III. Approximation Algorithms: Covering Problems

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UNIVERSITY OF CAMBRIDGE

Outline

Motivation

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Strategies to cope with NP-complete problems

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We will call these **approximation algorithms**.

Performance Ratios for Approximation Algorithms

Approximation Ratio

An algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size n, the cost C of the returned solution and optimal cost C^* satisfy:

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Outline

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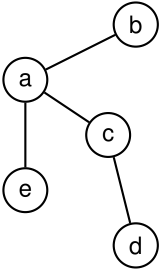
Vertex Cover Problem

- Given: Undirected graph $G = (V, E)$
- Goal: Find a minimum-cardinality subset $V' \subseteq V$ such that if $(u, v) \in E(G)$, then $u \in V'$ or $v \in V'$.



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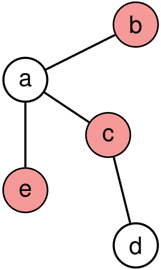

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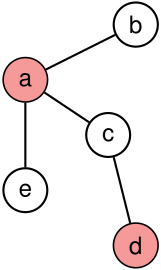

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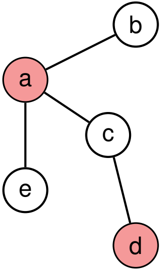

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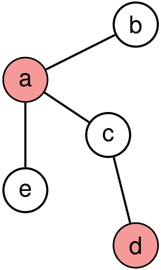

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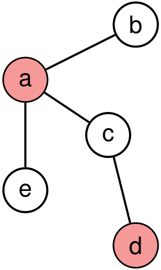

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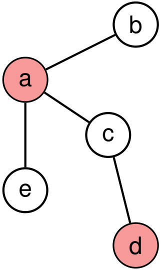

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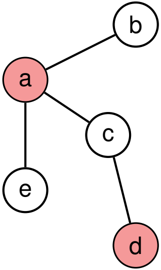

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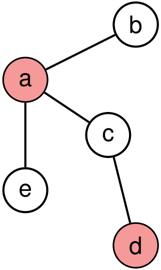

Applications:

- Every edge forms a task, and every vertex represents a person/machine which can execute that task
- Perform all tasks with the minimal amount of resources
- Extensions: weighted vertices or hypergraphs ($\rightsquigarrow$ Set-Covering Problem)

Exercise: Be creative and design your own algorithm for VERTEX-COVER!

An Approximation Algorithm based on Greedy

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APPROX-VERTEX-COVER(G)
C = ∅
E' = G.E
while E' ≠ ∅
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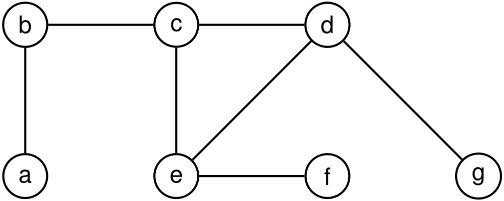

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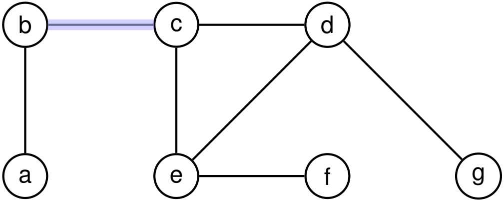

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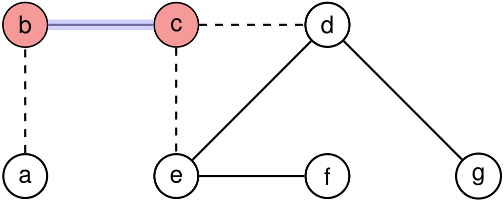

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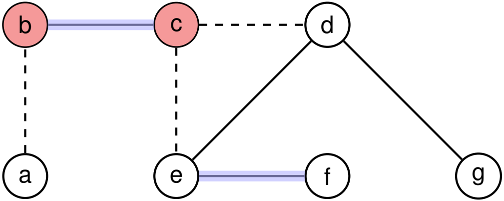

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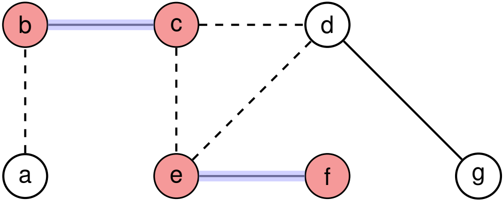

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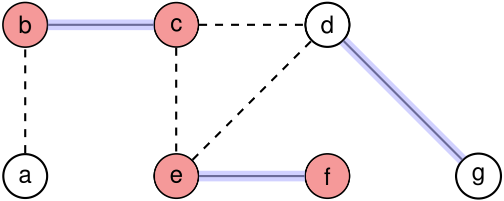

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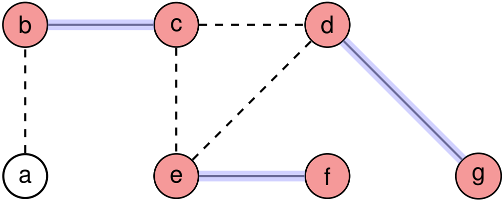

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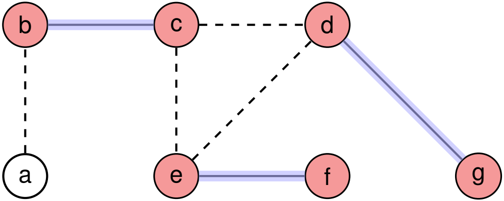

APPROX-VERTEX-COVER produces a set of size 6.

