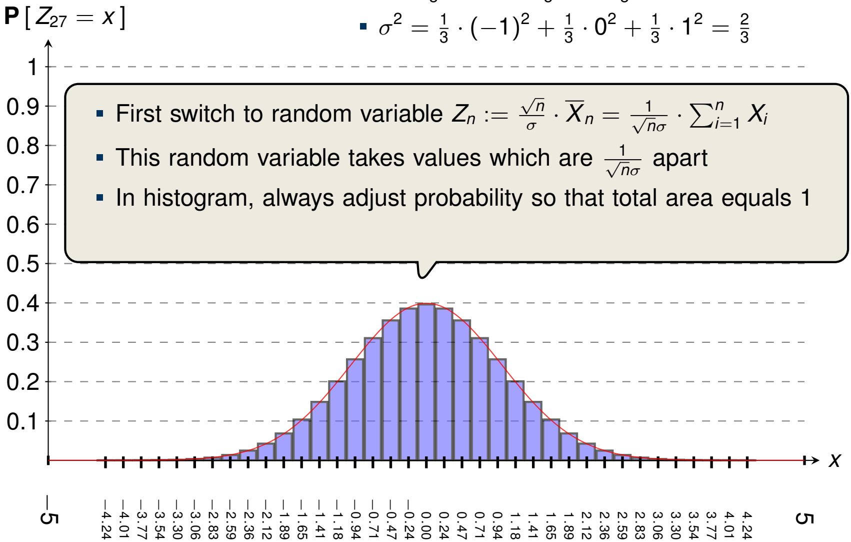


Illustration of CLT with Standardising (1/2)

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0^2 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$



- First switch to random variable $Z_n := \frac{\sqrt{n}}{\sigma} \cdot \overline{X}_n = \frac{1}{\sqrt{n}\sigma} \cdot \sum_{i=1}^{n} X_i$
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- In histogram, always adjust probability so that total area equals 1

Illustration of CLT with Standardising (1/2)

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- This random variable takes values which are $\frac{1}{\sqrt{n}\sigma}$ apart
- In histogram, always adjust probability so that total area equals 1

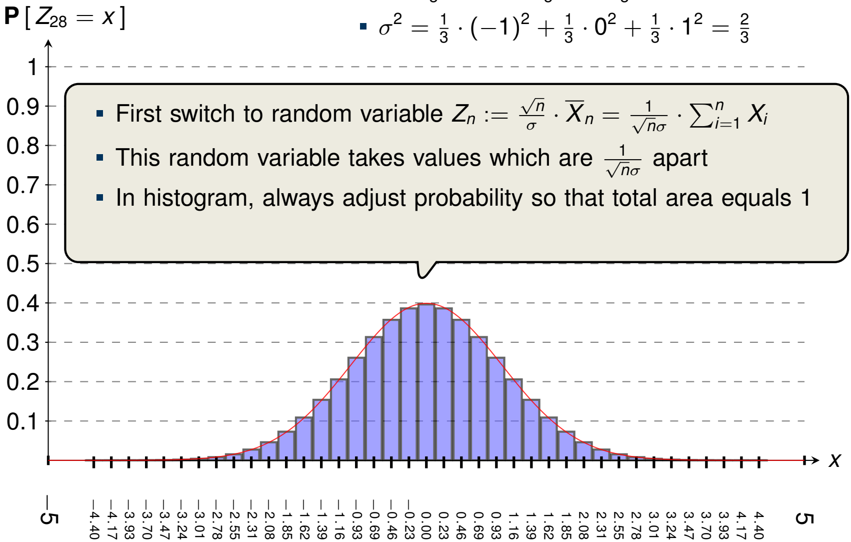

Illustration of CLT with Standardising (1/2)

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
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- First switch to random variable $Z_n := \frac{\sqrt{n}}{\sigma} \cdot \overline{X}_n = \frac{1}{\sqrt{n}\sigma} \cdot \sum_{i=1}^{n} X_i$
- This random variable takes values which are $\frac{1}{\sqrt{n}\sigma}$ apart
- In histogram, always adjust probability so that total area equals 1

