V. Approx. Algorithms: Travelling Salesman Problem

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UNIVERSITY OF CAMBRIDGE

Outline

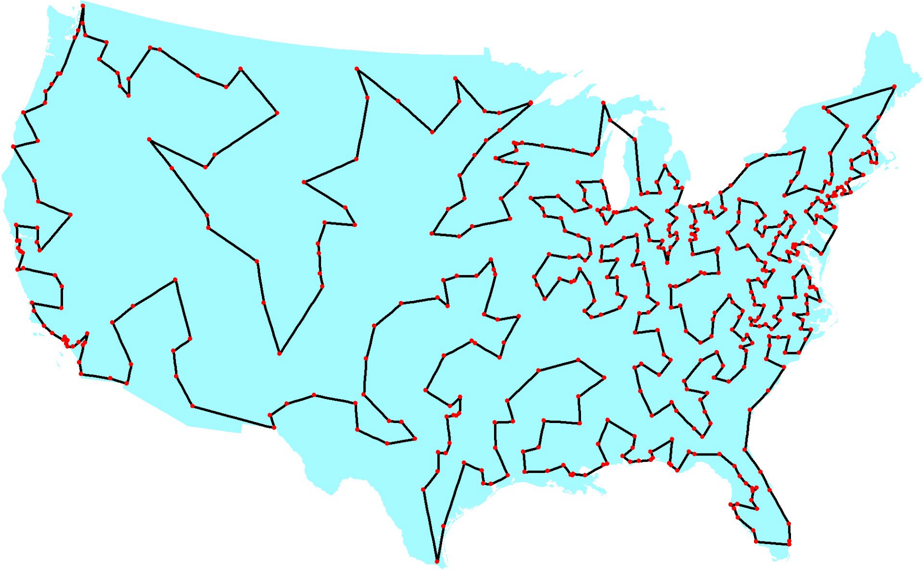
Introduction

General TSP

Metric TSP

33 city contest (1964)

532 cities (1987 [Padberg, Rinaldi])



13,509 cities (1999 [Applegate, Bixby, Chavatal, Cook])

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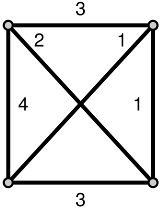

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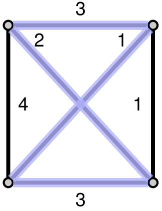

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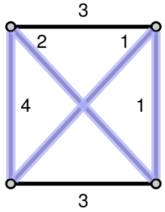

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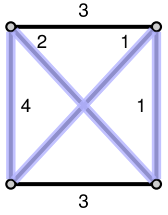

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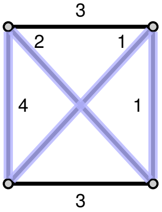

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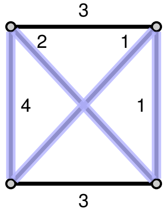

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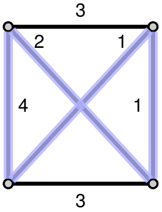

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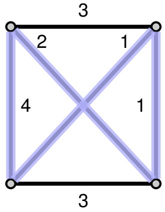

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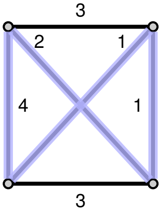

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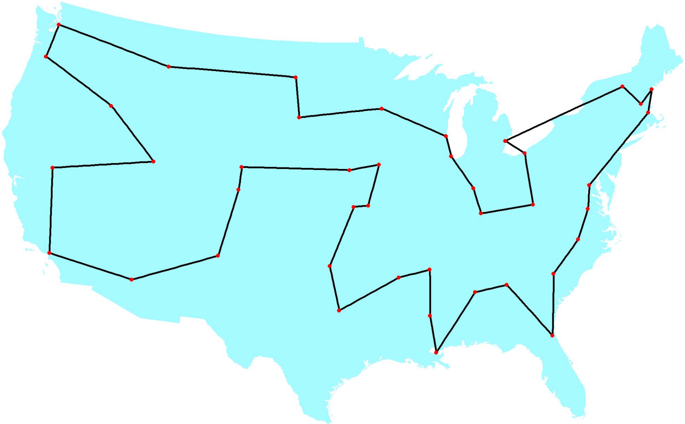
- Euclidean TSP: cities are points in the Euclidean space, costs are equal to their (rounded) Euclidean distance

History of the TSP problem (1954)

Dantzig, Fulkerson and Johnson found an optimal tour through 42 cities.



http://www.math.uwaterloo.ca/tsp/history/img/dantzig_big.html

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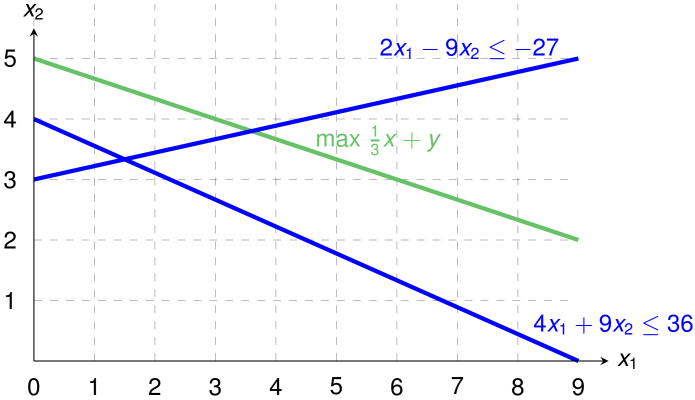

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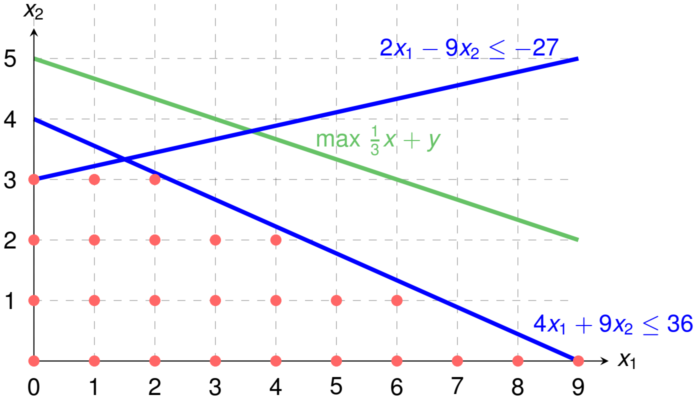

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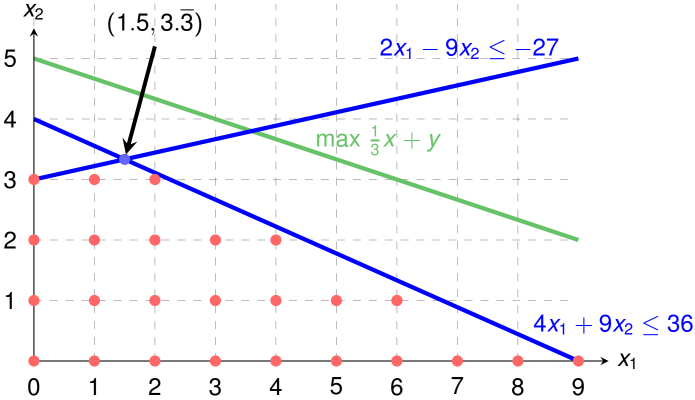

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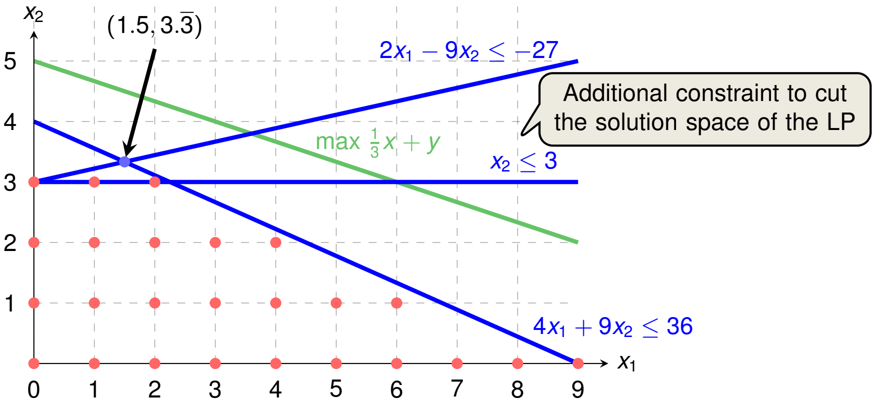

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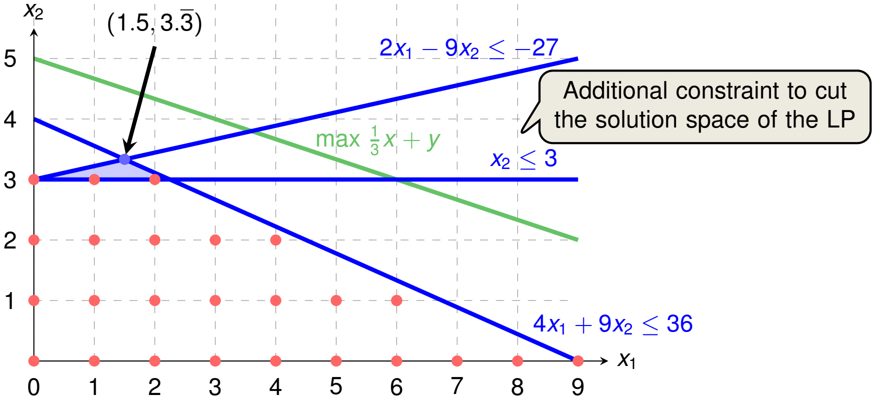