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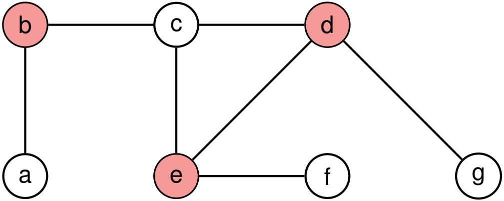

The optimal solution has size 3.

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- Every edge in A contributes 2 vertices to $|C|$:

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$\Rightarrow$ Every optimal cover C^* must include at least one endpoint: $|C^*| \geq |A|$

- Every edge in A contributes 2 vertices to $|C|$: $|C| = 2|A|$

Analysis of Greedy for Vertex Cover

APPROX-VERTEX-COVER(G)

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1  C = ∅
2  E' = G.E
3  while E' ≠ ∅
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Theorem 35.1

APPROX-VERTEX-COVER is a poly-time 2-approximation algorithm.

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- Running time is $O(V + E)$ (using adjacency lists to represent E')
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A "vertex-based" Greedy that adds **one** vertex at each iteration fails to achieve an approximation ratio of 2 (Supervision Exercise)!

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Solving Special Cases

Strategies to cope with NP-complete problems

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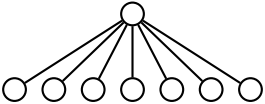

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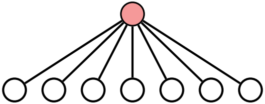

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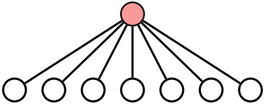

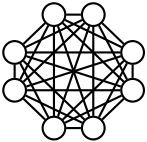

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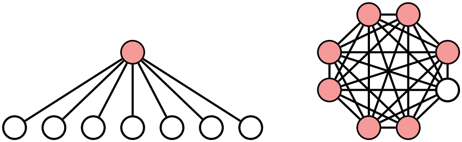

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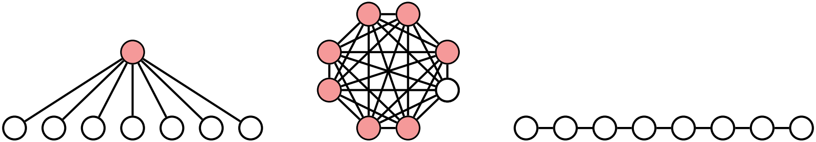

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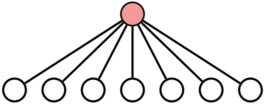
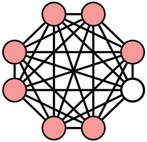
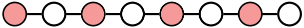
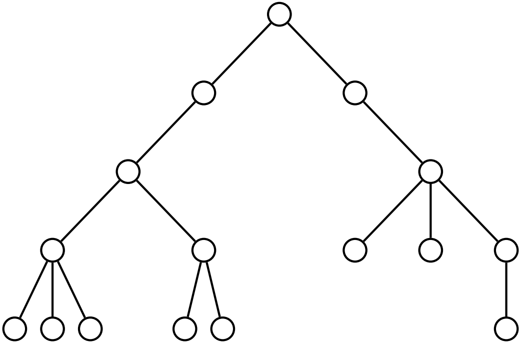






Vertex Cover on Trees



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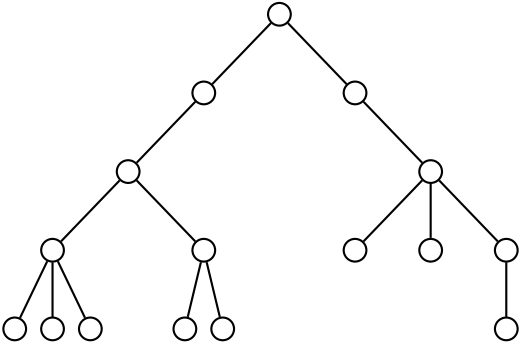

There exists an optimal vertex cover which does not include any leaves.

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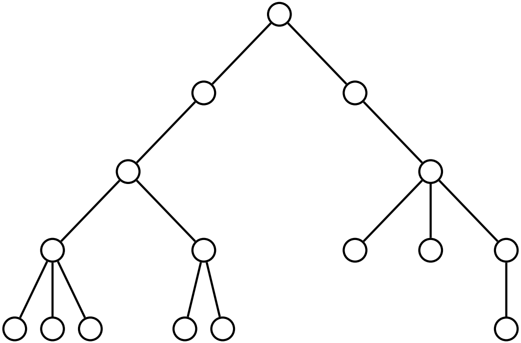

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Exchange-Argument: Replace any leaf in the cover by its parent.