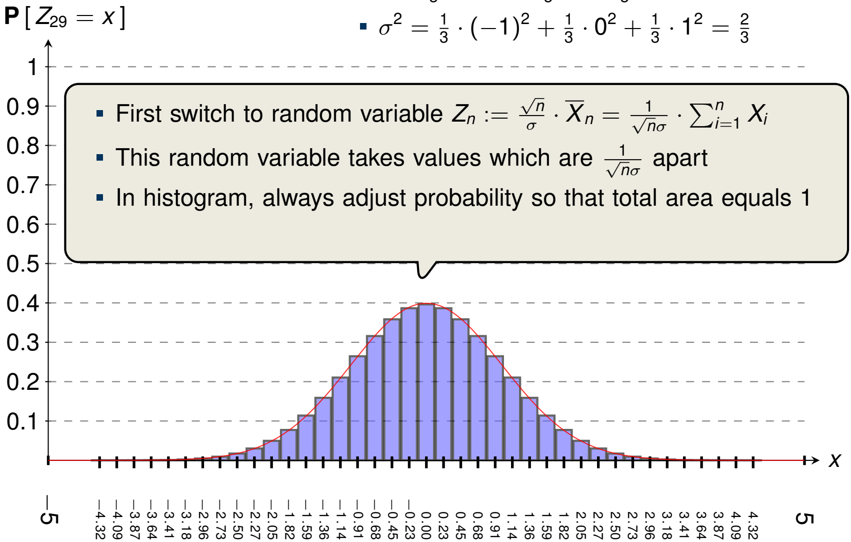
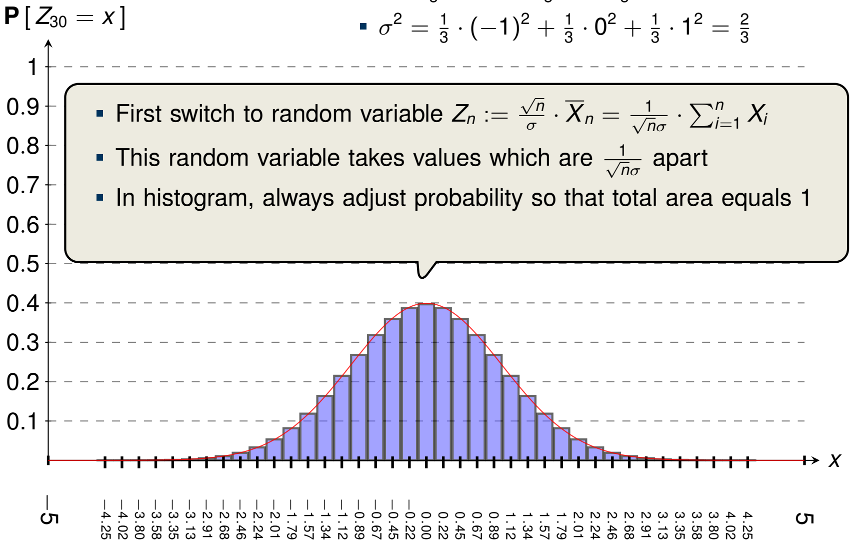


Illustration of CLT with Standardising (1/2)

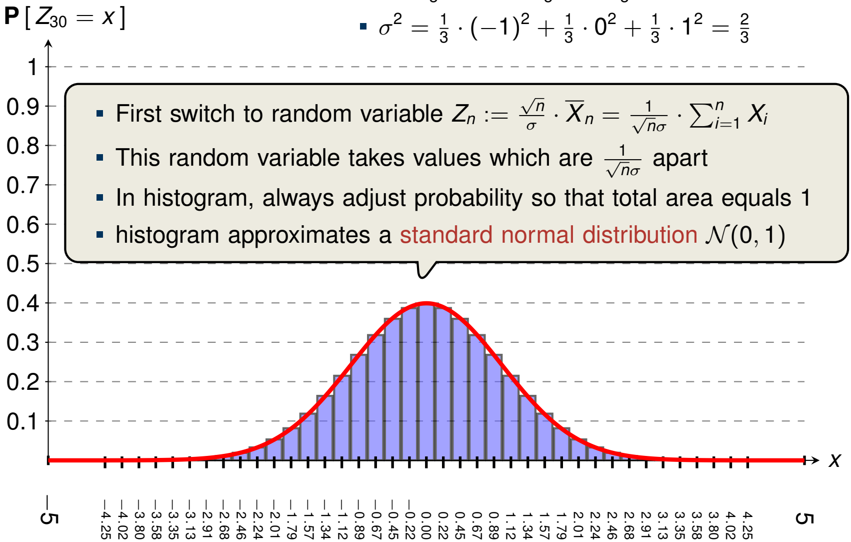
- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0^2 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$