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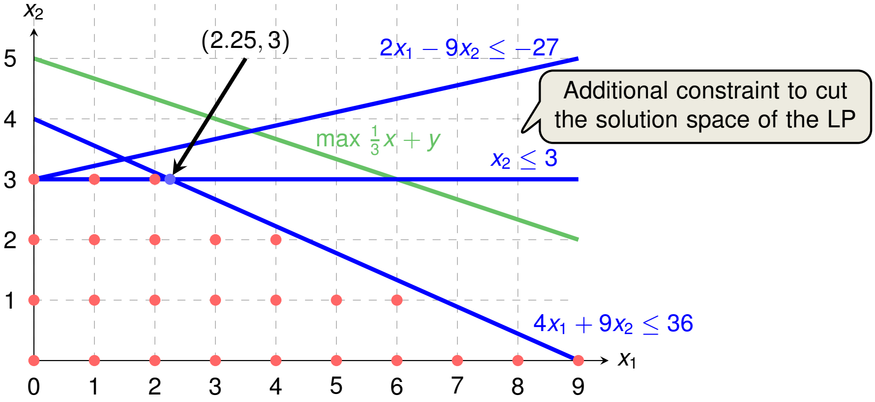

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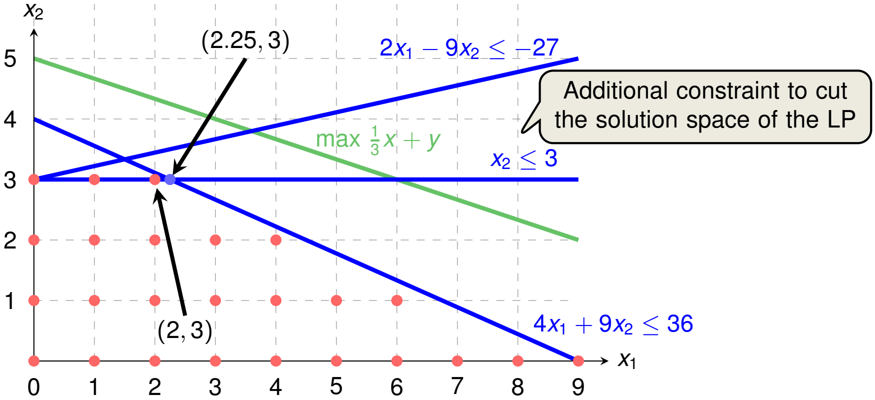

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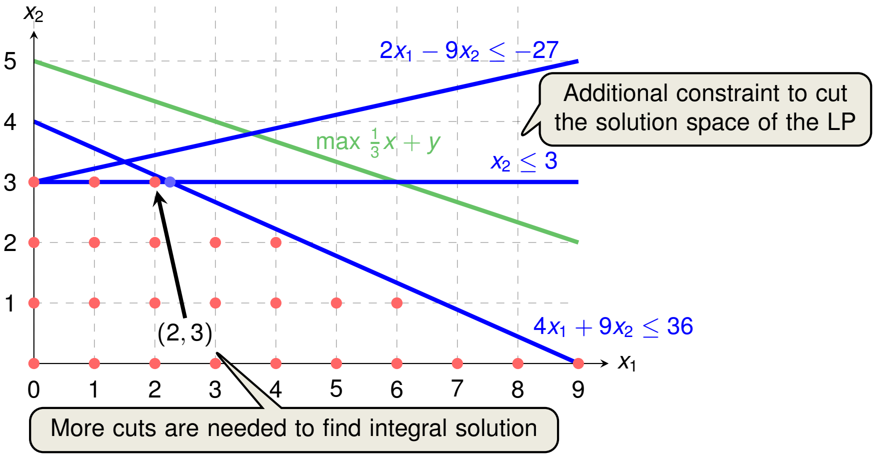

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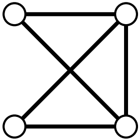

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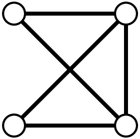

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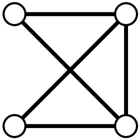

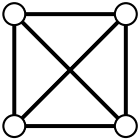

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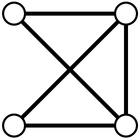

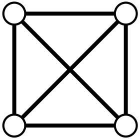

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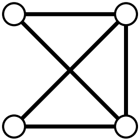

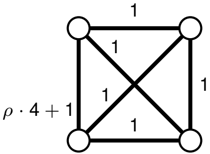

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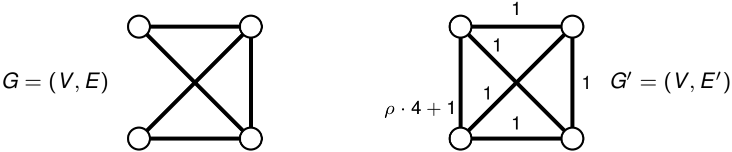

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Can create representations of G' and c in time polynomial in $|V|$ and $|E|$!

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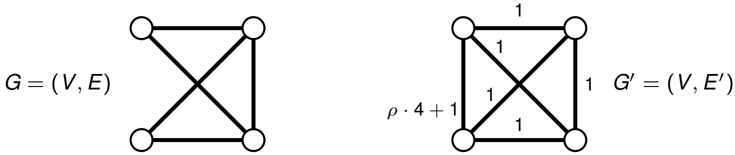

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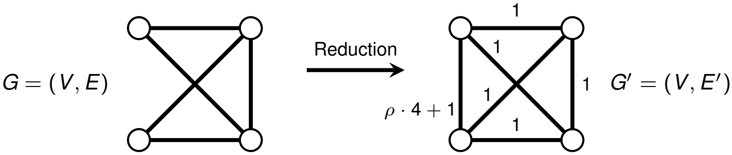

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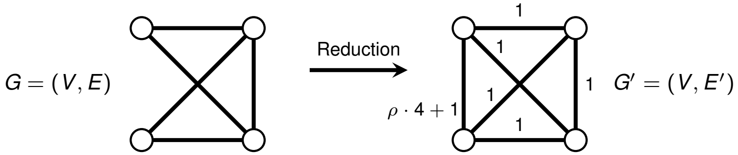

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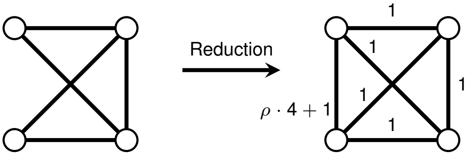

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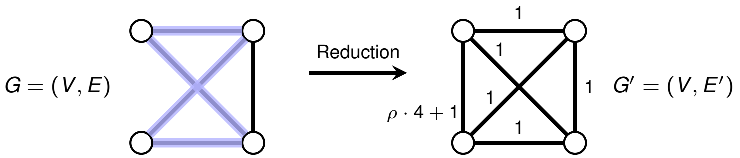

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If P $\neq$ NP, then for any constant $\rho \geq 1$, there is no polynomial-time approximation algorithm with approximation ratio ρ for the general TSP.

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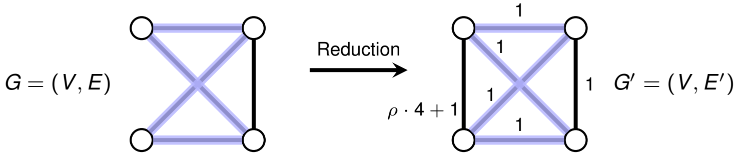

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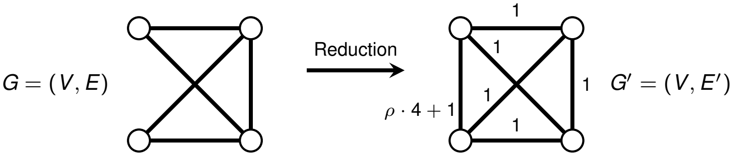

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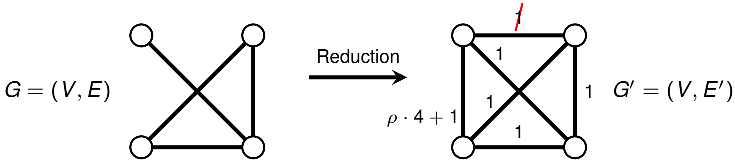

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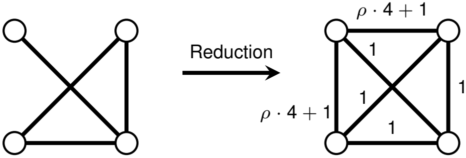

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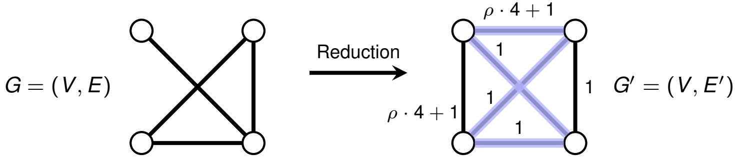

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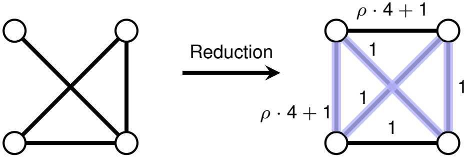

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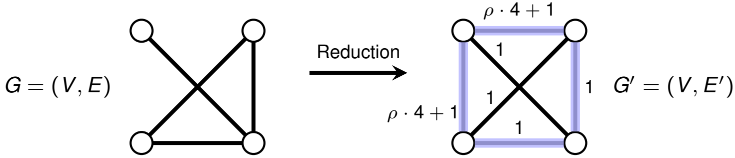

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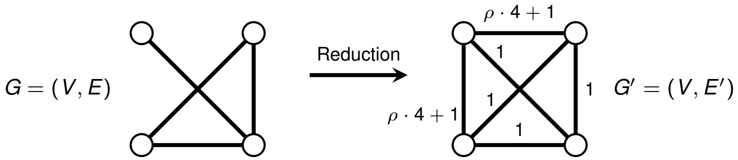