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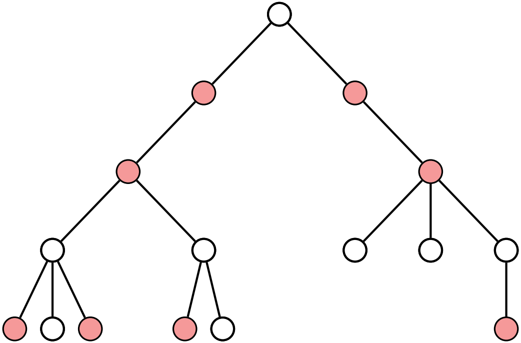

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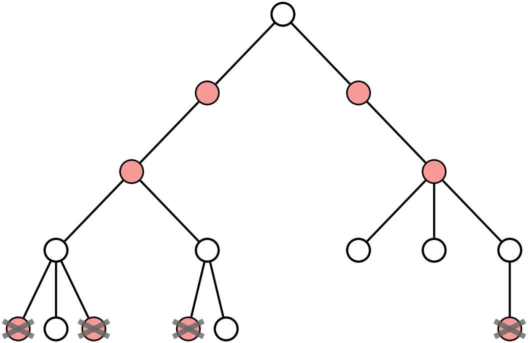

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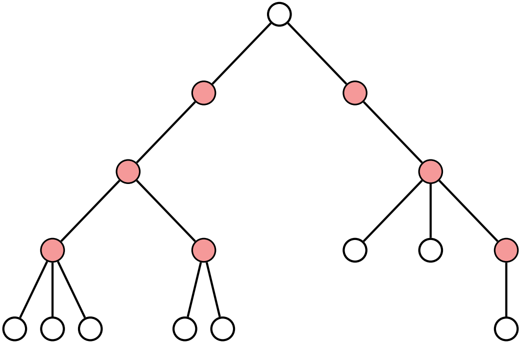

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$C = \emptyset$
while $\exists$ leaves in G
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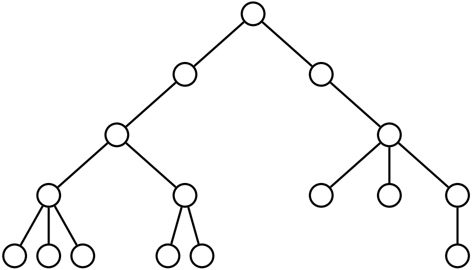
Solution is also optimal. (Use inductively the existence of an optimal vertex cover without leaves)

Execution on a Small Example



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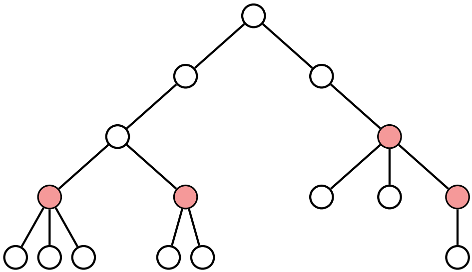

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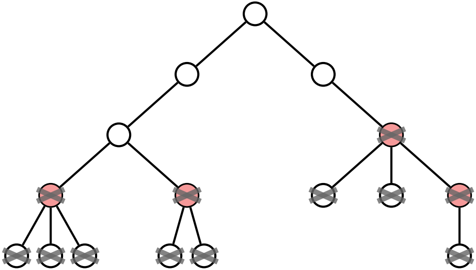

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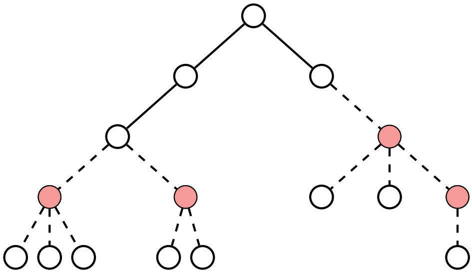

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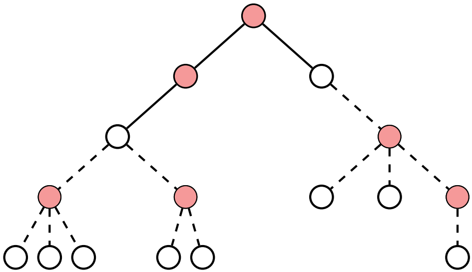

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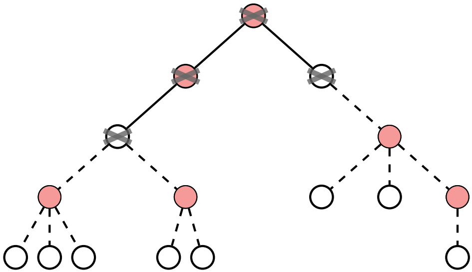

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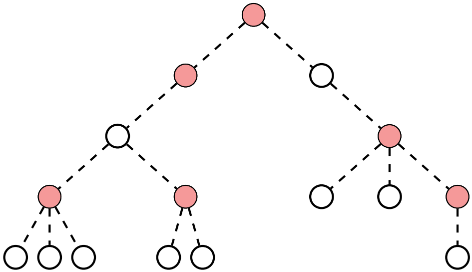

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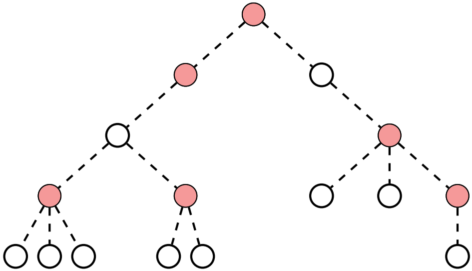

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Problem can be also solved on bipartite graphs, using Max-Flows and Min-Cuts.

Exact Algorithms

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Simple Brute-Force Search would take $\approx \binom{n}{k} = \Theta(n^k)$ time.

