

Lecture 14

Exercise Sheet 4 (graded)

anonymous results

highest = 100% (!)

lowest = 45% (:(

median = 85%

sample size = 29

(can't yet release model answers)

← expect
the take-
home test
to be
harder

Dependent Types

A brief look at some category theory for modelling type theories with **dependent types**.

Will restrict attention to the case of **Set**, rather than in full generality.

Further reading:

M. Hofmann, *Syntax and Semantics of Dependent Types*. In: A.M. Pitts and P. Dybjer (eds), *Semantics and Logics of Computation* (CUP, 1997).

Simple types

$$\diamond, x_1 : T_1, \dots, x_n : T_n \vdash t(x_1, \dots, x_n) : T$$

Dependent types

$$\diamond, x_1 : T_1, \dots, x_n : T_n \vdash t(x_1, \dots, x_n) : T(x_1, \dots, x_n)$$

and more generally

$$\begin{aligned} \diamond, x_1 : T_1, x_2 : T_2(x_1), x_3 : T_3(x_1, x_2), \dots \vdash \\ t(x_1, x_2, x_3, \dots) : T(x_1, x_2, x_3, \dots) \end{aligned}$$

If type expressions denote sets, then

a type $T_1(x)$ dependent upon $x : T$

should denote

an indexed family of sets $(E_i \mid i \in I)$
(where I is the set denoted by type T)

i.e. $E : I \rightarrow \mathbf{Set}$ is a set-valued function on a set I .

For each $I \in \mathbf{Set}$, let \mathbf{Set}^I be the category with

- ▶ $\mathbf{obj}(\mathbf{Set}^I) \triangleq (\mathbf{obj} \mathbf{Set})^I$, so objects are I -indexed families of sets, $X = (X_i \mid i \in I)$
- ▶ morphisms $f : X \rightarrow Y$ in \mathbf{Set}^I are I -indexed families of functions $f = (f_i \in \mathbf{Set}(X_i, Y_i) \mid i \in I)$
- ▶ composition: $(g \circ f) \triangleq (g_i \circ f_i \mid i \in I)$
(i.e. use composition of functions in \mathbf{Set} at each index $i \in I$)
- ▶ identity: $\mathbf{id}_X \triangleq (\mathbf{id}_{X_i} \mid i \in I)$
(i.e. use identity functions in \mathbf{Set} at each index $i \in I$)

For each $p : I \rightarrow J$ in **Set**, let $p^* : \mathbf{Set}^J \rightarrow \mathbf{Set}^I$ be the functor defined by:

$$p^* \left(\begin{array}{c} Y_j \\ \downarrow f_j \\ Y'_j \end{array} \middle| j \in J \right) \triangleq \left(\begin{array}{c} Y_{p i} \\ \downarrow f_{p i} \\ Y'_{p i} \end{array} \middle| i \in I \right)$$

i.e. p^* takes J -indexed families of sets/functions to I -indexed ones by precomposing with p

Dependent products

of families of sets

For $I, J \in \mathbf{Set}$, consider the functor $\pi_1^* : \mathbf{Set}^I \rightarrow \mathbf{Set}^{I \times J}$ induced by precomposition with the first projection function $\pi_1 : I \times J \rightarrow I$.

Theorem. π_1^* has a left adjoint $\Sigma : \mathbf{Set}^{I \times J} \rightarrow \mathbf{Set}^I$.

Proof. We apply the Theorem from Lecture 13: for each $E \in \mathbf{Set}^{I \times J}$ we define $\Sigma E \in \mathbf{Set}^I$ and $\eta_E : E \rightarrow \pi_1^*(\Sigma E)$ in $\mathbf{Set}^{I \times J}$ with the required universal property...

Theorem. π_1^* has a left adjoint $\Sigma : \mathbf{Set}^{I \times J} \rightarrow \mathbf{Set}^I$.

For each $E \in \mathbf{Set}^{I \times J}$, define $\Sigma E \in \mathbf{Set}^I$ to be the function mapping each $i \in I$ to the set

$$(\Sigma E)_i \triangleq \sum_{j \in J} E_{(i,j)} = \{(j, e) \mid j \in J \wedge e \in E_{(i,j)}\}$$

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$$(\Sigma E)_i \triangleq \sum_{j \in J} E_{(i,j)} = \{(j, e) \mid j \in J \wedge e \in E_{(i,j)}\}$$

and define $\eta_E : E \rightarrow \pi_1^*(\Sigma E)$ in $\mathbf{Set}^{I \times J}$ to be the function mapping each $(i, j) \in I \times J$ to the function $(\eta_E)_{(i,j)} : E_{(i,j)} \rightarrow (\Sigma E)_i$ given by $e \mapsto (j, e)$.

Universal property–

Theorem. π_1^* has a left adjoint $\Sigma : \mathbf{Set}^{I \times J} \rightarrow \mathbf{Set}^I$.

