# IV. Approximation Algorithms via Exact Algorithms

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Easter 2021

UNIVERSITY OF CAMBRIDGE

# Outline

The Subset-Sum Problem

Parallel Machine Scheduling

Bonus Material: A PTAS for Parallel Machine Scheduling (non-examinable)

# The Subset-Sum Problem

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- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer $t$
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\ x_i \in S'} x_i \leq t$.

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This problem is NP-hard

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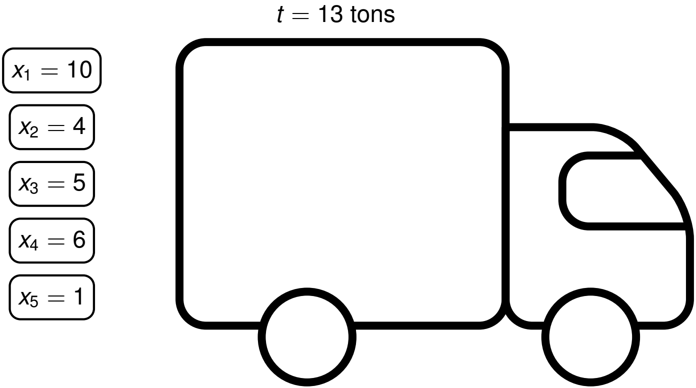

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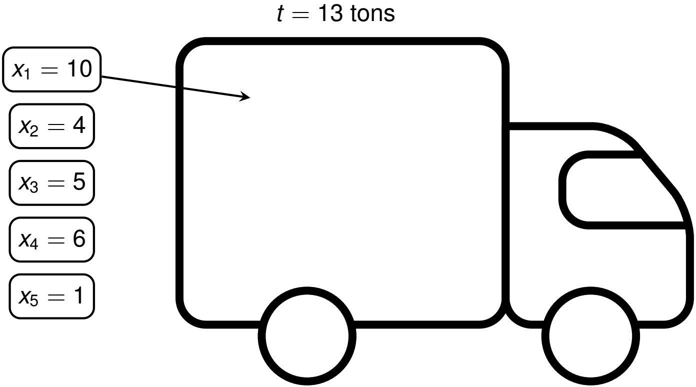

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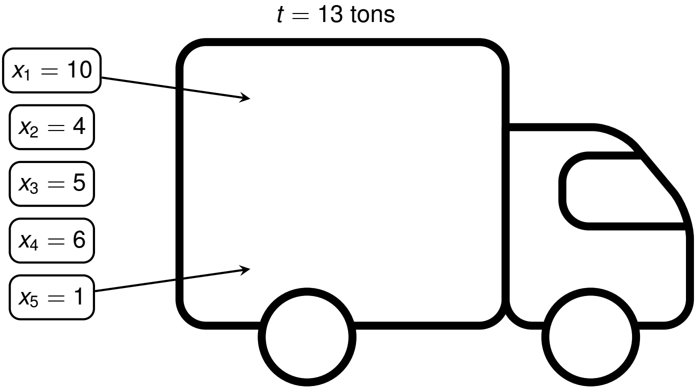

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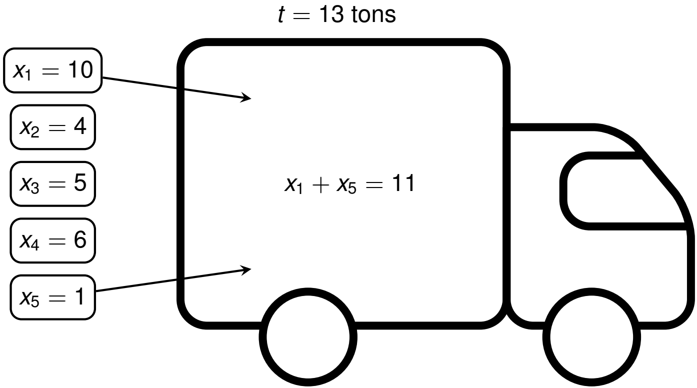

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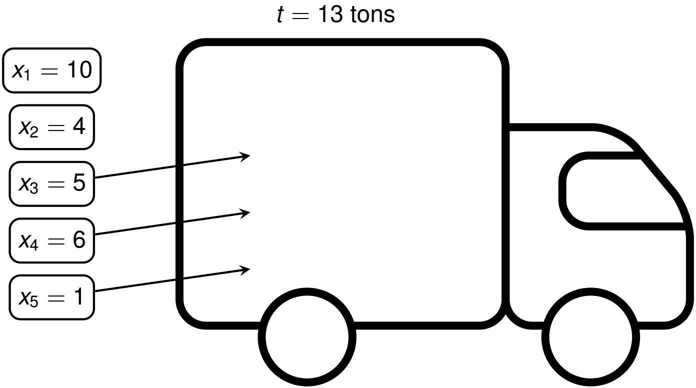

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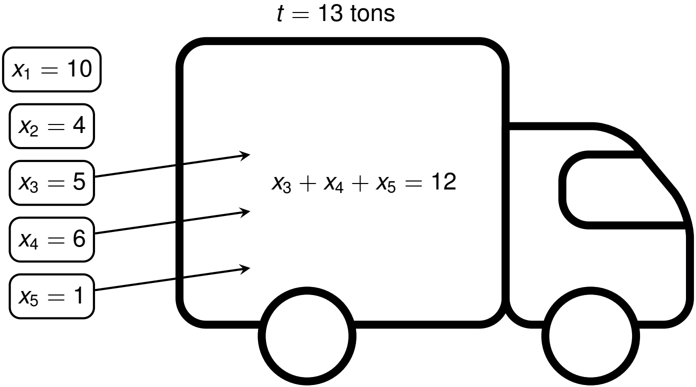

# An Exact (Exponential-Time) Algorithm

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implementable in time $O(|L_{i-1}|)$ (like Merge-Sort)

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can be shown by induction on $n$

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Better runtime if $t$ and/or $|L_i|$ are small.

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let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ to $m$
    if $y_i > last \cdot (1+\delta)$        // $y_i \geq last$ because $L$ is sorted
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TRIM works in time $\Theta(m)$, if $L$ is given in sorted order.