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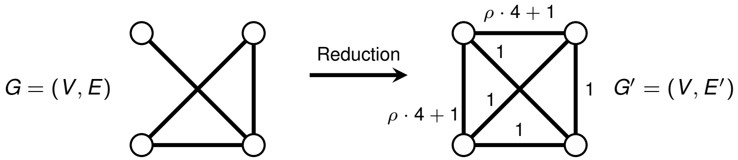

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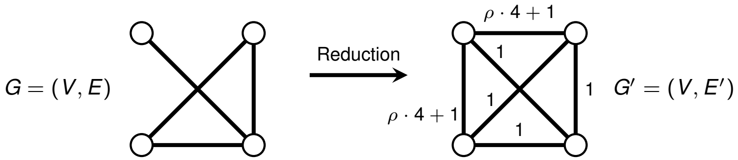

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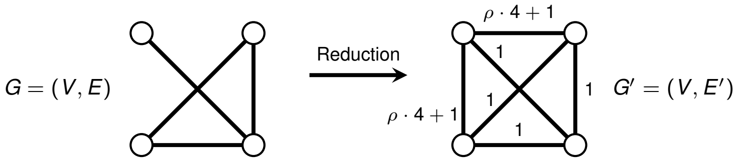

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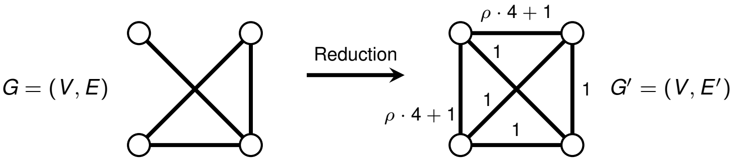
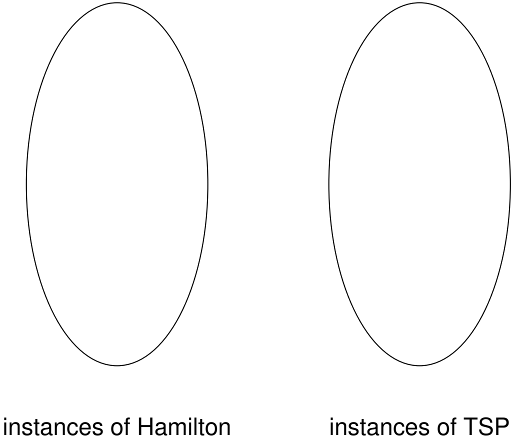


Proof of Theorem 35.3 from a higher perspective



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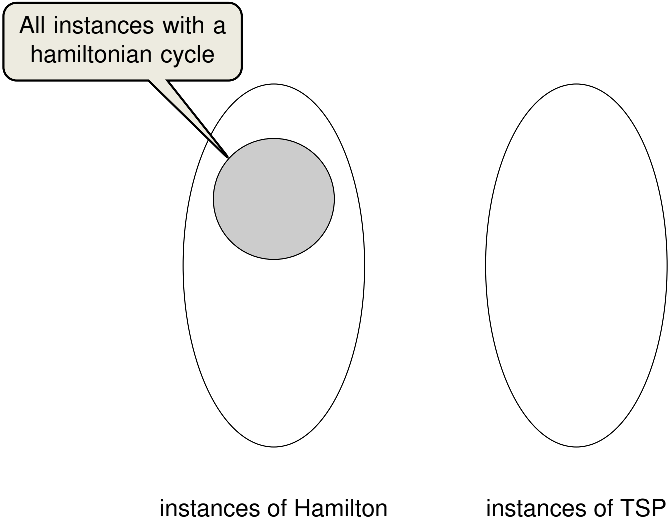

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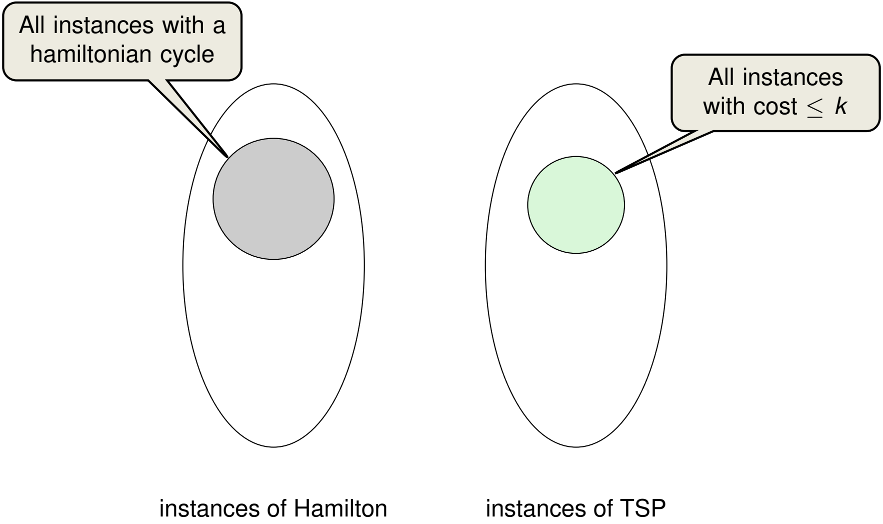

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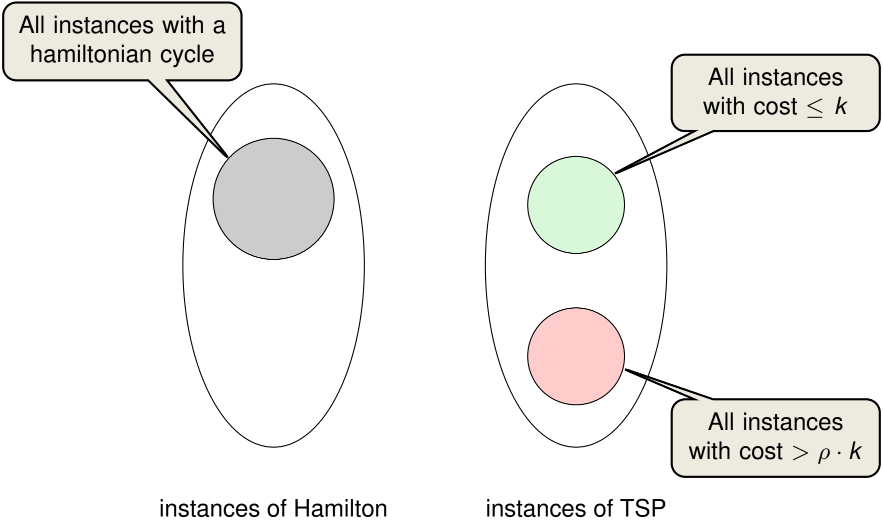

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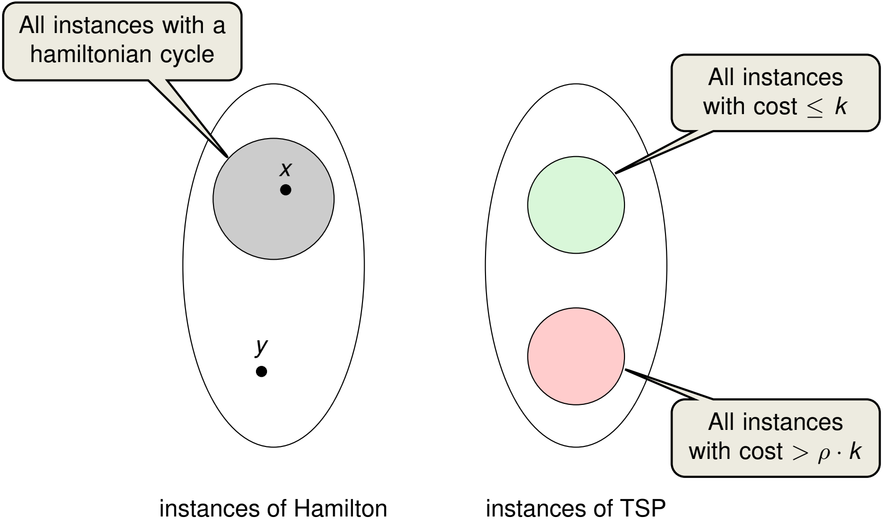

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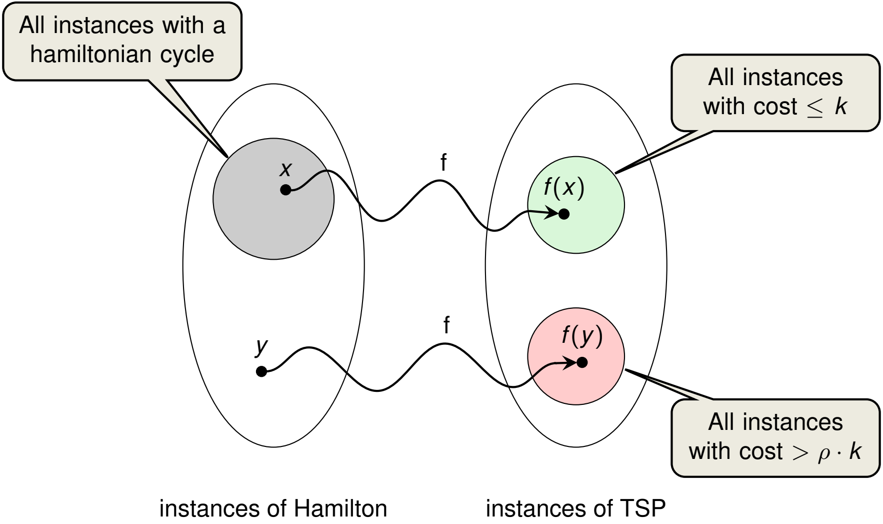