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Universal property–existence part: given any $X \in \mathbf{Set}^I$ and $f : E \rightarrow \pi_1^*(X)$ in

$\mathbf{Set}^{I \times J}$, we have

$$\begin{array}{ccc}
 E & \xrightarrow{\eta_E} & \pi_1^*(\Sigma E) & & \Sigma E \\
 & \searrow f & \downarrow \pi_1^*(\bar{f}) & & \downarrow \bar{f} \\
 & & \pi_1^*(X) & & X
 \end{array}$$

where for all $i \in I, j \in J$ and $e \in E_{(i,j)}$ $\bar{f}_i(j, e) \triangleq f_{(i,j)}(e)$

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Universal property–uniqueness part: given $g : \Sigma E \rightarrow X$ in \mathbf{Set}^I making

$$\begin{array}{ccc} E & \xrightarrow{\eta_E} & \pi_1^*(\Sigma E) \\ & \searrow f & \downarrow \pi_1^*(g) \\ & & \pi_1^*(X) \end{array} \text{ commute in } \mathbf{Set}^{I \times J},$$

then for all $i \in I$, and $(j, e) \in (\Sigma E)_i$ we have

$$\bar{f}_i(j, e) \triangleq f_{(i,j)}(e) = (\pi_1^*g \circ \eta_E)_{(i,j)} e = (\pi_1^*g)_{(i,j)}((\eta_E)_{(i,j)} e) \triangleq g_i(j, e)$$

so $g = \bar{f}$. \square

Dependent functions

of families of sets

We have seen that the left adjoint to $\pi_1^* : \mathbf{Set}^I \rightarrow \mathbf{Set}^{I \times J}$ is given by dependent products of sets.

Dually, dependent function sets give:

Theorem. π_1^* has a right adjoint $\Pi : \mathbf{Set}^{I \times J} \rightarrow \mathbf{Set}^I$.

Proof. We apply the Theorem from Lecture 13: for each $E \in \mathbf{Set}^{I \times J}$ we define $\Pi E \in \mathbf{Set}^I$ and $\varepsilon_E : \pi_1^*(\Pi E) \rightarrow E$ in $\mathbf{Set}^{I \times J}$ with the required universal property...

Theorem. π_1^* has a right adjoint $\Pi : \mathbf{Set}^{I \times J} \rightarrow \mathbf{Set}^I$.

For each $E \in \mathbf{Set}^{I \times J}$, define $\Pi E \in \mathbf{Set}^I$ to be the function mapping each $i \in I$ to the set

$$(\Pi E)_i \triangleq \prod_{j \in J} E_{(i,j)} = \{f \subseteq (\Sigma E)_i \mid f \text{ is single-valued and total}\}$$

where $f \subseteq (\Sigma E)_i$ is

single-valued if $\forall j \in J, \forall e, e' \in E_{(i,j)}, (j, e) \in f \wedge (j, e') \in f \Rightarrow e = e'$

total if $\forall j \in J, \exists e \in E_{(i,j)} (j, e) \in f$

Thus each $f \in (\Pi E)_i$ is a **dependently typed function** mapping elements $j \in J$ to elements of $E_{(i,j)}$ (result set depends on the argument j).

Theorem. π_1^* has a right adjoint $\Pi : \mathbf{Set}^{I \times J} \rightarrow \mathbf{Set}^I$.

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$$(\Pi E)_i \triangleq \prod_{j \in J} E_{(i,j)} = \{f \subseteq (\Sigma E)_i \mid f \text{ is single-value and total}\}$$

and define $\varepsilon_E : \pi_1^*(\Pi E) \rightarrow E$ in $\mathbf{Set}^{I \times J}$ to be the function mapping each $(i, j) \in I \times J$ to the function $(\varepsilon_E)_{(i,j)} : (\Pi E)_i \rightarrow E_{(i,j)}$ given by $f \mapsto f j =$ unique $e \in E_{(i,j)}$ such that $(j, e) \in f$.

Universal property–

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For each $E \in \mathbf{Set}^{I \times J}$, define $\Pi E \in \mathbf{Set}^I$ to be the function mapping each $i \in I$ to the set

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Universal property–existence part: given any $X \in \mathbf{Set}^I$ and $f : \pi_1^*(X) \rightarrow E$ in $\mathbf{Set}^{I \times J}$, we have

$$\begin{array}{ccc}
 \Pi E & & \pi_1^*(\Pi E) \xrightarrow{\varepsilon_E} E \\
 \uparrow & & \uparrow \\
 \bar{f} \mid & & \pi_1^*(\bar{f}) \mid \\
 \mid & & \mid \\
 X & & \pi_1^*(X) \xrightarrow{f} E
 \end{array}$$

where for all $i \in I$ and $x \in X_i$ $\bar{f}_i x \triangleq \{(j, f_{(i,j)} x) \mid j \in J\}$

Theorem. π_1^* has a right adjoint $\Pi : \mathbf{Set}^{I \times J} \rightarrow \mathbf{Set}^I$.