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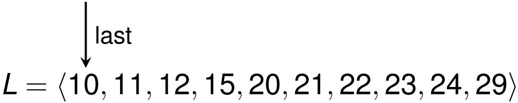

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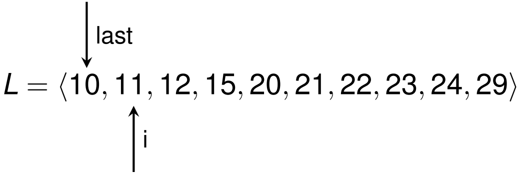

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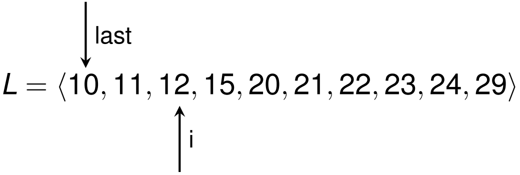

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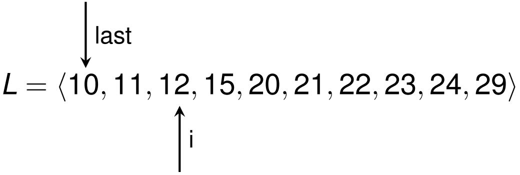

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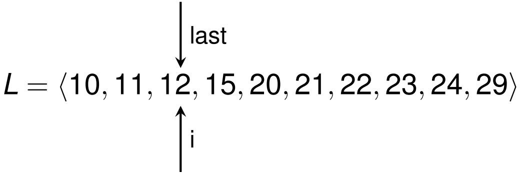


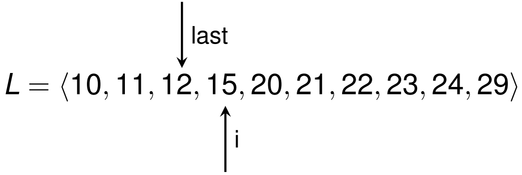
$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

$L' = \langle 10, 12 \rangle$

# Illustration of the Trim Operation

```
TRIM($L, \delta$)
let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ to $m$
    if $y_i > last \cdot (1 + \delta)$        // $y_i \geq last$ because $L$ is sorted
        append $y_i$ onto the end of $L'$
        $last = y_i$
return $L'$
```

$\delta = 0.1$



$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

$L' = \langle 10, 12 \rangle$

# Illustration of the Trim Operation

```
TRIM($L, \delta$)
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        append $y_i$ onto the end of $L'$
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return $L'$
```
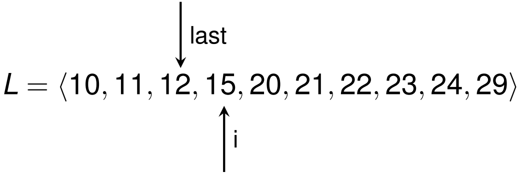

$\delta = 0.1$



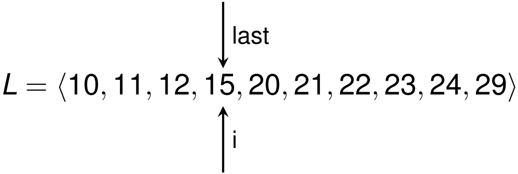
$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

$L' = \langle 10, 12, 15 \rangle$

# Illustration of the Trim Operation

```
TRIM$(L, \delta)$
let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ to $m$
    if $y_i > last \cdot (1 + \delta)$        // $y_i \geq last$ because $L$ is sorted
        append $y_i$ onto the end of $L'$
        $last = y_i$
return $L'$
```

$\delta = 0.1$



last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

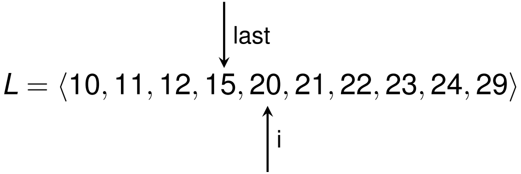
i

$L' = \langle 10, 12, 15 \rangle$

# Illustration of the Trim Operation

```
TRIM($L, \delta$)
let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ to $m$
    if $y_i > last \cdot (1 + \delta)$        // $y_i \geq last$ because $L$ is sorted
        append $y_i$ onto the end of $L'$
        $last = y_i$
return $L'$
```

$$\delta = 0.1$$



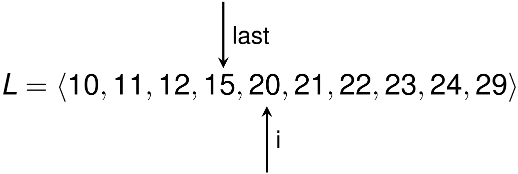
$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

$$L' = \langle 10, 12, 15 \rangle$$

# Illustration of the Trim Operation

```
TRIM($L, \delta$)
let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ to $m$
    if $y_i > last \cdot (1 + \delta)$        // $y_i \geq last$ because $L$ is sorted
        append $y_i$ onto the end of $L'$
        $last = y_i$
return $L'$
```

$\delta = 0.1$



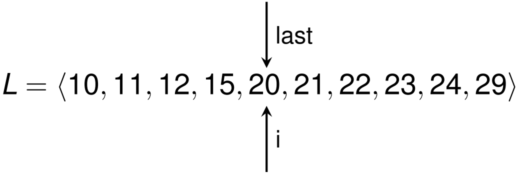
$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

$L' = \langle 10, 12, 15, 20 \rangle$

# Illustration of the Trim Operation

```
$\textsc{Trim}(L, \delta)$
let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ to $m$
    if $y_i > last \cdot (1 + \delta)$        // $y_i \geq last$ because $L$ is sorted
        append $y_i$ onto the end of $L'$
        $last = y_i$
return $L'$
```

$\delta = 0.1$



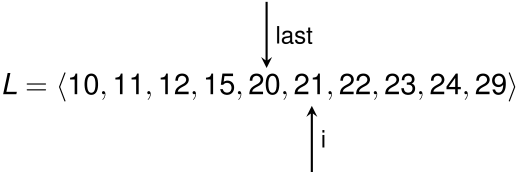
$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