Towards a more efficient Search

Substructure Lemma

Consider a graph $G = (V, E)$, edge $\{u, v\} \in E(G)$ and integer $k \geq 1$. Let G_u be the graph obtained by deleting u and its incident edges (G_v is defined similarly). Then G has a vertex cover of size k if and only if G_u or G_v (or both) have a vertex cover of size $k - 1$.

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Reminiscent of Dynamic Programming.

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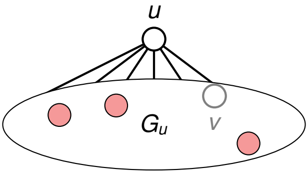

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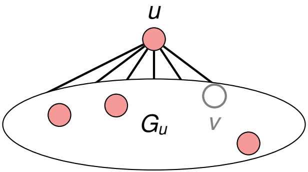
Adding u yields a vertex cover of G which is of size k



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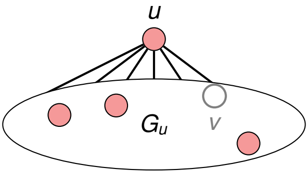

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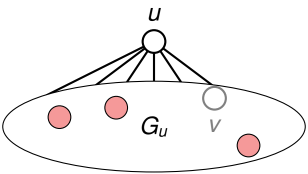
Removing u from C yields a vertex cover of G_u which is of size $k - 1$. □



A More Efficient Search Algorithm

VERTEX-COVER-SEARCH(G, k)

if $E = \emptyset$ **return** $\emptyset$
if $k = 0$ and $E \neq \emptyset$ **return** $\perp$
Pick an arbitrary edge $(u, v) \in E$
$S_1 =$ VERTEX-COVER-SEARCH$(G_u, k-1)$
$S_2 =$ VERTEX-COVER-SEARCH$(G_v, k-1)$
if $S_1 \neq \perp$ **return** $S_1 \cup \{u\}$
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Correctness follows by the Substructure Lemma and induction.

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Running time:

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Running time:

- Depth k, branching factor 2 $\Rightarrow$ total number of calls is $O(2^k)$

A More Efficient Search Algorithm

$\text{VERTEX-COVER-SEARCH}(G, k)$

```
if E = ∅ return ∅
if k = 0 and E ≠ ∅ return ⊥
Pick an arbitrary edge (u, v) ∈ E
S_1 = VERTEX-COVER-SEARCH(G_u, k − 1)
S_2 = VERTEX-COVER-SEARCH(G_v, k − 1)
if S_1 ≠ ⊥ return S_1 ∪ {u}
if S_2 ≠ ⊥ return S_2 ∪ {v}
return ⊥
```

Running time:

- Depth k, branching factor 2 $\Rightarrow$ total number of calls is $O(2^k)$
- $O(E)$ worst-case time for one call (computing G_u or G_v could take $\Theta(E)$!)

A More Efficient Search Algorithm

VERTEX-COVER-SEARCH(G, k)

if $E = \emptyset$ **return** $\emptyset$
if $k = 0$ and $E \neq \emptyset$ **return** $\bot$
Pick an arbitrary edge $(u, v) \in E$
$S_1 =$ VERTEX-COVER-SEARCH$(G_u, k-1)$
$S_2 =$ VERTEX-COVER-SEARCH$(G_v, k-1)$
if $S_1 \neq \bot$ **return** $S_1 \cup \{u\}$
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Running time:

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exponential in k, but much better than $\Theta(n^k)$ (i.e., still polynomial for $k = O(\log n)$)

Outline

Introduction

Vertex Cover

The Set-Covering Problem

The Set-Covering Problem

Set Cover Problem

- Given: set X of size n and family of subsets $\mathcal{F}$
- Goal: Find a minimum-size subset $\mathcal{C} \subseteq \mathcal{F}$

$$\text{s.t.} \qquad X = \bigcup_{S \in \mathcal{C}} S.$$

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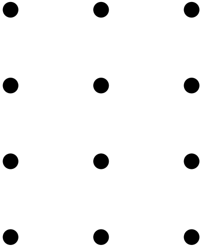

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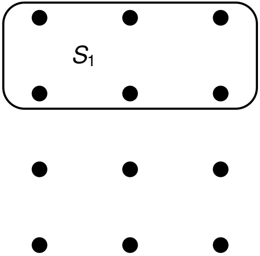

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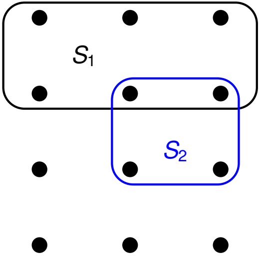

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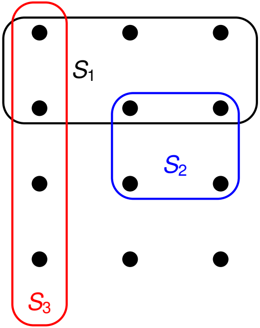

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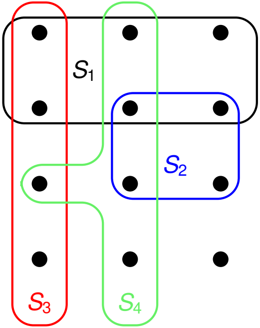

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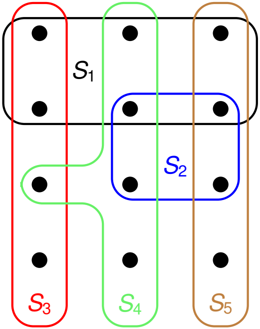

