Lecture 11

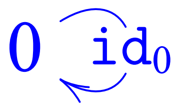
The category of small categories

Recall definition of **Cat**:

- objects are all small categories
- morphisms in $\mathbf{Cat}(\mathbf{C}, \mathbf{D})$ are all functors $\mathbf{C} \to \mathbf{D}$
- composition and identity morphisms as for functors

Cat has a terminal object

The category



one object, one morphism

is terminal in **Cat**

$\mathbf{Cat}$ has binary products

Given small categories $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}$, their product $\mathbf{C} \xleftarrow{\pi_1} \mathbf{C} \times \mathbf{D} \xrightarrow{\pi_2} \mathbf{D}$ is:

Cat has binary products

Given small categories $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}$, their product $\mathbf{C} \xleftarrow{\pi_1} \mathbf{C} \times \mathbf{D} \xrightarrow{\pi_2} \mathbf{D}$ is:

- objects of $\mathbf{C} \times \mathbf{D}$ are pairs (X, Y) where $X \in \mathbf{C}$ and $Y \in \mathbf{D}$
- morphisms $(X, Y) \to (X', Y')$ in $\mathbf{C} \times \mathbf{D}$ are pairs (f, g) where $f \in \mathbf{C}(X, X')$ and $g \in \mathbf{D}(Y, Y')$
- composition and identity morphisms are given by those of $\mathbf{C}$ (in the first component) and $\mathbf{D}$ (in the second component)
- $$\begin{cases} \pi_1\left((X, Y) \xrightarrow{(f,g)} (X', Y')\right) = X \xrightarrow{f} X' \\ \pi_2\left((X, Y) \xrightarrow{(f,g)} (X', Y')\right) = Y \xrightarrow{g} Y' \end{cases}$$

Cat not only has finite products, it is also cartesian closed.

Exponentials in **Cat** are called functor categories.

To define them we need to consider natural transformations, which are the appropriate notion of morphism between functors.

Natural transformations

Motivating example: fix a set $S \in \mathbf{Set}$ and consider the two functors $F, G : \mathbf{Set} \to \mathbf{Set}$ given by

$$F\left(X \xrightarrow{f} Y\right) = S \times X \xrightarrow{\mathtt{id}_S \times f} S \times Y$$

$$G\left(X \xrightarrow{f} Y\right) = X \times S \xrightarrow{f \times \mathtt{id}_S} Y \times S$$

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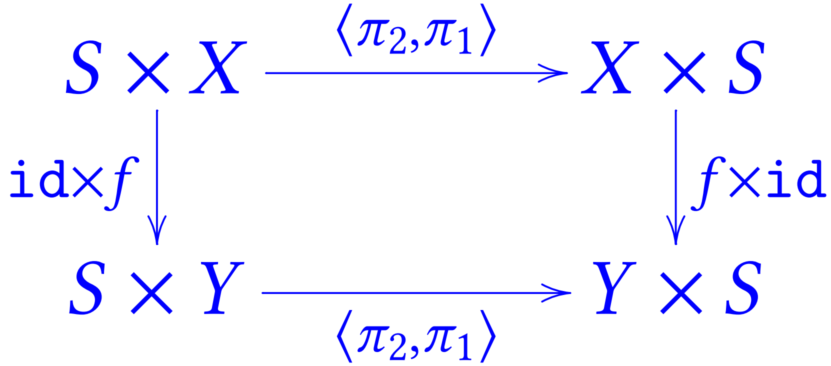
For each $X \in \mathbf{Set}$ there is an isomorphism (bijection) $\theta_X : F\,X \cong G\,X$ in $\mathbf{Set}$ given by $\langle \pi_2\,,\, \pi_1 \rangle : S \times X \to X \times S$.

These isomorphisms do not depend on the particular nature of each set X (they are "polymorphic in X"). One way to make this precise is...