$L' = \langle 10, 12, 15, 20 \rangle$

# Illustration of the Trim Operation

```
TRIM($L, \delta$)
let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ to $m$
    if $y_i > last \cdot (1 + \delta)$        // $y_i \geq last$ because $L$ is sorted
        append $y_i$ onto the end of $L'$
        $last = y_i$
return $L'$
```

$$\delta = 0.1$$



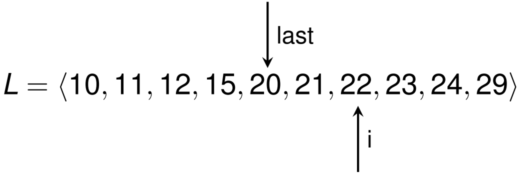
$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

$$L' = \langle 10, 12, 15, 20 \rangle$$

# Illustration of the Trim Operation

```
TRIM($L, \delta$)
let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ to $m$
    if $y_i > last \cdot (1 + \delta)$        // $y_i \geq last$ because $L$ is sorted
        append $y_i$ onto the end of $L'$
        $last = y_i$
return $L'$
```

$$\delta = 0.1$$



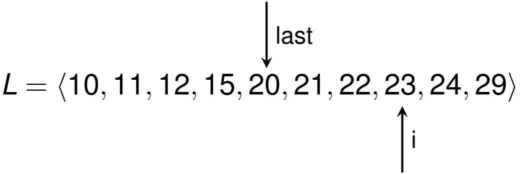
$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

$$L' = \langle 10, 12, 15, 20 \rangle$$

# Illustration of the Trim Operation

```
TRIM($L, \delta$)
let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ to $m$
    if $y_i > last \cdot (1 + \delta)$        // $y_i \geq last$ because $L$ is sorted
        append $y_i$ onto the end of $L'$
        $last = y_i$
return $L'$
```

$$\delta = 0.1$$



last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20 \rangle$$

# Illustration of the Trim Operation

```
TRIM($L, \delta$)
let $m$ be the length of $L$
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ to $m$
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        append $y_i$ onto the end of $L'$
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```
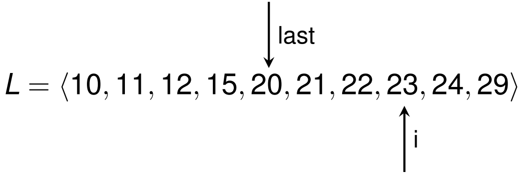

$\delta = 0.1$



$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

$L' = \langle 10, 12, 15, 20, 23 \rangle$

# Illustration of the Trim Operation

```
TRIM($L, \delta$)
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```
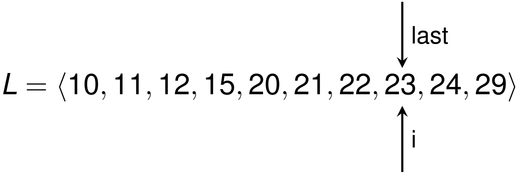

$$\delta = 0.1$$



$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

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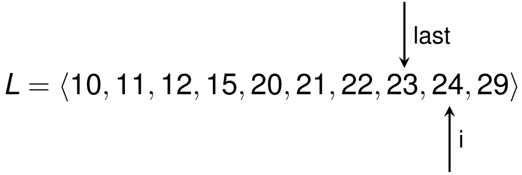
$$\delta = 0.1$$



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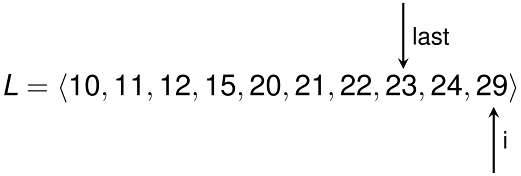
$\delta = 0.1$



last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15, 20, 23 \rangle$

# Illustration of the Trim Operation

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TRIM($L, \delta$)
let $m$ be the length of $L$
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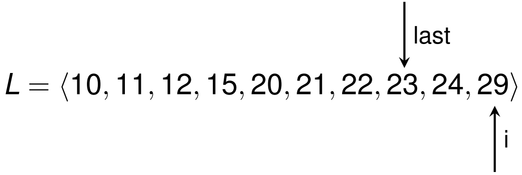

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

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$$L' = \langle 10, 12, 15, 20, 23, 29 \rangle$$

# Illustration of the Trim Operation

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```

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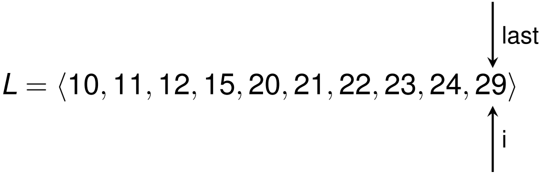

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

$$L' = \langle 10, 12, 15, 20, 23, 29 \rangle$$

# The FPTAS

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    $L_i = $ TRIM$(L_i, \epsilon/2n)$
    remove from $L_i$ every element that is greater than $t$
let $z^*$ be the largest value in $L_n$
return $z^*$
```

# The FPTAS

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    remove from $L_i$ every element that is greater than $t$
let $z^*$ be the largest value in $L_n$
return $z^*$
```

```
EXACT-SUBSET-SUM$(S, t)$
$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    remove from $L_i$ every element that is greater than $t$
return the largest element in $L_n$
```

# The FPTAS

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    $L_i = $ TRIM$(L_i, \epsilon/2n)$
    remove from $L_i$ every element that is greater than $t$
let $z^*$ be the largest value in $L_n$
return $z^*$
```

Repeated application of TRIM to make sure $L_i$'s remain short.

```
EXACT-SUBSET-SUM$(S, t)$
$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    remove from $L_i$ every element that is greater than $t$
return the largest element in $L_n$
```

# The FPTAS

```
APPROX-SUBSET-SUM($S, t, \epsilon$)
$n = |S|$
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return the largest element in $L_n$
```

- We must bound the inaccuracy introduced by repeated trimming

# The FPTAS

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
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$L_0 = \langle 0 \rangle$
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    $L_i$ = MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
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return $z^*$
```

Repeated application of TRIM to make sure $L_i$'s remain short.

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EXACT-SUBSET-SUM$(S, t)$
$n = |S|$
$L_0 = \langle 0 \rangle$
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    $L_i$ = MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    remove from $L_i$ every element that is greater than $t$
return the largest element in $L_n$
```

- We must bound the inaccuracy introduced by repeated trimming
- We must show that the algorithm is polynomial time

# The FPTAS

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
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    remove from $L_i$ every element that is greater than $t$
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return $z^*$
```

Repeated application of TRIM to make sure $L_i$'s remain short.

```
EXACT-SUBSET-SUM$(S, t)$
$n = |S|$
$L_0 = \langle 0 \rangle$
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    remove from $L_i$ every element that is greater than $t$
return the largest element in $L_n$
```

- We must bound the inaccuracy introduced by repeated trimming
- We must show that the algorithm is polynomial time

Solution is a careful choice of $\delta$!