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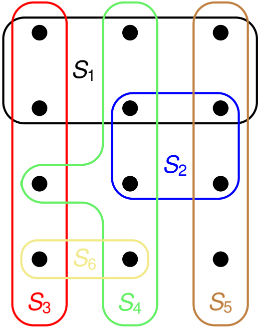

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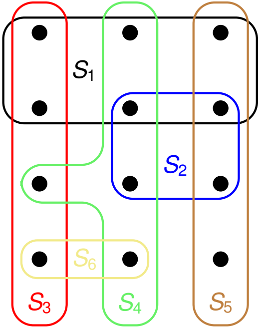

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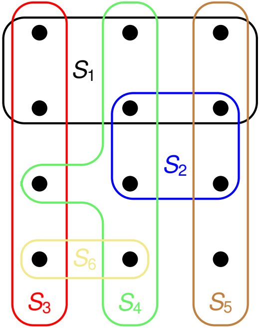

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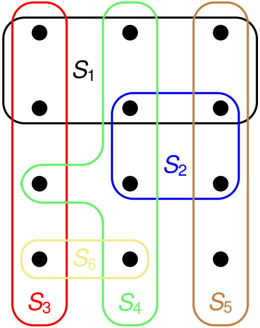

Remarks:

- generalisation of the vertex-cover problem and hence also NP-hard.
- models resource allocation problems, e.g., wireless coverage

Greedy

Strategy: Pick the set S that covers the largest number of uncovered elements.

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$U = X$
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while $U \neq \emptyset$
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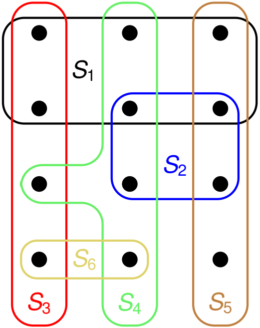

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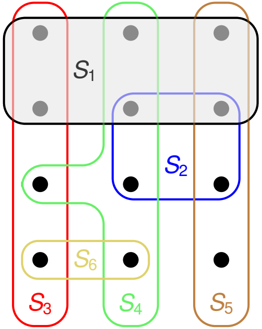

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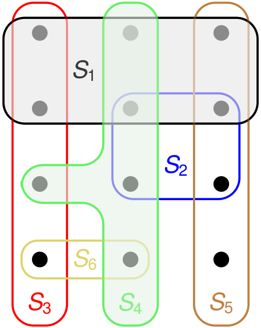

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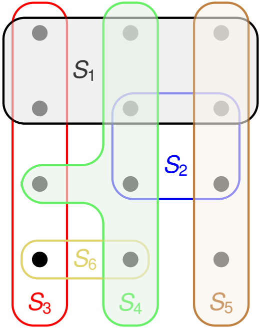

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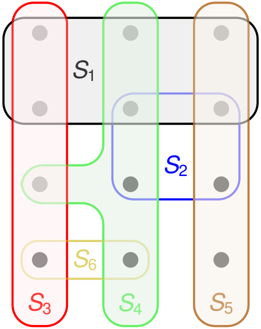

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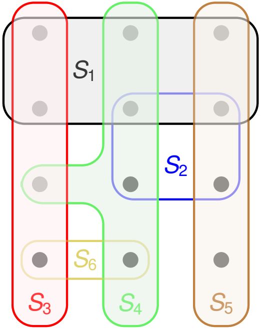

Greedy chooses S_1, S_4, S_5 and S_3 (or S_6), which is a cover of size 4.

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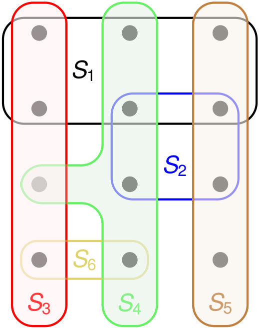

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Optimal cover is $\mathcal{C} = \{S_3, S_4, S_5\}$

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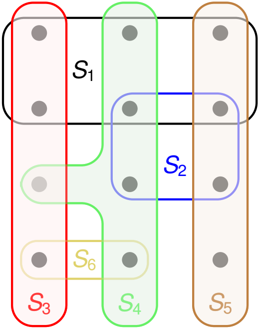

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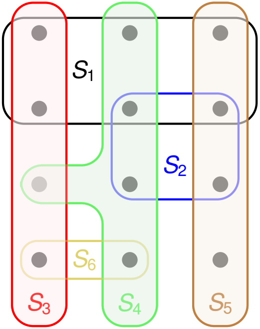

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How good is the approximation ratio?

Approximation Ratio of Greedy

Theorem 35.4

GREEDY-SET-COVER is a polynomial-time $\rho(n)$-algorithm, where

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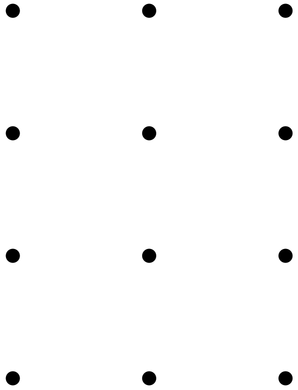
Notice that in the mathematical analysis, S_i is the set chosen in iteration i - not to be confused with the sets $S_1, S_2, \ldots, S_6$ in the example.