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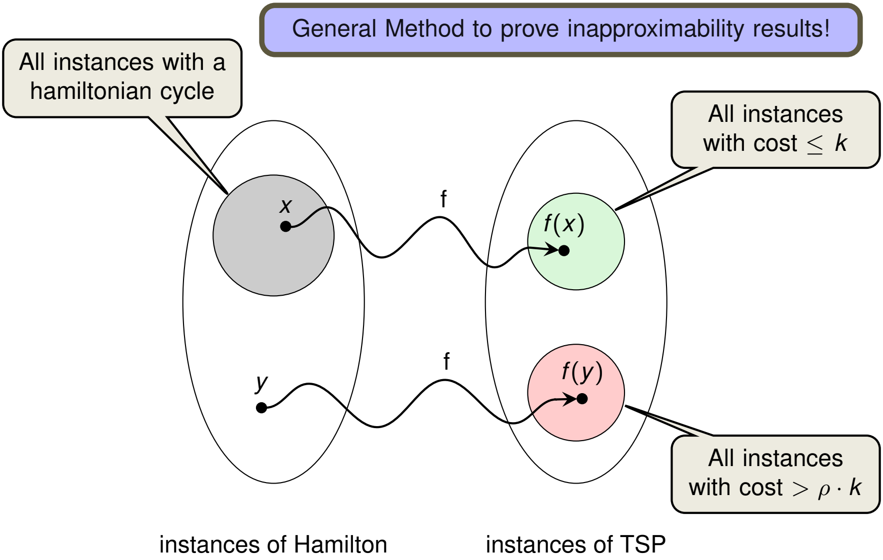

Outline

Introduction

General TSP

Metric TSP

Metric TSP (TSP Problem with the Triangle Inequality)

Idea: First compute an MST, and then create a tour based on the tree.

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APPROX-TSP-TOUR(G, c)
select a vertex r ∈ G.V to be a "root" vertex
compute a minimum spanning tree T_min for G from root r
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let H be a list of vertices, ordered according to when they are first visited
        in a preorder walk of T_min
return the hamiltonian cycle H
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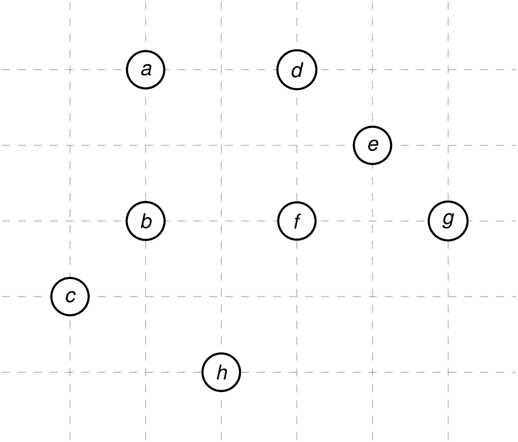
Remember: In the Metric-TSP problem, G is a complete graph.

Run of APPROX-TSP-TOUR



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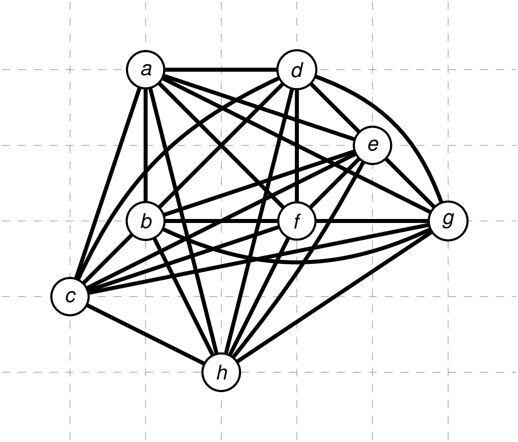

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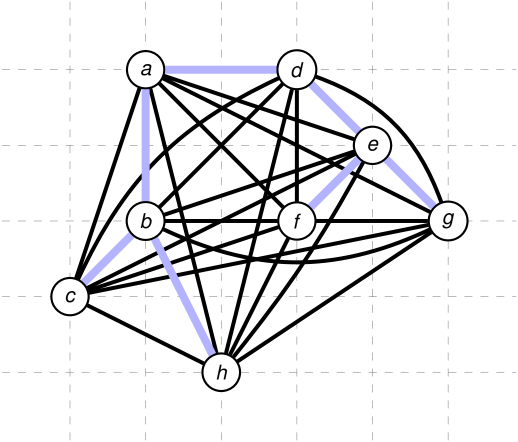

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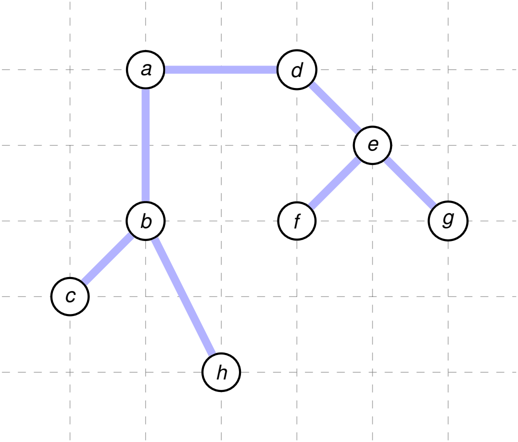

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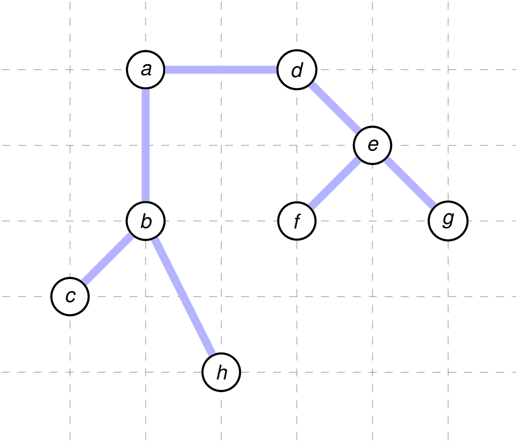

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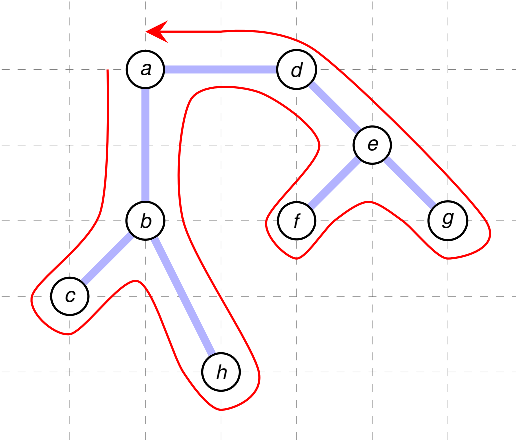

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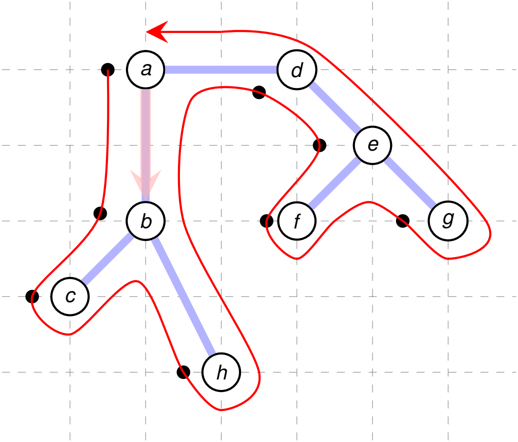

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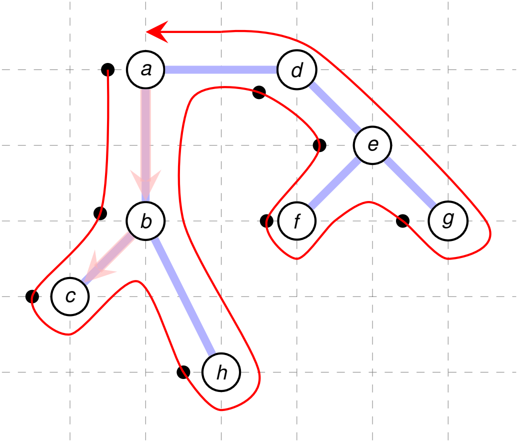

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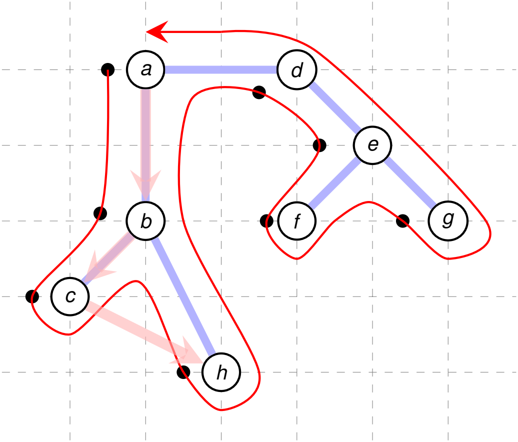

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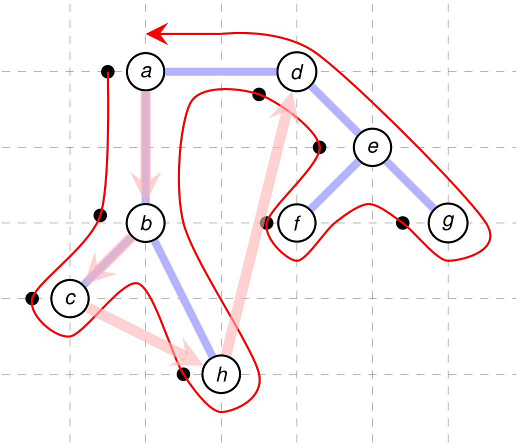