…if we change from X to Y along a function $f : X \to Y$, then we get a commutative diagram in **Set**:



The square commutes because for all $s \in S$ and $x \in X$

$$
\begin{aligned}
\langle \pi_2 \, , \, \pi_1 \rangle((\mathtt{id} \times f)(s, x)) &= \langle \pi_2 \, , \, \pi_1 \rangle(s, f\,x) \\
&= (f\,x, s) \\
&= (f \times \mathtt{id})(x, s) \\
&= (f \times \mathtt{id})(\langle \pi_2 \, , \, \pi_1 \rangle(s, x))
\end{aligned}
$$

…if we change from X to Y along a function $f : X \to Y$, then we get a commutative diagram in **Set**:

$$\begin{array}{ccc} F\,X & \xrightarrow{\;\theta_X\;} & G\,X \\ {\scriptstyle F\,f}\big\downarrow & & \big\downarrow{\scriptstyle G\,f} \\ F\,Y & \xrightarrow[\;\theta_Y\;]{} & G\,Y \end{array}$$

We say that the family $(\theta_X \mid X \in \mathbf{Set})$ is natural in X.

Natural transformations

Definition. Given categories and functors $F, G : \mathbf{C} \longrightarrow \mathbf{D}$, a natural transformation $\boxed{\theta : F \to G}$ is a family of $\mathbf{D}$-morphisms $\theta_X \in \mathbf{D}(F\,X, G\,X)$, one for each $X \in \mathbf{C}$, such that for all $\mathbf{C}$-morphisms $f : X \to Y$, the diagram

$$\begin{array}{ccc} F\,X & \xrightarrow{\ \theta_X\ } & G\,X \\ {\scriptstyle F\,f}\downarrow & & \downarrow{\scriptstyle G\,f} \\ F\,Y & \xrightarrow[\ \theta_Y\ ]{} & G\,Y \end{array}$$

commutes in $\mathbf{D}$, that is, $\theta_Y \circ F\,f = G\,f \circ \theta_X$.

Example

Recall forgetful (U) and free (F) functors:

$$\mathbf{Set} \underset{F}{\overset{U}{\rightleftarrows}} \mathbf{Mon}$$

There is a natural transformation $\eta : \mathtt{id}_{\mathbf{Set}} \longrightarrow U \circ F$, where for each $\Sigma \in \mathbf{Set}$

$$\eta_\Sigma : \Sigma \longrightarrow U(F\,\Sigma) = \mathtt{List}\,\Sigma$$

$$a \in \Sigma \mapsto [a] \in \mathtt{List}\,\Sigma \quad \text{(one-element list)}$$

(Easy to see that
$$\begin{array}{ccc} \Sigma & \xrightarrow{\eta_\Sigma} & U(F\,\Sigma) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle U(F\,f)} \\ \Sigma' & \xrightarrow[\eta_{\Sigma'}]{} & U(F\,\Sigma') \end{array}$$
commutes.)

Example

The covariant powerset functor $\mathscr{P} : \mathbf{Set} \to \mathbf{Set}$ is

$$\mathscr{P}\,X \triangleq \{S \mid S \subseteq X\}$$

$$\mathscr{P}\left(X \xrightarrow{f} Y\right) \triangleq \mathscr{P}\,X \xrightarrow{\mathscr{P}\,f} \mathscr{P}\,Y$$

$$S \mapsto \mathscr{P}\,f\,S \triangleq \{f\,x \mid x \in S\}$$

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There is a natural transformation $\cup : \mathscr{P} \circ \mathscr{P} \to \mathscr{P}$ whose component at $X \in \mathbf{Set}$ sends $\mathcal{S} \in \mathscr{P}(\mathscr{P}\, X)$ to

$$\cup_X \mathcal{S} \triangleq \{x \in X \mid \exists S \in \mathcal{S},\ x \in S\} \in \mathscr{P}\, X$$

(check that $\cup_X$ is natural in X)

Non-example

The classic example of an "un-natural transformation" (the one that caused Eilenburg and MacLane to invent the concept of naturality) is the linear isomorphism between a finite dimensional real vectorspace V and its dual V^* (= vectorspace of linear functions $V \to \mathbb{R}$).