# Running through an Example (CLRS3)

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i$ = MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    $L_i$ = TRIM$(L_i, \epsilon/2n)$
    remove from $L_i$ every element that is greater than $t$
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return $z^*$
```

# Running through an Example (CLRS3)

```
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$n = |S|$
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for $i = 1$ to $n$
    $L_i =$ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    $L_i =$ TRIM$(L_i, \epsilon/2n)$
    remove from $L_i$ every element that is greater than $t$
let $z^*$ be the largest value in $L_n$
return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

# Running through an Example (CLRS3)

```
APPROX-SUBSET-SUM($S, t, \epsilon$)
$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ to $n$
    $L_i$ = MERGE-LISTS($L_{i-1}, L_{i-1} + x_i$)
    $L_i$ = TRIM($L_i, \epsilon/2n$)
    remove from $L_i$ every element that is greater than $t$
let $z^*$ be the largest value in $L_n$
return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

⇒ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

# Running through an Example (CLRS3)

```
APPROX-SUBSET-SUM($S, t, \epsilon$)
1  $n = |S|$
2  $L_0 = \langle 0 \rangle$
3  for $i = 1$ to $n$
4      $L_i$ = MERGE-LISTS($L_{i-1}, L_{i-1} + x_i$)
5      $L_i$ = TRIM($L_i, \epsilon/2n$)
6      remove from $L_i$ every element that is greater than $t$
7  let $z^*$ be the largest value in $L_n$
8  return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

⇒ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$

# Running through an Example (CLRS3)

```
APPROX-SUBSET-SUM$(S, t, \epsilon)$
1  $n = |S|$
2  $L_0 = \langle 0 \rangle$
3  for $i = 1$ to $n$
4      $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
5      $L_i = $ TRIM$(L_i, \epsilon/2n)$
6      remove from $L_i$ every element that is greater than $t$
7  let $z^*$ be the largest value in $L_n$
8  return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

⇒ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$

# Running through an Example (CLRS3)

```
APPROX-SUBSET-SUM($S, t, \epsilon$)
1  $n = |S|$
2  $L_0 = \langle 0 \rangle$
3  for $i = 1$ to $n$
4      $L_i$ = MERGE-LISTS($L_{i-1}, L_{i-1} + x_i$)
5      $L_i$ = TRIM($L_i, \epsilon/2n$)
6      remove from $L_i$ every element that is greater than $t$
7  let $z^*$ be the largest value in $L_n$
8  return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

⇒ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$

# Running through an Example (CLRS3)

```
APPROX-SUBSET-SUM($S, t, \epsilon$)
1  $n = |S|$
2  $L_0 = \langle 0 \rangle$
3  for $i = 1$ to $n$
4      $L_i$ = MERGE-LISTS($L_{i-1}, L_{i-1} + x_i$)
5      $L_i$ = TRIM($L_i, \epsilon/2n$)
6      remove from $L_i$ every element that is greater than $t$
7  let $z^*$ be the largest value in $L_n$
8  return $z^*$
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

⇒ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

# Running through an Example (CLRS3)

```
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- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

⇒ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

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Returned solution $z^* = 302$, which is 2% within the optimum $307 = 104 + 102 + 101$

# Reminder: Performance Ratios for Approximation Algorithms

Approximation Ratio

An algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size $n$, the cost $C$ of the returned solution and optimal cost $C^*$ satisfy:

$$\max\left(\frac{C}{C^*}, \frac{C^*}{C}\right) \leq \rho(n).$$

For many problems: tradeoff between runtime and approximation ratio.

Approximation Schemes

An approximation scheme is an approximation algorithm, which given any input and $\epsilon > 0$, is a $(1+\epsilon)$-approximation algorithm.

- It is a polynomial-time approximation scheme (PTAS) if for any fixed $\epsilon > 0$, the runtime is polynomial in $n$. For example, $O(n^{2/\epsilon})$.
- It is a fully polynomial-time approximation scheme (FPTAS) if the runtime is polynomial in both $1/\epsilon$ and $n$. For example, $O((1/\epsilon)^2 \cdot n^3)$.

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$$\leq 1+\epsilon/2+(\epsilon/2)^2 \leq 1+\epsilon$$

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Need $\log(t)$ bits to represent $t$ and $n$ bits to represent $S$

# Concluding Remarks

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer $t$
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Algorithm very similar to APPROX-SUBSET-SUM

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# Outline

The Subset-Sum Problem

Parallel Machine Scheduling

Bonus Material: A PTAS for Parallel Machine Scheduling (non-examinable)

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Machine Scheduling Problem

- Given: $n$ jobs $J_1, J_2, \ldots, J_n$ with processing times $p_1, p_2, \ldots, p_n$, and $m$ identical machines $M_1, M_2, \ldots, M_m$

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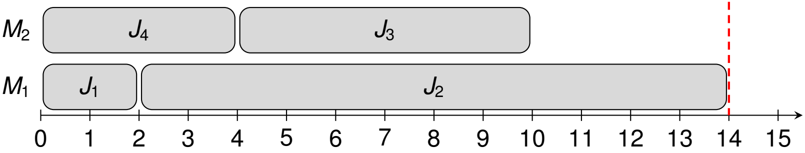

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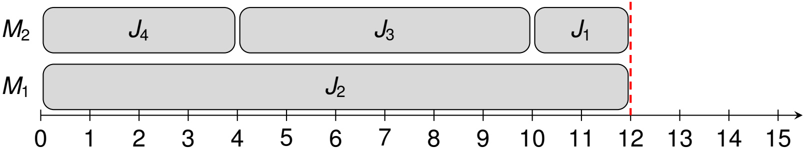

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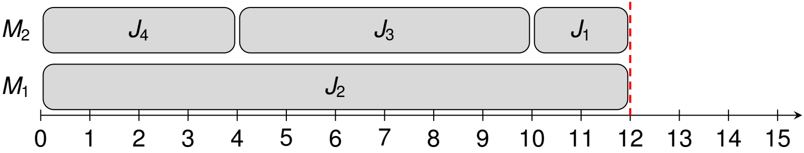
For the analysis, it will be convenient to denote by $C_i$ the completion time of a machine $i$.