- First switch to random variable $Z_n := \frac{\sqrt{n}}{\sigma} \cdot \overline{X}_n = \frac{1}{\sqrt{n}\sigma} \cdot \sum_{i=1}^{n} X_i$
- This random variable takes values which are $\frac{1}{\sqrt{n}\sigma}$ apart
- In histogram, always adjust probability so that total area equals 1

Illustration of CLT with Standardising (1/2)

- $\mu = \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0$
- $\sigma^2 = \frac{1}{3} \cdot (-1)^2 + \frac{1}{3} \cdot 0^2 + \frac{1}{3} \cdot 1^2 = \frac{2}{3}$



- First switch to random variable $Z_n := \frac{\sqrt{n}}{\sigma} \cdot \overline{X}_n = \frac{1}{\sqrt{n}\sigma} \cdot \sum_{i=1}^{n} X_i$
- This random variable takes values which are $\frac{1}{\sqrt{n}\sigma}$ apart
- In histogram, always adjust probability so that total area equals 1
- histogram approximates a standard normal distribution $\mathcal{N}(0, 1)$

Illustration of CLT with Standardising (2/2)

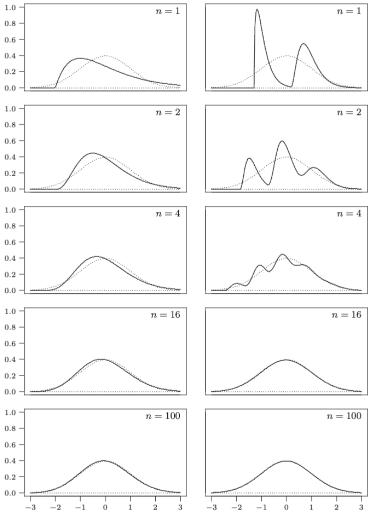


Fig. 14.2. Densities of standardized averages Z_n. Left column: from a gamma density; right column: from a bimodal density. Dotted line: $N(0,1)$ probability density.

Source: Dekking et al., Modern Introduction to Statistics

Outline

Recall: Standard Normal Table

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF X

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Source: Ross, Probability 8th ed.

$$Z \sim \mathcal{N}(0,1) \qquad \mathbf{P}[Z \leq x] = \Phi(x)$$

Recall: Standard Normal Table

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF X

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.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
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1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
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2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Source: Ross, Probability 8th ed.

Question: What if we need $\Phi(x)$ for negative x?

$$Z \sim \mathcal{N}(0,1) \qquad \mathbf{P}[Z \leq x] = \Phi(x)$$

Recall: Standard Normal Table

Section 5.4 Normal Random Variables **201**

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF X

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
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.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
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.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
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1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Source: Ross, Probability 8th ed.

Question: What if we need $\Phi(x)$ for negative x?

$$Z \sim \mathcal{N}(0,1) \qquad \mathbf{P}[Z \leq x] = \Phi(x)$$

Due to symmetry of density we have $\Phi(x) = 1 - \Phi(-x)$.

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True value is 0.0197. Error lies in the discretisation!

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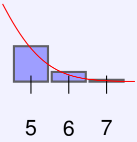

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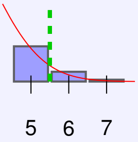

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continuity correction: a better approximation is obtained by $\mathbf{P}\left[\sum_{i=1}^{n} X_i \geq 5.5\right] \rightsquigarrow \approx 0.0143$

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A "Reverse" Application of the CLT

Example 2

Suppose we are sequentially loading one container with packets, whose weights are i.i.d. exponential variables with parameter $\lambda = 1/2$. The container has a capacity of 100 weight units. How many packets can we load so that we meet the capacity threshold with at least .95 probability?