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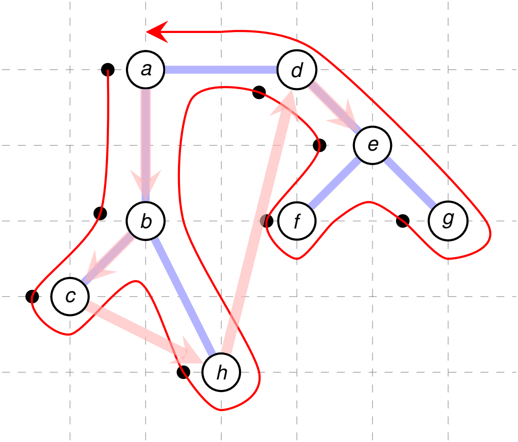

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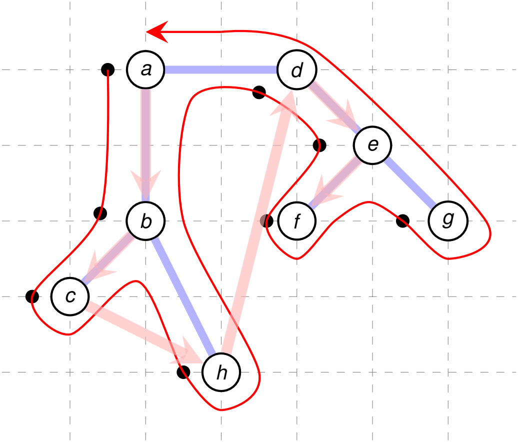

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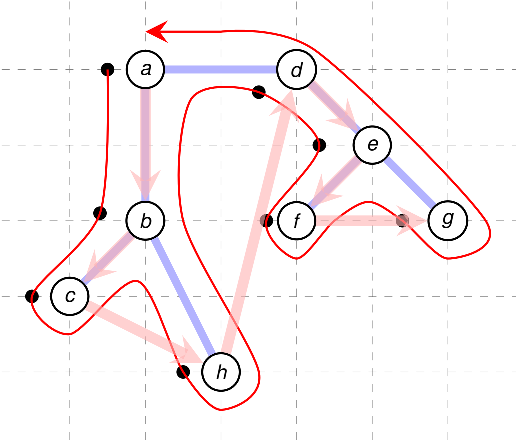

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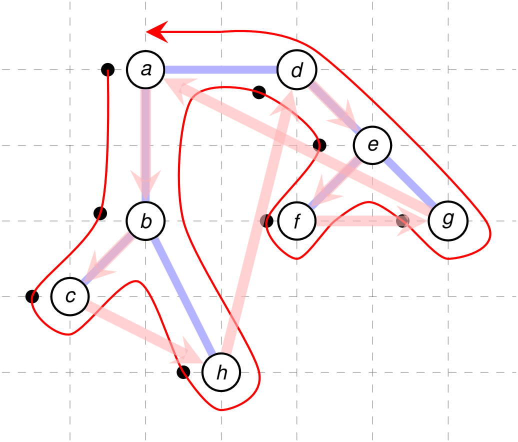

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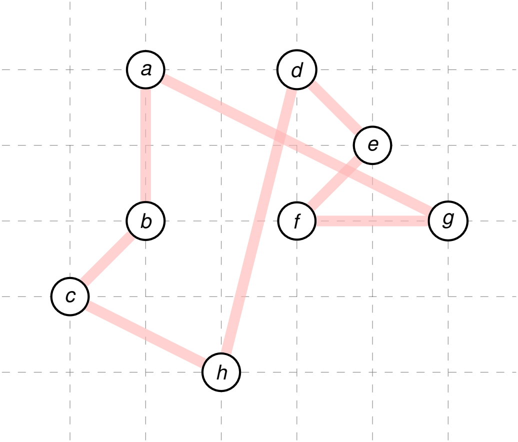

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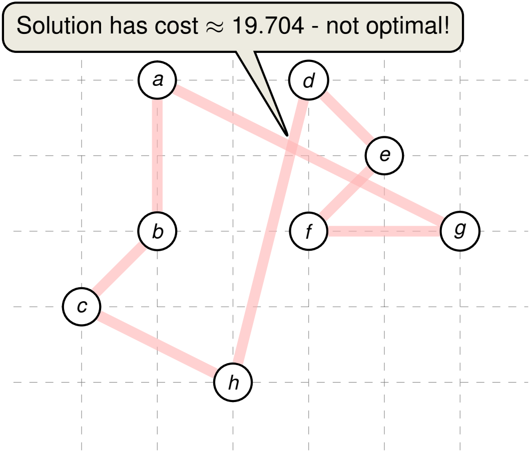
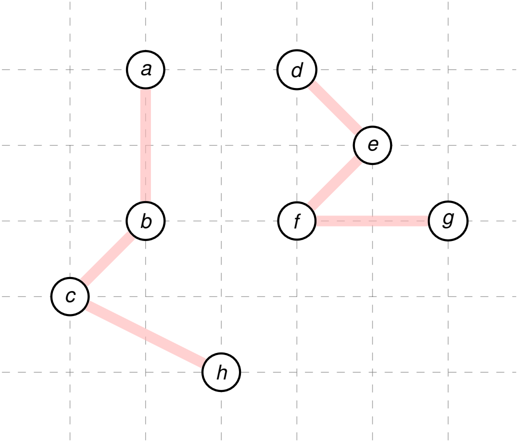


1. Compute MST T_{min} ✓
2. Perform preorder walk on MST T_{min} ✓
3. Return list of vertices according to the preorder tree walk ✓

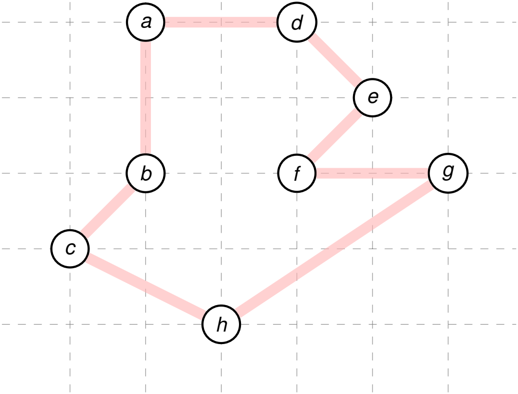
Run of APPROX-TSP-TOUR



1. Compute MST T_{min} ✓
2. Perform preorder walk on MST T_{min} ✓
3. Return list of vertices according to the preorder tree walk ✓

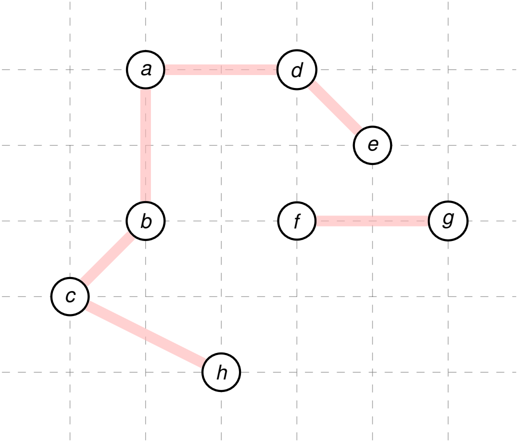
Run of APPROX-TSP-TOUR

Better solution, yet still not optimal!



1. Compute MST T_{min} ✓
2. Perform preorder walk on MST T_{min} ✓
3. Return list of vertices according to the preorder tree walk ✓

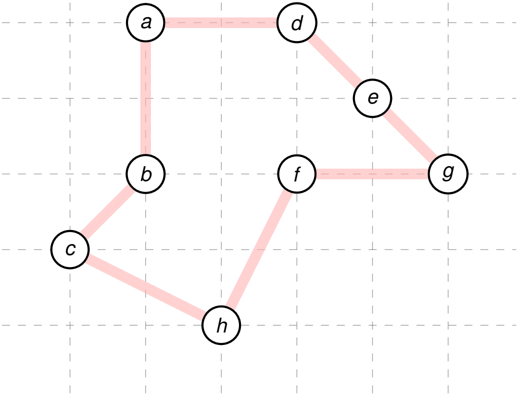
Run of APPROX-TSP-TOUR



1. Compute MST $T_{\min}$ ✓
2. Perform preorder walk on MST $T_{\min}$ ✓
3. Return list of vertices according to the preorder tree walk ✓

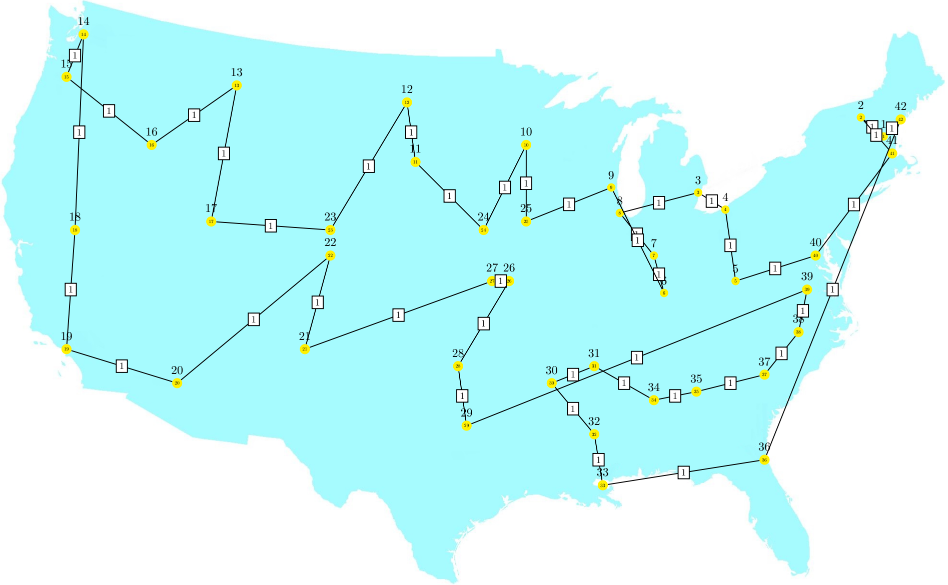
Run of APPROX-TSP-TOUR

This is the optimal solution (cost $\approx$ 14.715).