# NP-Completeness of Parallel Machine Scheduling

**Lemma**

Parallel Machine Scheduling is NP-complete even if there are only two machines.

Proof Idea: Polynomial time reduction from NUMBER-PARTITIONING.

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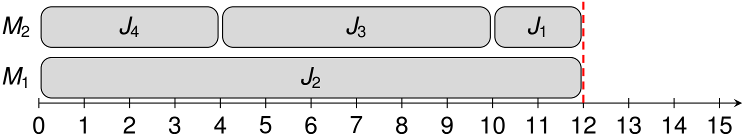

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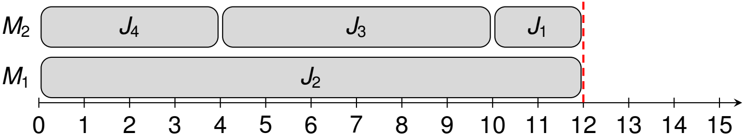

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LIST SCHEDULING($J_1, J_2, \ldots, J_n, m$)
while there exists an unassigned job
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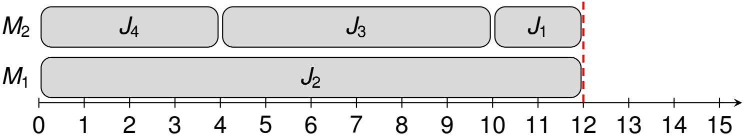

Equivalent to the following Online Algorithm [CLRS3]:
Whenever a machine is idle, schedule the next job on that machine.

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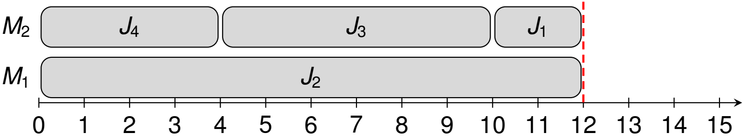

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How good is this most basic Greedy Approach?

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Ex 35-5 a.&b.

a. The optimal makespan is at least as large as the greatest processing time, that is,

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$\Rightarrow$ One machine must have a load of at least $\frac{1}{m} \cdot \sum_{k=1}^{n} p_k$ $\square$

# List Scheduling Analysis (Final Step)

Ex 35-5 d. (Graham 1966)

For the schedule returned by the greedy algorithm it holds that

$$C_{\max} \leq \frac{1}{m}\sum_{k=1}^{n} p_k + \max_{1\leq k\leq n} p_k.$$

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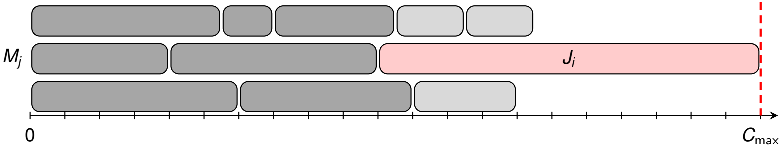

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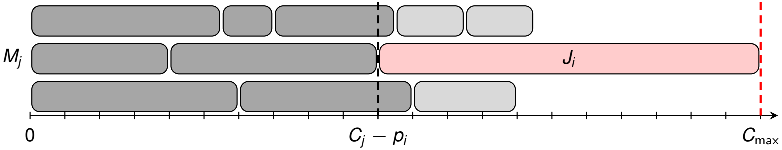

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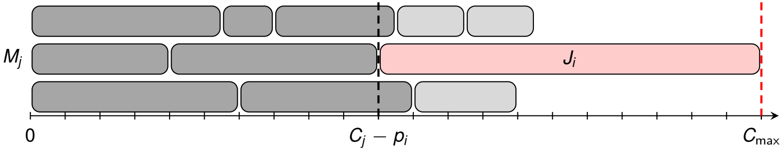

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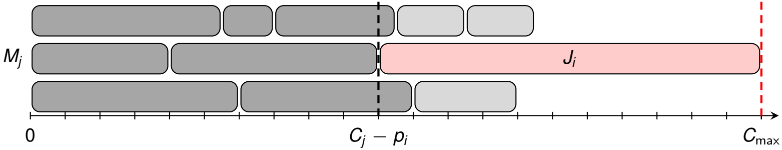

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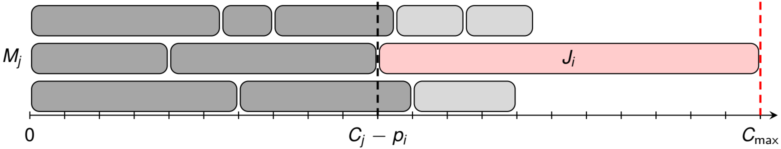

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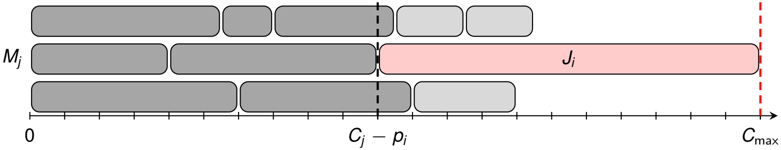

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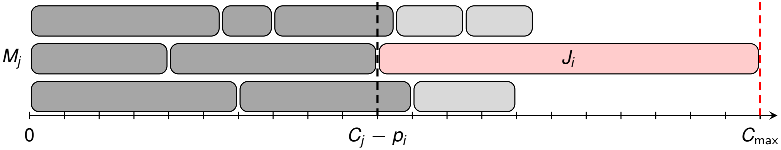


# List Scheduling Analysis (Final Step)

Ex 35-5 d. (Graham 1966)

For the schedule returned by the greedy algorithm it holds that

$$C_{\max} \leq \frac{1}{m} \sum_{k=1}^{n} p_k + \max_{1 \leq k \leq n} p_k.$$

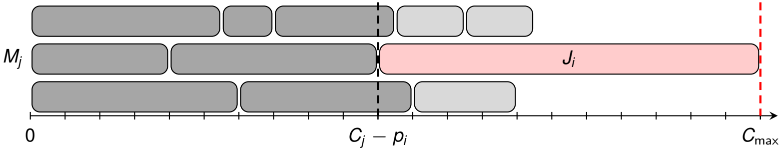
Hence list scheduling is a poly-time 2-approximation algorithm.