Illustration of Costs for Greedy picking S_1, S_4, S_5 and S_3



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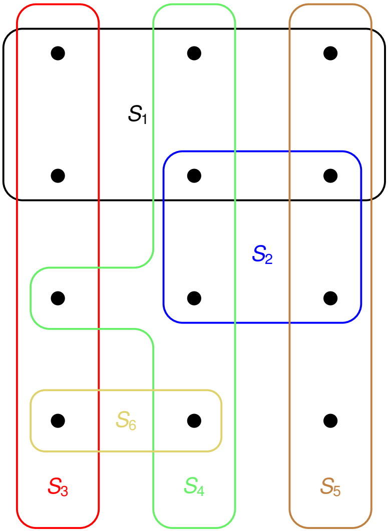

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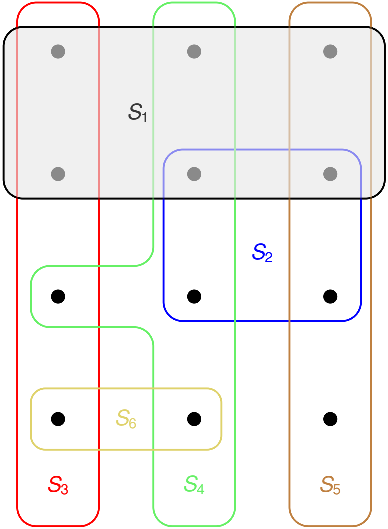

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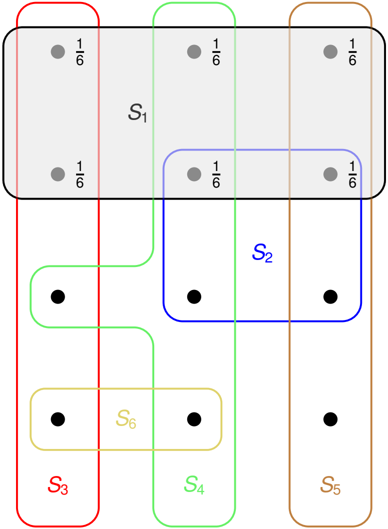

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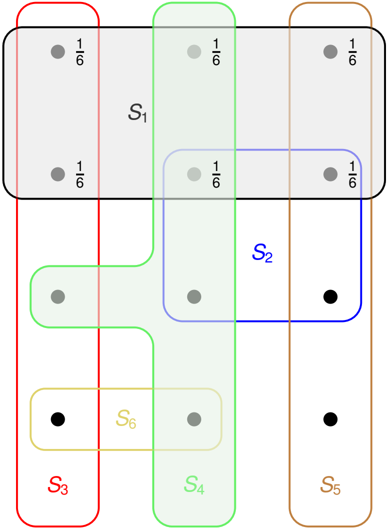

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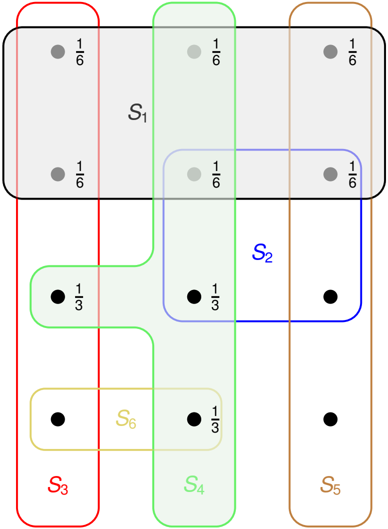

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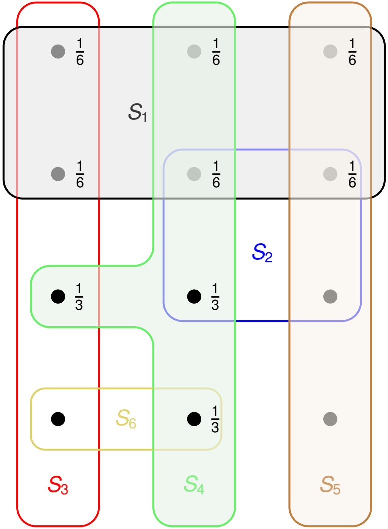

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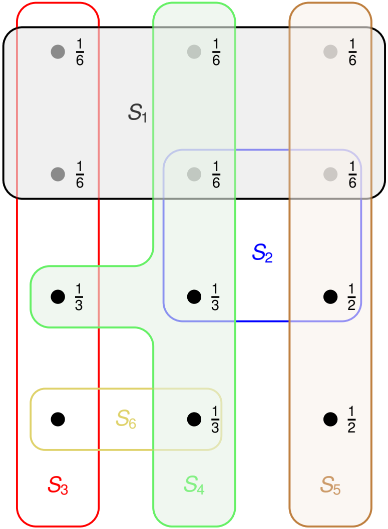

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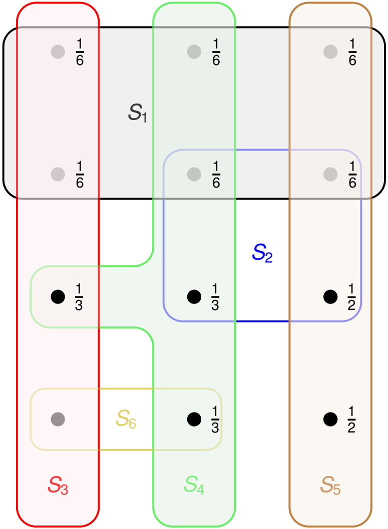