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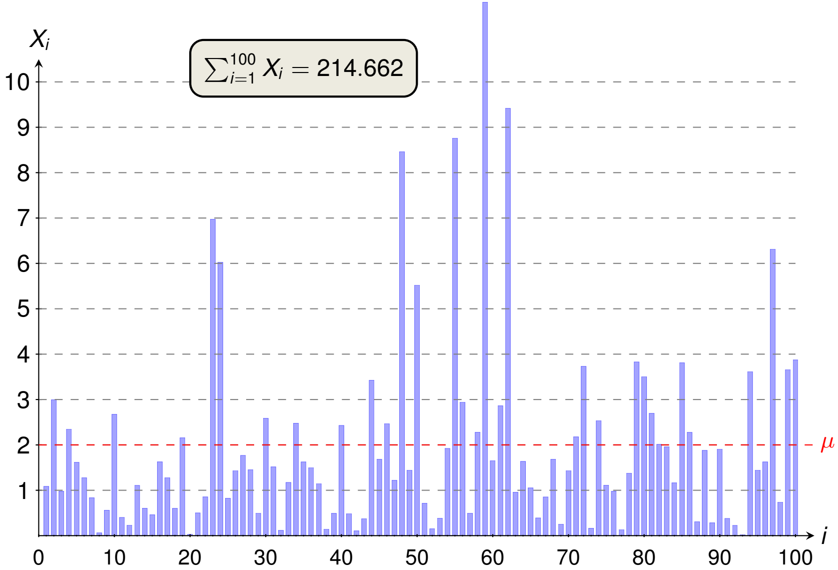
$\Rightarrow$ Solving the quadratic gives $n \leq 39.6$ (so $n \leq 39$)

A Sample of 100 Exponential Random Variables $Exp(1/2)$



Comparison between Markov, Chebyshev and CLT

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- Side Note: without continuity correction, we have 75 instead 74.5:

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Comparison between Markov, Chebyshev and CLT

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Consider $n = 100$ independent coin flips. Estimate the probability that the number of heads is greater or equal than 75.

Answer

- Markov: $X = \sum_{i=1}^{100} X_i$, $X_i \in \{0,1\}$ and $\mathbf{E}[X] = 100 \cdot \frac{1}{2} = 50$.

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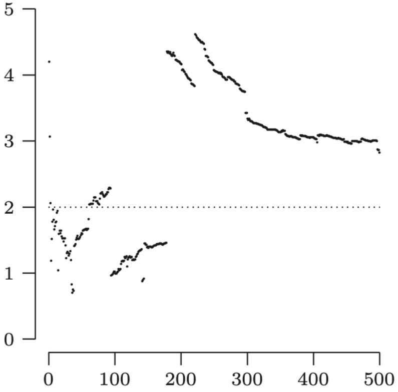
  - In this region, 75 gives a better approximation than 74.5, but for smaller values (e.g., ≤ 63) the continuity corrections gives significantly better results.

A Distribution whose Average does not converge



Cau(2, 1) distribution, Source: Dekking et al., Modern Introduction to Statistics

The Cauchy distribution has "too heavy" tails (no expectation), in particular the average does not converge.

Outline

Recap: Weak Law of Large Numbers

Central Limit Theorem

Illustrations

Examples

Bonus Material (non-examinable)

Towards a Proof of CLT: Moment Generating Functions

Moment-Generating Function

The moment-generating function of a random variable X is

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1. If X and Y are two r.v.'s with $M_X(t) = M_Y(t)$ for all $t \in (-\delta, +\delta)$ for some $\delta > 0$, then the distributions X and Y are identical.
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Proof of 2: (Proof of 1 is quite non-trivial!)

$$M_{X+Y}(t) = \mathbf{E}\left[e^{t(X+Y)}\right] = \mathbf{E}\left[e^{tX} \cdot e^{tY}\right] \overset{(!)}{=} \mathbf{E}\left[e^{tX}\right] \cdot \mathbf{E}\left[e^{tY}\right] = M_X(t) M_Y(t) \qquad \square$$

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- Differentiating (details ommitted here, see book by Ross) shows $L(0) = 0, L'(0) = \mu = 0$ and $L''(0) = \mathbf{E}\left[X^2 \right] = 1$.

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- To prove the theorem, we must show that

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&= \frac{t^2}{2}.
\end{aligned}$$

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We proved that the MGF of Z_n converges to that one of $\mathcal{N}(0,1)$.