Proof:

- Let $J_i$ be the last job scheduled on machine $M_j$ with $C_{\max} = C_j$
- When $J_i$ was scheduled to machine $M_j$, $C_j - p_i \leq C_k$ for all $1 \leq k \leq m$
- Averaging over $k$ yields:

Using Ex 35-5 a. & b.

$$C_j - p_i \leq \frac{1}{m} \sum_{k=1}^{m} C_k = \frac{1}{m} \sum_{k=1}^{n} p_k \quad \Rightarrow \quad C_j \leq \frac{1}{m} \sum_{k=1}^{n} p_k + \max_{1 \leq k \leq n} p_k$$



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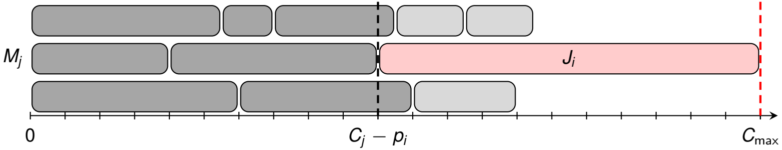

# Improving Greedy

Analysis can be shown to be almost tight. Is there a better algorithm?

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The problem of the List-Scheduling Approach were the large jobs

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LEAST PROCESSING TIME($J_1, J_2, \ldots, J_n, m$)
Sort jobs decreasingly in their processing times
for $i = 1$ to $m$
    $C_i = 0$
    $S_i = \emptyset$
end for
for $j = 1$ to $n$
    $i = \text{argmin}_{1 \leq k \leq m} C_k$
    $S_i = S_i \cup \{j\}$, $C_i = C_i + p_j$
end for
return $S_1, \ldots, S_m$
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Runtime:

- $O(n \log n)$ for sorting
- $O(n \log m)$ for extracting (and re-inserting) the minimum (use priority queue).

# Analysis of Improved Greedy

**Graham 1966**

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

This can be shown to be tight (see next slide).

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Proof (of approximation ratio $3/2$).

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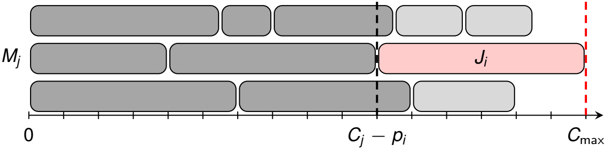

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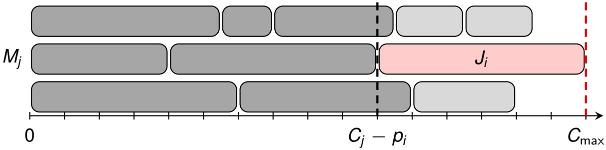
$$C_{\max} = C_j = (C_j - p_i) + p_i$$



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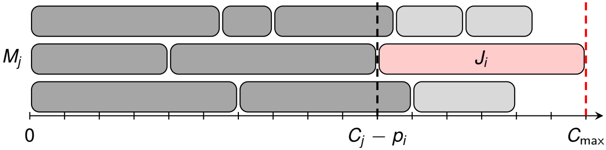
$$C_{\max} = C_j = (C_j - p_i) + p_i \leq C^*_{\max} + \frac{1}{2} C^*_{\max}$$

This is for the case $i \geq m+1$ (otherwise, an even stronger inequality holds)



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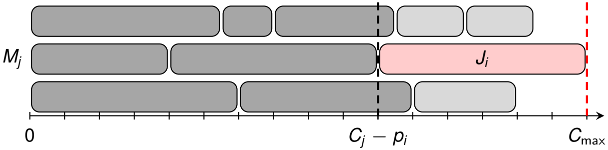

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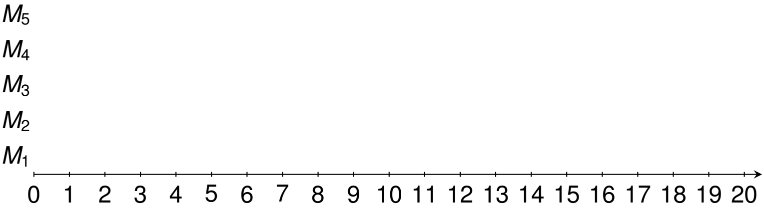

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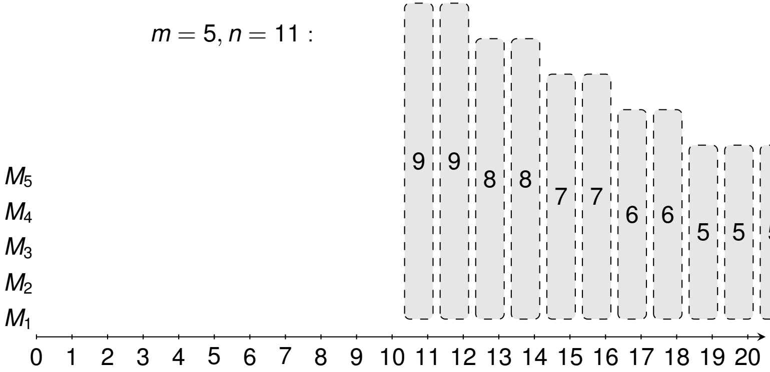

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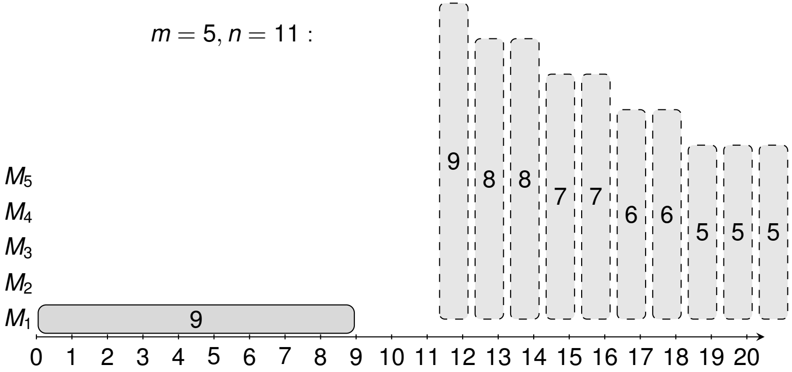