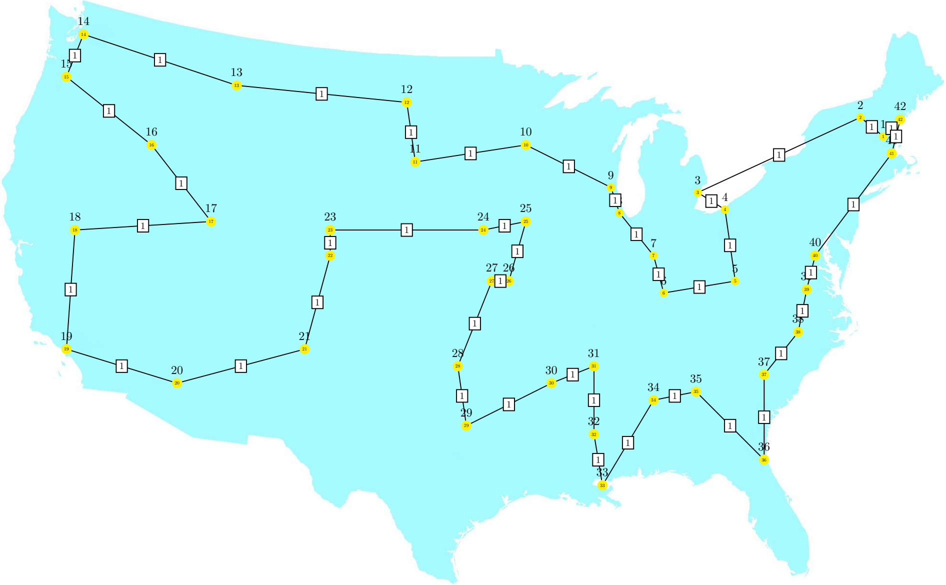


1. Compute MST T_{min} ✓
2. Perform preorder walk on MST T_{min} ✓
3. Return list of vertices according to the preorder tree walk ✓

Approximate Solution: Objective 921



Optimal Solution: Objective 699



Proof of the Approximation Ratio

Theorem 35.2

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

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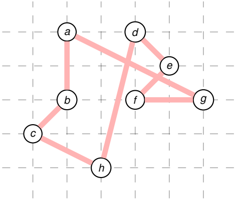

solution H of APPROX-TSP

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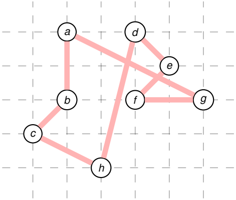

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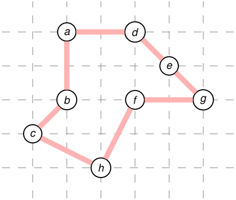

optimal solution H^*

Proof of the Approximation Ratio

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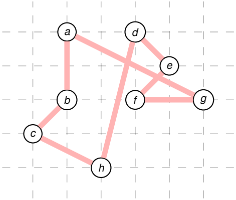
Proof:

- Consider the optimal tour H^* and remove an arbitrary edge



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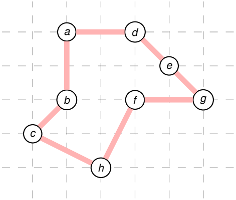

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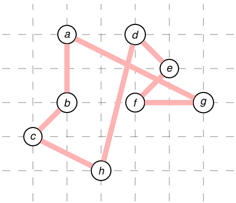
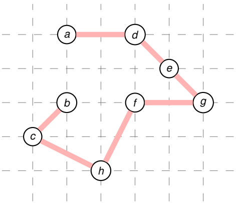


solution H of APPROX-TSP



spanning tree T as a subset of H^*

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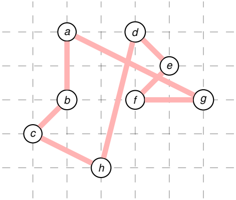

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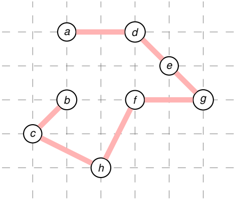

spanning tree T as a subset of H^*

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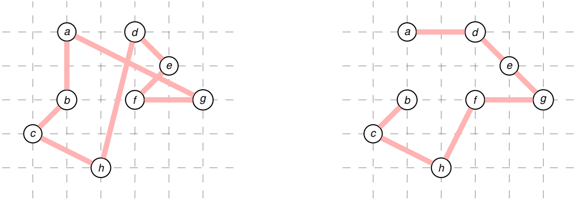

solution H of APPROX-TSP — spanning tree T as a subset of H^*

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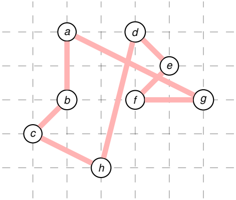
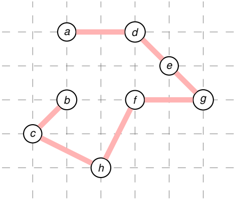
exploiting that all edge costs are non-negative!



solution H of APPROX-TSP



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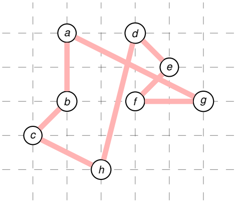
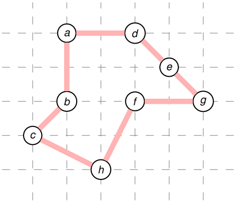


solution H of APPROX-TSP



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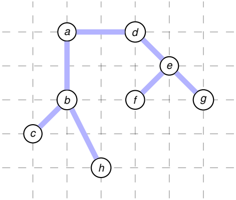
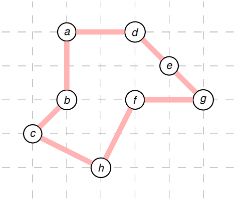


minimum spanning tree $T_{\min}$



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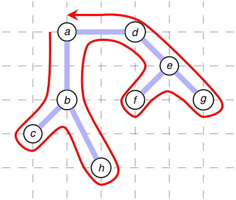
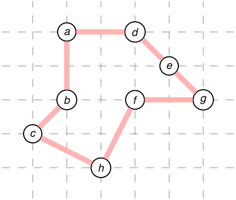


Walk $W = (a, b, c, b, h, b, a, d, e, f, e, g, e, d, a)$



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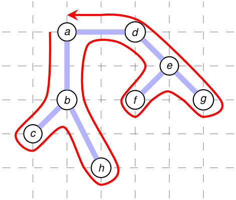

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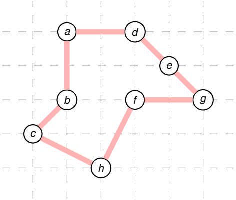

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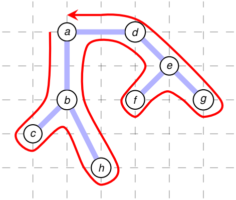

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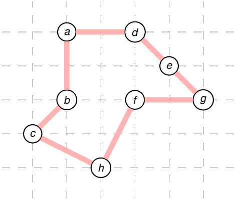

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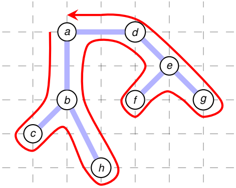

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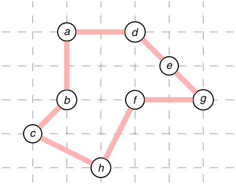

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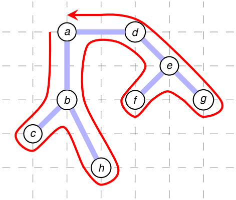
- Deleting duplicate vertices from W yields a tour H



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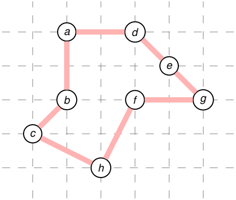

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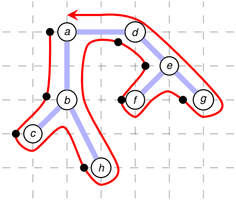

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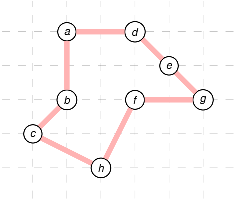

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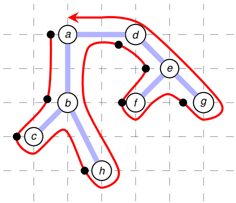

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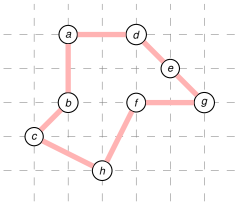

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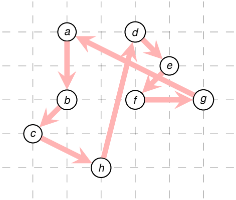
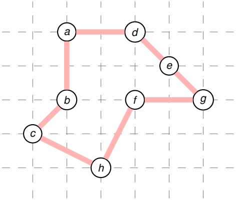


Tour $H = (a, b, c, h, d, e, f, g, a)$



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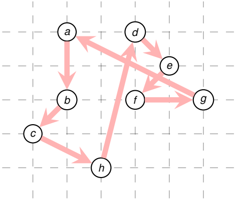
exploiting triangle inequality!