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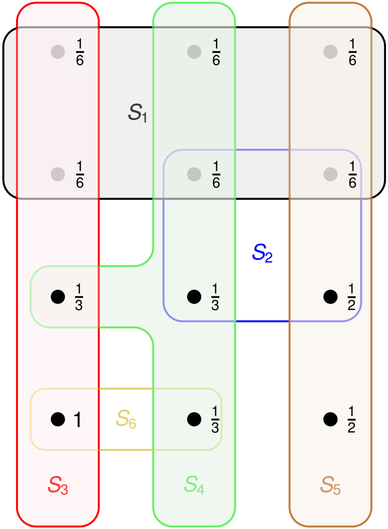

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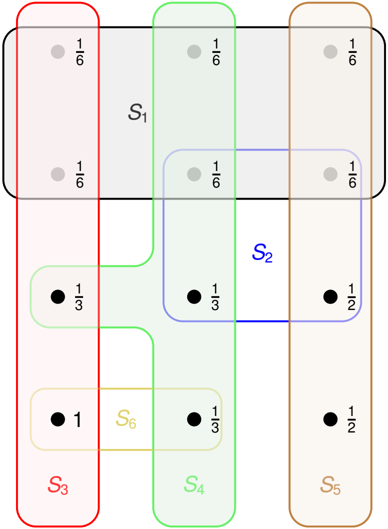

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{2} + \frac{1}{2} + 1 = ??$$

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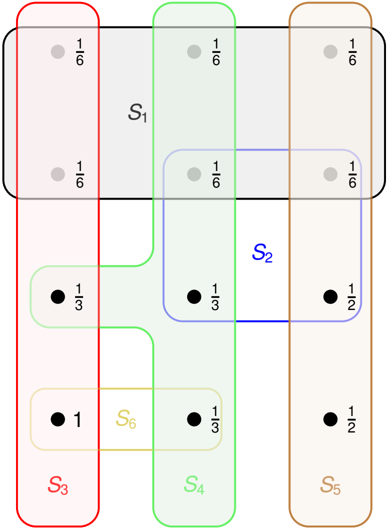

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Proof of Theorem 35.4 (1/2)

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Remaining uncovered elements in S

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Sets chosen by the algorithm

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Set-Covering Problem (Summary)

Theorem 35.4

GREEDY-SET-COVER is a polynomial-time $\rho(n)$-algorithm, where

$$\rho(n) = H(\max\{|S|: S \in \mathcal{F}\}) \leq \ln(n) + 1.$$

Set-Covering Problem (Summary)

The same approach also gives an approximation ratio of $O(\ln(n))$ if there exists a cost function $c : \mathcal{F} \to \mathbb{R}^+$

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Lower Bound

Unless P=NP, there is no $c \cdot \ln(n)$ polynomial-time approximation algorithm for some constant $0 < c < 1$.

Example where the solution of Greedy is bad

Instance

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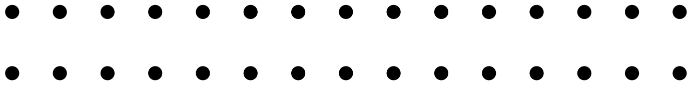
$k = 4, n = 30$:



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- There are $n = 2^{k+1} - 2$ elements overall (so $k \approx \log_2 n$)
- Sets $S_1, S_2, \ldots, S_k$ are pairwise disjoint and each set contains $2, 4, \ldots, 2^k$ elements

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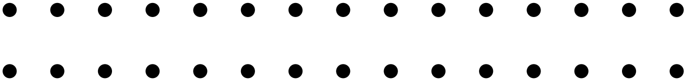

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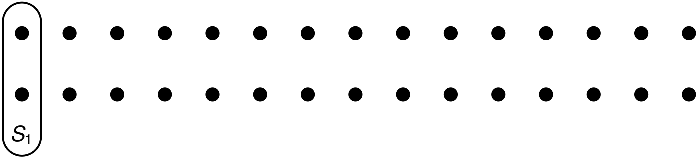

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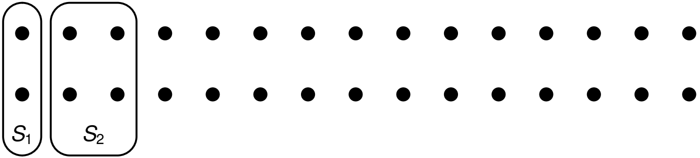

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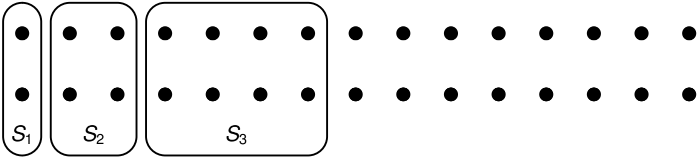

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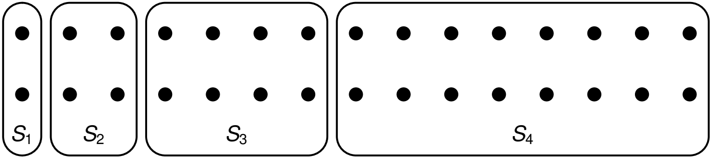