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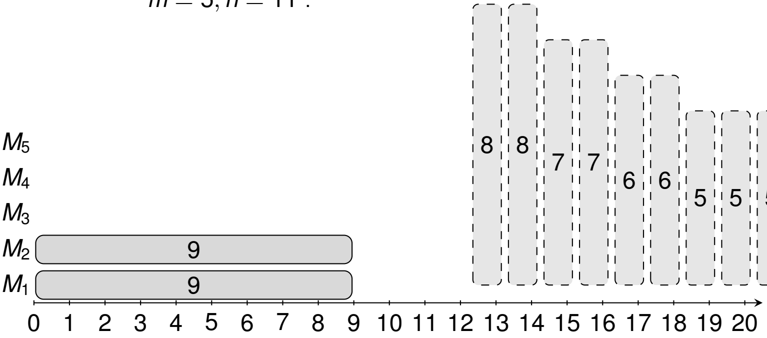

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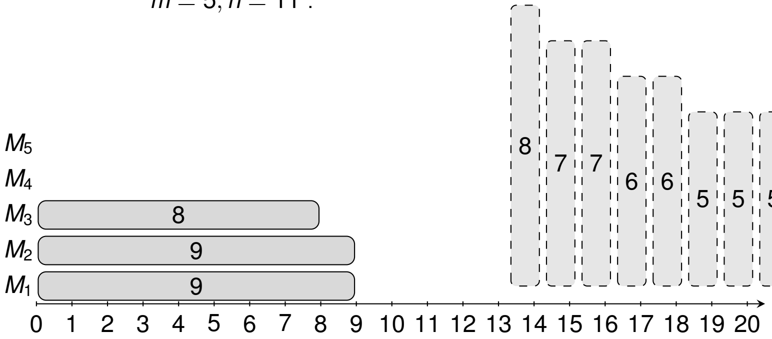

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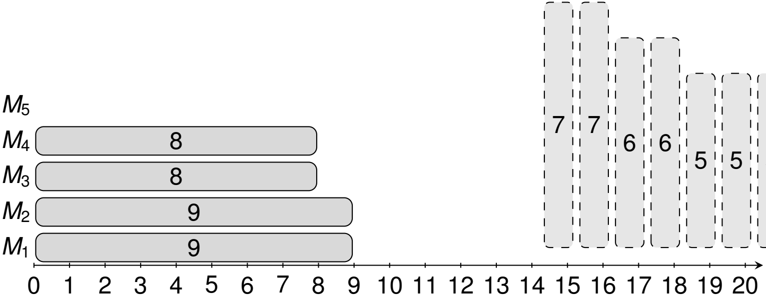

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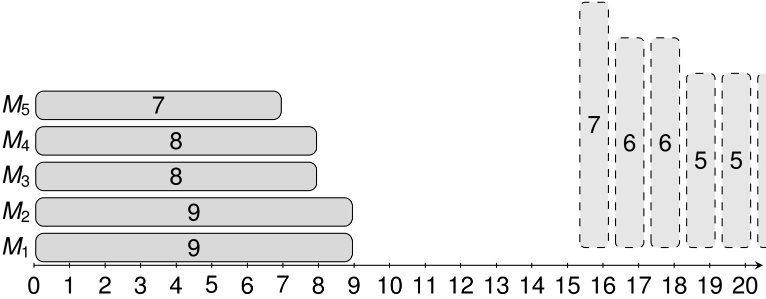

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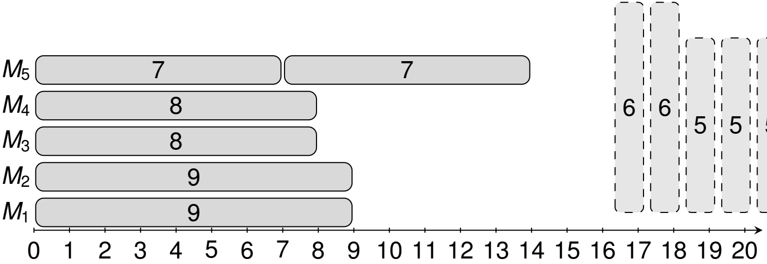

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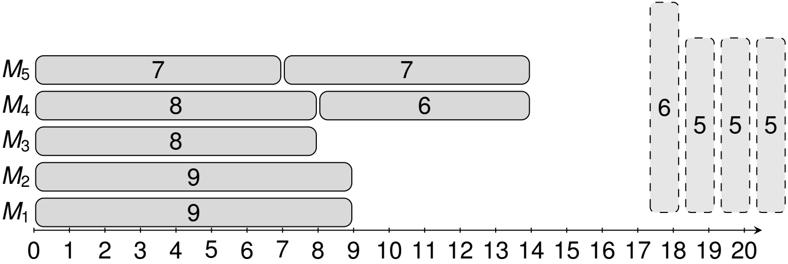

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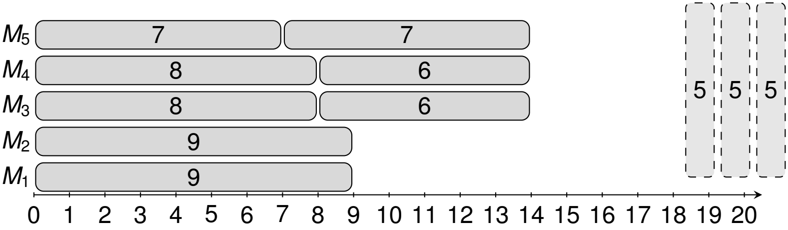

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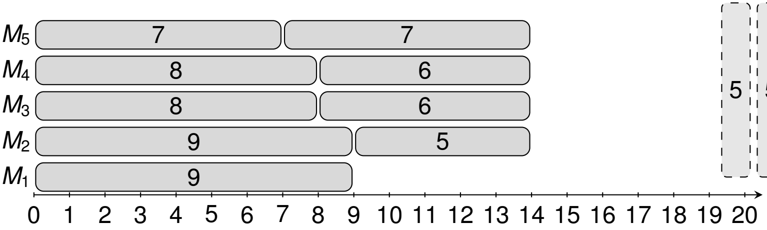