Both V and V^* have the same finite dimension, so are isomorphic by choosing bases; but there is no choice of basis for each V that makes the family of isomorphisms natural in V.

For a similar, more elementary non-example, see Ex. Sh. 5, question 4.

Composing natural transformations

Given functors $F, G, H : \mathbf{C} \to \mathbf{D}$ and natural transformations $\theta : F \to G$ and $\varphi : G \to H$,

we get $\boxed{\varphi \circ \theta} : F \to H$ with

$$(\varphi \circ \theta)_X = \left(F\,X \xrightarrow{\theta_X} G\,X \xrightarrow{\varphi_X} H\,X \right)$$

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Check naturality:

$$\begin{aligned}
H\,f \circ (\varphi \circ \theta)_X &\triangleq H\,f \circ \varphi_X \circ \theta_X \\
&= \varphi_Y \circ G\,f \circ \theta_X && \text{naturality of } \varphi \\
&= \varphi_Y \circ \theta_Y \circ F\,f && \text{naturality of } \theta \\
&\triangleq (\varphi \circ \theta)_Y \circ F\,f
\end{aligned}$$

Identity natural transformation

Given a functor $F : \mathbf{C} \longrightarrow \mathbf{D}$, we get a natural transformation $\boxed{\mathtt{id}_F : F \longrightarrow F}$ with

$$(\mathtt{id}_F)_X = F\,X \xrightarrow{\mathtt{id}_{F\,X}} F\,X$$

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Check naturality:

$$F\,f \circ (\mathtt{id}_F)_X \triangleq F\,f \circ \mathtt{id}_{F\,X} = F\,f = \mathtt{id}_{F\,Y} \circ F\,f \triangleq (\mathtt{id}_F)_Y \circ F\,f$$

Functor categories

It is easy to see that composition and identities for natural transformations satisfy

$$(\psi \circ \varphi) \circ \theta = \psi \circ (\varphi \circ \theta)$$
$$\mathtt{id}_G \circ \theta = \theta \circ \mathtt{id}_F$$

so that we get a category:

Definition. Given categories $\mathbf{C}$ and $\mathbf{D}$, the functor category $\boxed{\mathbf{D}^{\mathbf{C}}}$ has

- objects are all functors $\mathbf{C} \to \mathbf{D}$
- given $F, G : \mathbf{C} \to \mathbf{D}$, morphism from F to G in $\mathbf{D}^{\mathbf{C}}$ are the natural transformations $F \to G$
- composition and identity morphisms as above

If $\mathscr{U}$ is a Grothendieck universe, then for each $X \in \mathscr{U}$ and $F \in \mathscr{U}^X$ we have that their dependent product and dependent function sets

$$\sum_{x \in X} F\,x \triangleq \{(x, y) \mid x \in X \wedge y \in F\,x\}$$

$$\prod_{x \in X} F\,x \triangleq \{f \subseteq \sum_{x \in X} F\,x \mid f \text{ is single-valued and total}\}$$

are also in $\mathscr{U}$; and as a special case (of $\prod$, when F is a constant function with value Y) we also have that $X, Y \in \mathscr{U}$ implies $Y^X \in \mathscr{U}$.

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If $\mathbf{C}$ and $\mathbf{D}$ are small categories, then so is $\mathbf{D}^{\mathbf{C}}$.

because

$$\texttt{obj}(\mathbf{D}^{\mathbf{C}}) \subseteq \textstyle\sum_{F \in (\texttt{obj}\,D)^{\texttt{obj}\,\mathbf{C}}} \prod_{X, Y \in \texttt{obj}\,\mathbf{C}} \mathbf{D}(F\,X, F\,Y)^{\mathbf{C}(X,Y)}$$
$$\mathbf{D}^{\mathbf{C}}(F, G) \subseteq \textstyle\prod_{X \in \texttt{obj}\,\mathbf{C}} \mathbf{D}(F\,X, G\,X)$$

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Aim to show that functor category $\mathbf{D}^{\mathbf{C}}$ is the exponential of $\mathbf{C}$ and $\mathbf{D}$ in $\mathbf{Cat}$…

$\mathbf{Cat}$ is cartesian closed

Theorem. There is an application functor

$$\mathrm{app} : \mathbf{D}^{\mathbf{C}} \times \mathbf{C} \to \mathbf{D}$$

that makes $\mathbf{D}^{\mathbf{C}}$ the exponential for $\mathbf{C}$ and $\mathbf{D}$ in $\mathbf{Cat}$.