- Deleting duplicate vertices from W yields a tour H with smaller cost:



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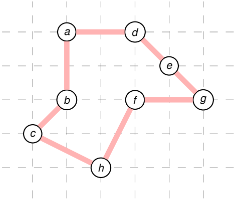

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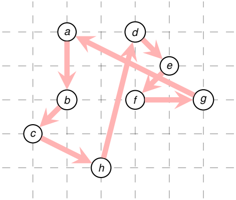
- Deleting duplicate vertices from W yields a tour H with smaller cost:

$$c(H) \leq c(W)$$



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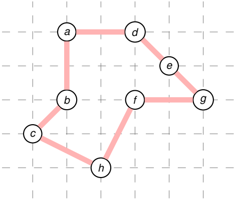

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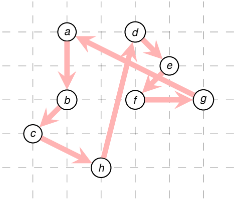

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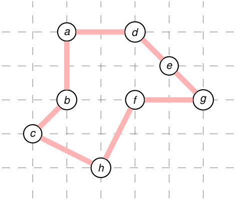

optimal solution H^*

Proof of the Approximation Ratio

Theorem 35.2

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

Proof:

- Consider the optimal tour H^* and remove an arbitrary edge

$\Rightarrow$ yields a spanning tree T and $c(T_{\min}) \leq c(T) \leq c(H^*)$

- Let W be the full walk of the minimum spanning tree $T_{\min}$ (including repeated visits)

$\Rightarrow$ Full walk traverses every edge exactly twice, so

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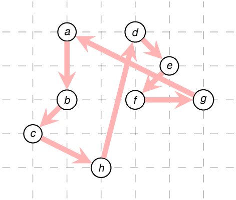

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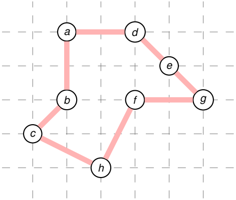

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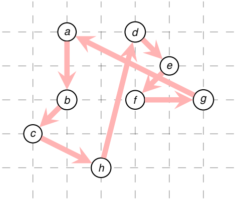

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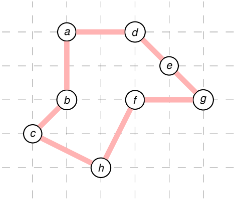

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Can we get a better approximation ratio?

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```
CHRISTOFIDES(G, c)
select a vertex r ∈ G.V to be a "root" vertex
compute a minimum spanning tree T_min for G from root r
        using MST-PRIM(G, c, r)
compute a perfect matching M_min with minimum weight in the complete graph
        over the odd-degree vertices in T_min
let H be a list of vertices, ordered according to when they are first visited
        in a Eulearian circuit of T_min ∪ M_min
return the hamiltonian cycle H
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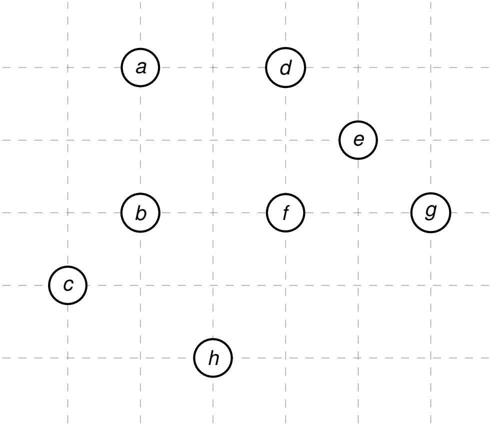
Theorem (Christofides'76)

There is a polynomial-time $\frac{3}{2}$-approximation algorithm for the travelling salesman problem with the triangle inequality.

Run of CHRISTOFIDES



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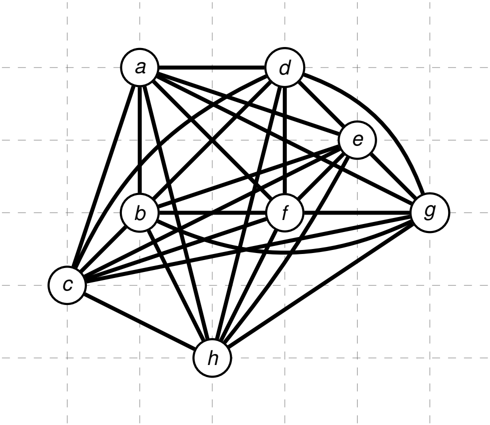

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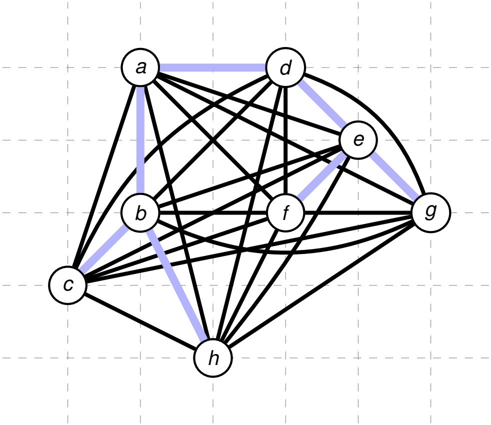
Run of CHRISTOFIDES



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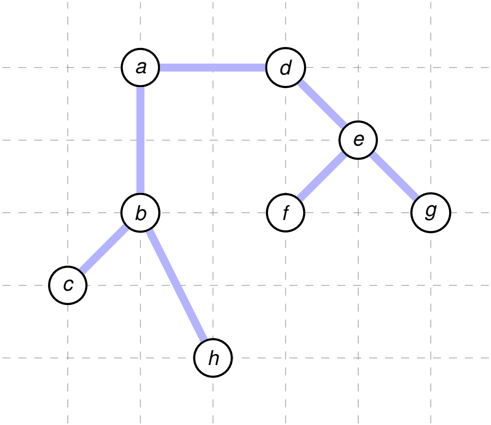

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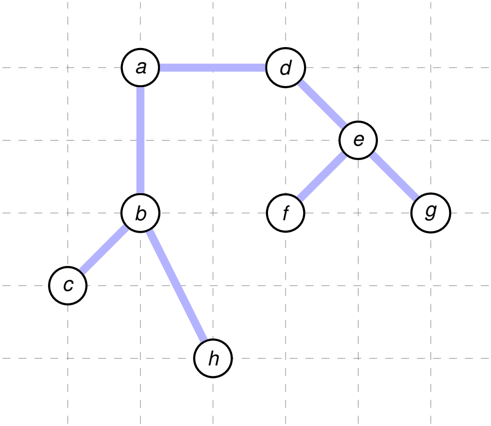

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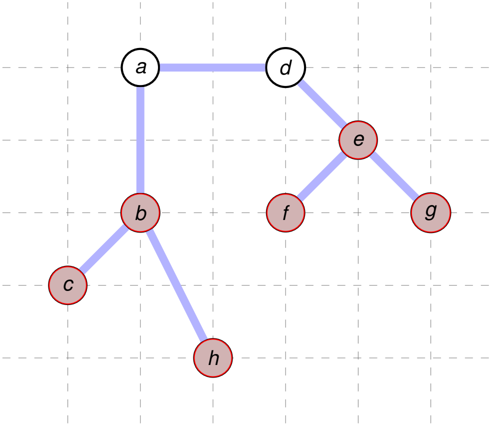

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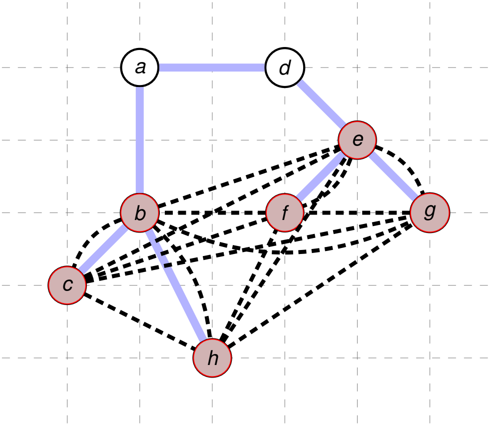

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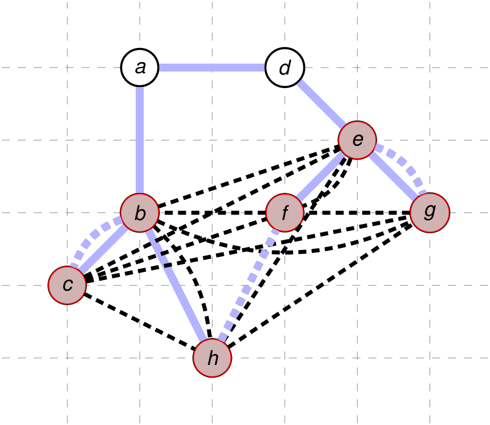

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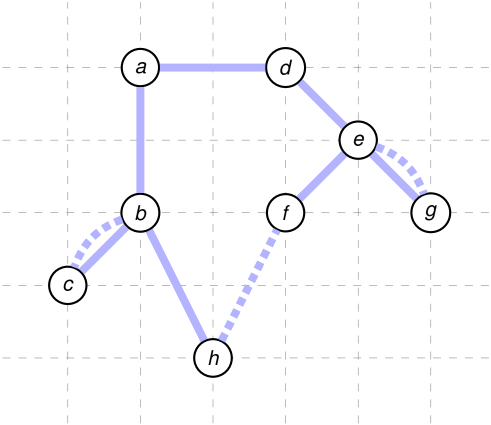

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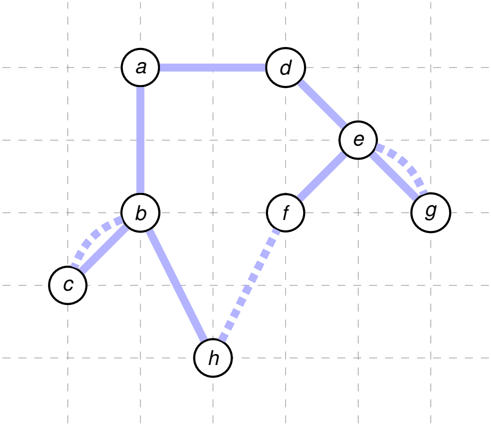

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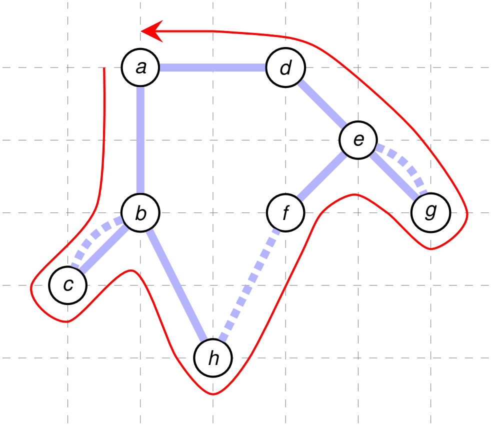
All vertices in $T_{\text{min}} \cup M_{\text{min}}$ have even degree!