For each $E \in \mathbf{Set}^{I \times J}$, define $\Pi E \in \mathbf{Set}^I$ to be the function mapping each $i \in I$ to the set

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Universal property–uniqueness part: given $g : X \rightarrow \Pi E$ in \mathbf{Set}^I making

$$\begin{array}{ccc} \pi_1^*(\Pi E) & \xrightarrow{\varepsilon_E} & E \\ \pi_1^*(g) \uparrow & \nearrow f & \\ \pi_1^*(X) & & \end{array} \text{ commute in } \mathbf{Set}^{I \times J},$$

then for all $i \in I, j \in J$ and $x \in X_i$ we have

$$\overline{f}_i x j \triangleq f_{(i,j)} x = (\varepsilon_E \circ \pi_1^* g)_{(i,j)} x = (\varepsilon_E)_{(i,j)} (g_i x) \triangleq g_i x j$$

so $g = \overline{f}$. \square

Isomorphism of categories

Two categories **C** and **D** are **isomorphic** if they are isomorphic objects in the category of all categories of some given size, that is, if there are functors

$$\mathbf{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathbf{D} \text{ with } \text{id}_{\mathbf{C}} = G \circ F \text{ and } F \circ G = \text{id}_{\mathbf{D}}.$$

In which case, as usual, we write $\mathbf{C} \cong \mathbf{D}$.

Equivalence of categories

Two categories **C** and **D** are **equivalent** if there are

functors $C \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} D$ and natural isomorphisms

$\eta : \text{id}_C \cong G \circ F$ and $\varepsilon : F \circ G \cong \text{id}_D$.

In which case, one writes $C \simeq D$.

Equivalence of categories

Two categories \mathbf{C} and \mathbf{D} are **equivalent** if there are

functors $\mathbf{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathbf{D}$ and natural isomorphisms

$\eta : \text{id}_{\mathbf{C}} \cong G \circ F$ and $\varepsilon : F \circ G \cong \text{id}_{\mathbf{D}}$.

In which case, one writes $\mathbf{C} \simeq \mathbf{D}$.

Some deep results in mathematics take the form of equivalences of categories.

E.g.

Stone duality: $\left(\begin{array}{c} \text{category of} \\ \text{Boolean algebras} \end{array} \right)^{\text{op}} \simeq \left(\begin{array}{c} \text{category of compact} \\ \text{totally disconnected} \\ \text{Hausdorff spaces} \end{array} \right)$

Gelfand duality: $\left(\begin{array}{c} \text{category of} \\ \text{abelian } C^* \text{ algebras} \end{array} \right)^{\text{op}} \simeq \left(\begin{array}{c} \text{category of compact} \\ \text{Hausdorff spaces} \end{array} \right)$

Example: $\mathbf{Set}^I \simeq \mathbf{Set}/I$

\mathbf{Set}/I is a **slice category**:

- ▶ objects are pairs (E, p) where $E \in \text{obj } \mathbf{Set}$ and $p \in \mathbf{Set}(E, I)$
- ▶ morphisms $g : (E, p) \rightarrow (E', p')$ are $f \in \mathbf{Set}(E, E')$ satisfying $p' \circ f = p$ in \mathbf{Set}
- ▶ composition and identities – as for \mathbf{Set}

Example: $\mathbf{Set}^I \simeq \mathbf{Set}/I$

There are functors $F : \mathbf{Set}^I \rightarrow \mathbf{Set}/I$ and $G : \mathbf{Set}/I \rightarrow \mathbf{Set}^I$, given on objects and morphisms by:

$$F X \triangleq (\{(i, x) \mid i \in I \wedge x \in X_i\}, \text{fst})$$

$$F f (i, x) \triangleq (i, f_i x)$$

$$G(E, p) \triangleq (\{e \in E \mid p e = i\} \mid i \in I)$$

$$(G f)_i e \triangleq f e$$

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$$F X \triangleq (\{(i, x) \mid i \in I \wedge x \in X_i\}, \text{fst})$$

$$F f (i, x) \triangleq (i, f_i x)$$

$$G(E, p) \triangleq (\{e \in E \mid p e = i\} \mid i \in I)$$

$$(G f)_i e \triangleq f e$$

There are natural isomorphisms

$$\eta : \text{id}_{\mathbf{Set}^I} \cong G \circ F \text{ and } \varepsilon : F \circ G \cong \text{id}_{\mathbf{Set}/I}$$

defined by... [exercise]

FACT Given $p : I \rightarrow J$ in **Set**, the composition

$$\mathbf{Set}/J \simeq \mathbf{Set}^J \xrightarrow{p^*} \mathbf{Set}^I \simeq \mathbf{Set}/I$$

is the functor “**pullback** along p ”.

One can generalize from **Set** to any category **C** with pullbacks and model Σ/Π types by left/right adjoints to pullback functors – see **locally cartesian closed** categories in the literature.