Given $(F, X) \in \mathbf{D}^{\mathbf{C}} \times \mathbf{C}$, we define

$$\mathrm{app}(F, X) \triangleq F\,X$$

and given $(\theta, f) : (F, X) \to (G, Y)$ in $\mathbf{D}^{\mathbf{C}} \times \mathbf{C}$, we define

$$\mathrm{app}\left((F, X) \xrightarrow{(\theta, f)} (G, Y)\right) \triangleq F\,X \xrightarrow{F\,f} F\,Y \xrightarrow{\theta_Y} G\,Y$$

$$= F\,X \xrightarrow{\theta_X} G\,X \xrightarrow{G\,f} G\,Y$$

Check: $\begin{cases} \mathrm{app}(\mathrm{id}_F, \mathrm{id}_X) & = \mathrm{id}_{F\,X} \\ \mathrm{app}(\varphi \circ \theta, g \circ f) & = \mathrm{app}(\varphi, g) \circ \mathrm{app}(\theta, f) \end{cases}$

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Definition of currying: given functor $F : \mathbf{E} \times \mathbf{C} \to \mathbf{D}$, we get a functor $\mathtt{cur}\, F : \mathbf{E} \to \mathbf{D}^{\mathbf{C}}$ as follows. For each $Z \in \mathbf{E}$, $\mathtt{cur}\, F\, Z \in \mathbf{D}^{\mathbf{C}}$ is the functor

$$\mathtt{cur}\, F\, Z \left(\begin{array}{c} X \\ \big\downarrow f \\ X' \end{array} \right) \triangleq \begin{array}{c} F(Z, X) \\ \big\downarrow F(\mathtt{id}_Z, f) \\ F(Z, X') \end{array}$$

For each $g : Z \to Z'$ in $\mathbf{E}$, $\mathtt{cur}\, F\, g : \mathtt{cur}\, F\, Z \to \mathtt{cur}\, F\, Z'$ is the natural transformation whose component at each $X \in \mathbf{C}$ is

$$(\mathtt{cur}\, F\, g)_X \triangleq F(g, \mathtt{id}_X) : F(Z, X) \to F(Z', X)$$

(Check that this is natural in X; and that $\mathtt{cur}\, F$ preserves composition and identities in $\mathbf{E}$.)

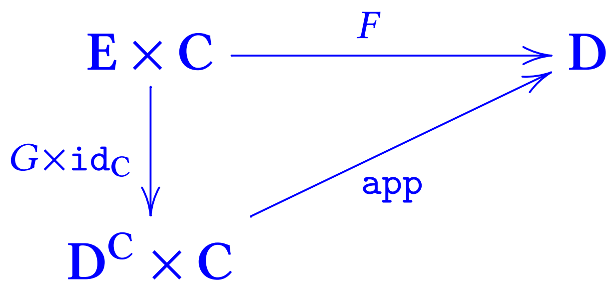
Cat is cartesian closed

Theorem. There is an application functor

$$\mathrm{app} : \mathbf{D}^{\mathbf{C}} \times \mathbf{C} \longrightarrow \mathbf{D}$$

that makes $\mathbf{D}^{\mathbf{C}}$ the exponential for $\mathbf{C}$ and $\mathbf{D}$ in **Cat**.

Have to check that $\mathtt{cur}\, F$ is the unique functor $G : \mathbf{E} \longrightarrow \mathbf{D}^{\mathbf{C}}$ that makes



commute in **Cat** (exercise).