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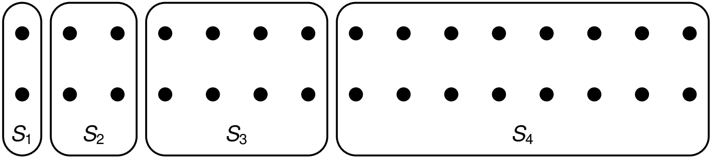

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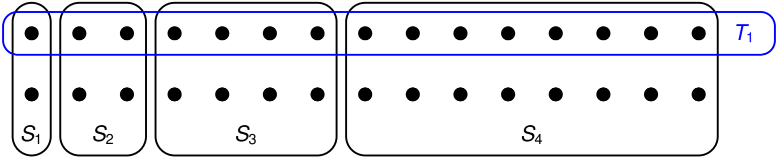

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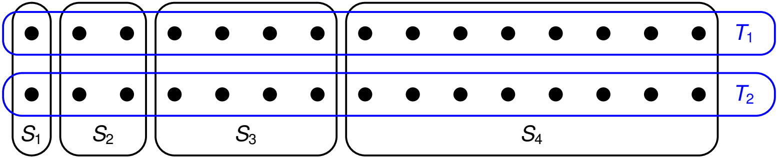

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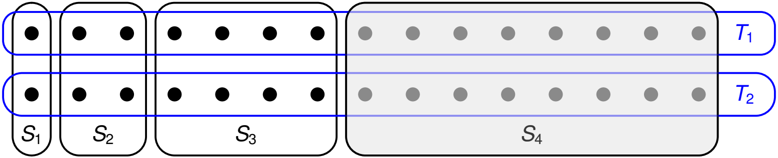

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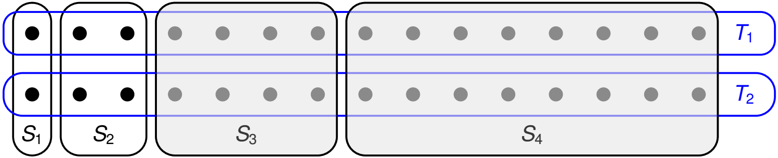

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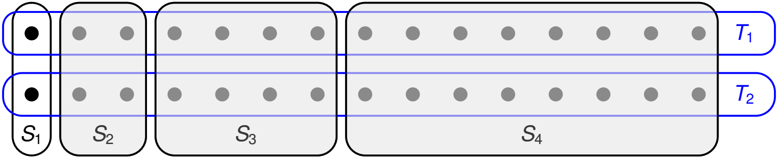

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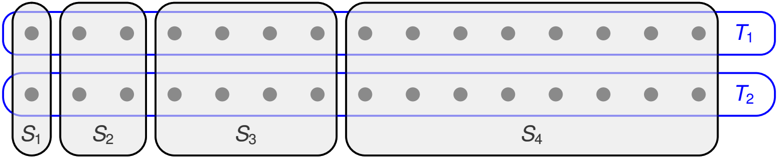

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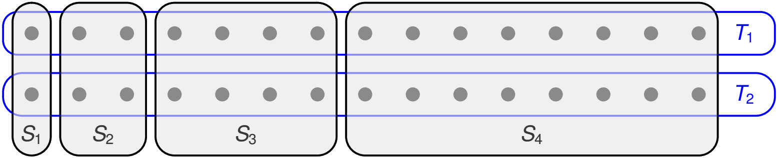

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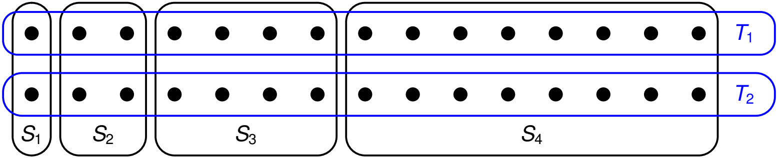

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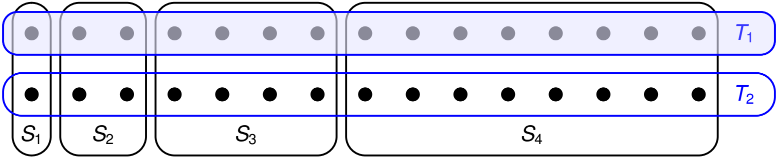

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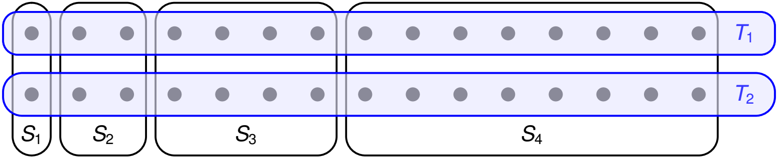

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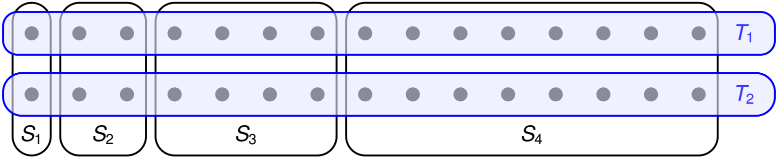

Solution of Greedy consists of k sets.

Optimum consists of 2 sets.

Exercise: Consider the vertex cover problem, restricted to a graph where every vertex has exactly 3 neighbours. Which approximation ratio can we obtain?

1. 1 (i.e., I can solve it exactly!!!)
2. 2
3. $11/6 = 2 - 1/6$
4. $H(n) \leq log(n)$