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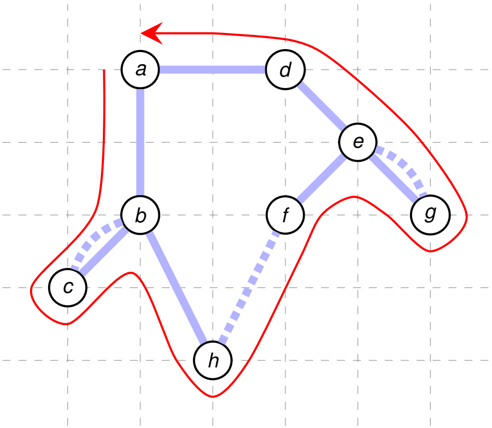

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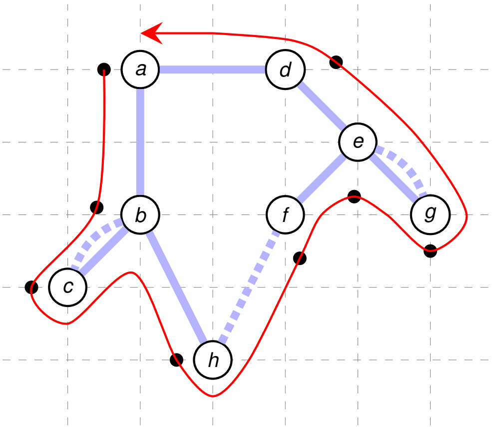

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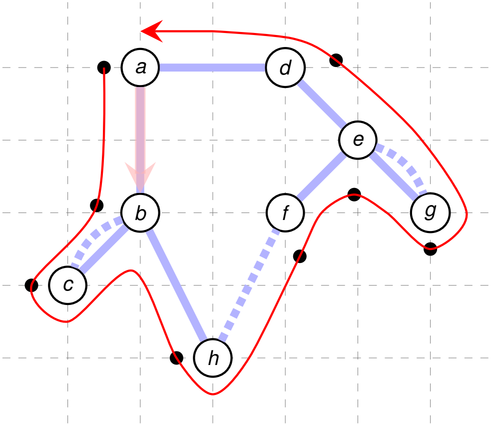

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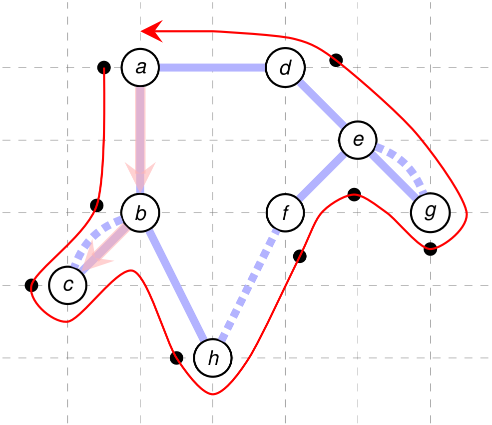

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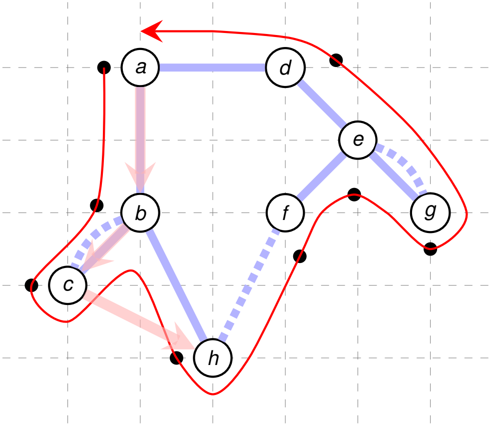

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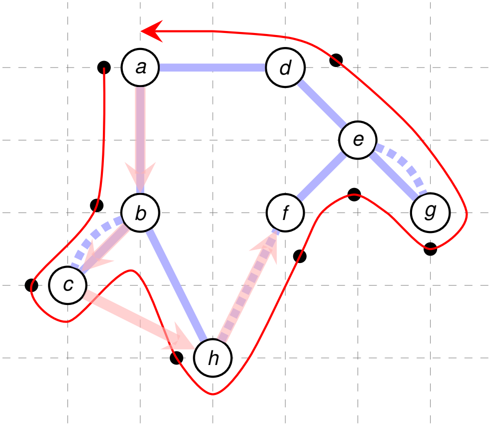

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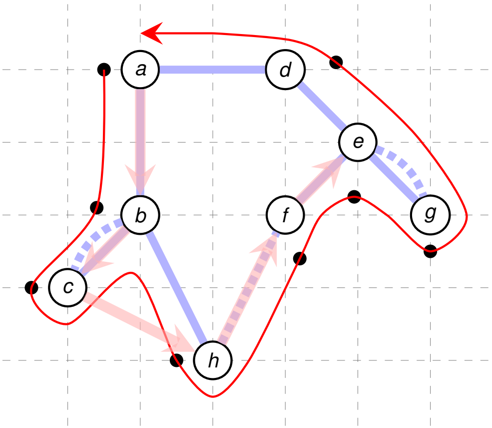

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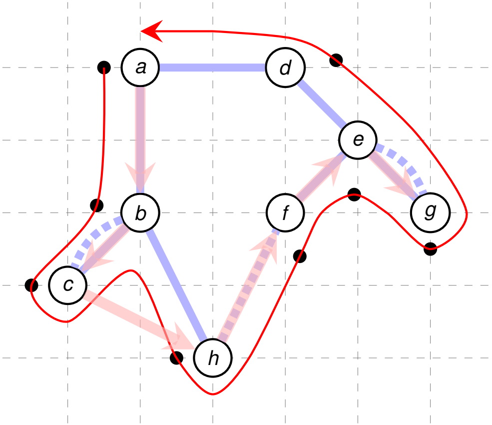

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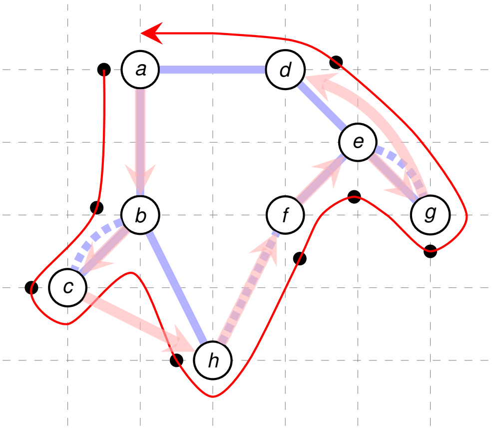

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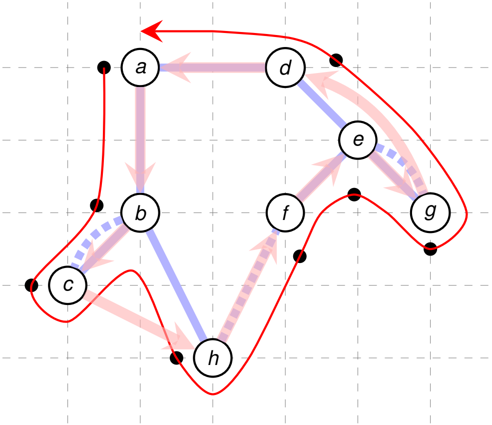

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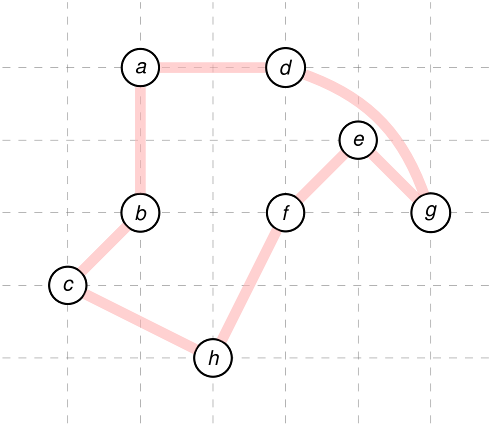

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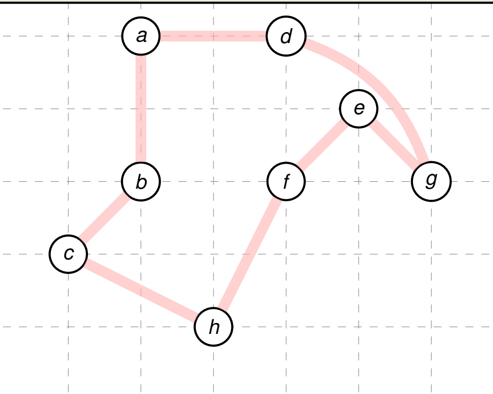
Solution has cost ≈ 15.54 - within 10% of the optimum!



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Theorem (Christofides'76)

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Theorem (Arora'96, Mitchell'96)

There is a PTAS for the Euclidean TSP Problem.

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Jon Bentley 1991

Concluding Remarks

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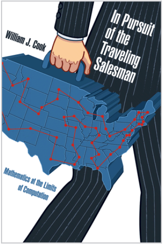

Exercise: Prove that the approximation ratio of APPROX-TSP-TOUR satisfies $\rho(n) < 2$.

Hint: Consider the effect of the shortcutting, but note that edge costs might be zero!