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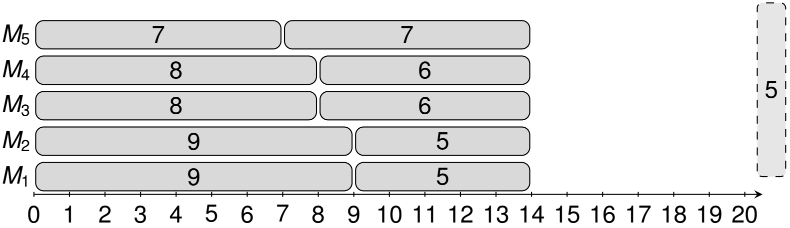

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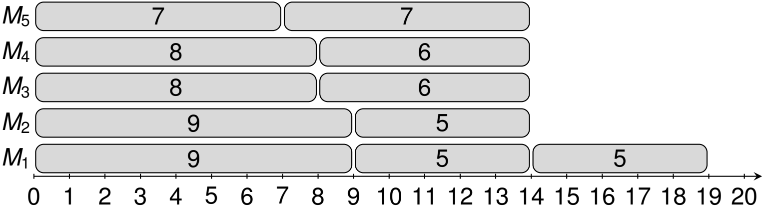

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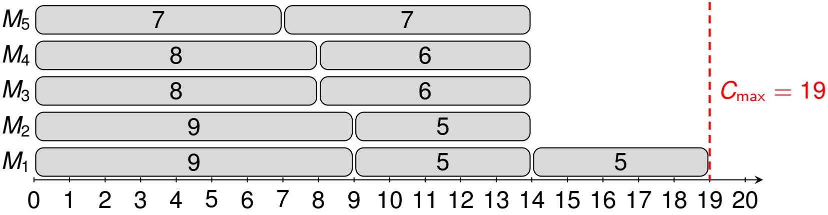

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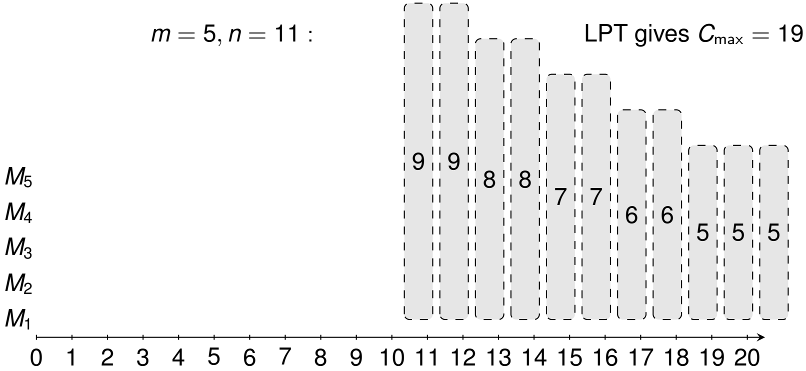

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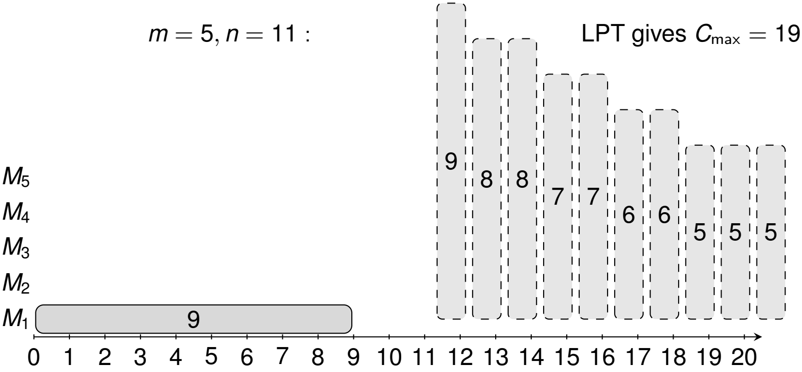

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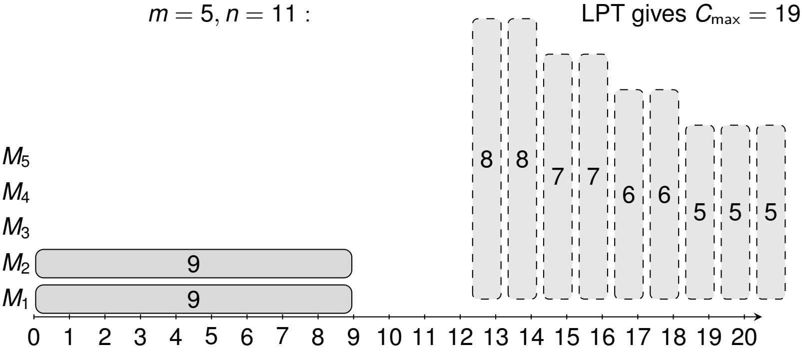

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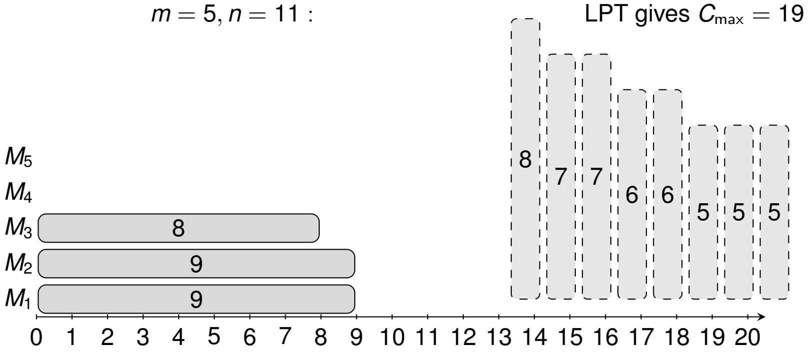

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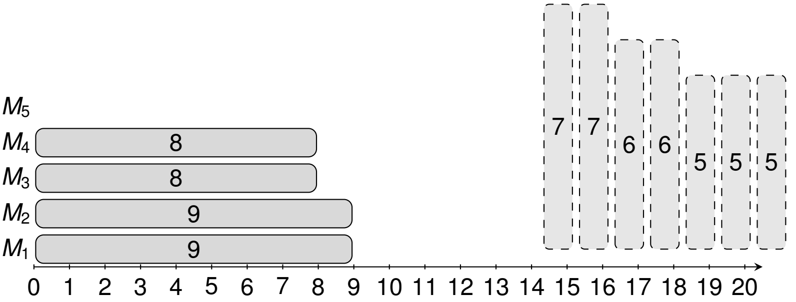

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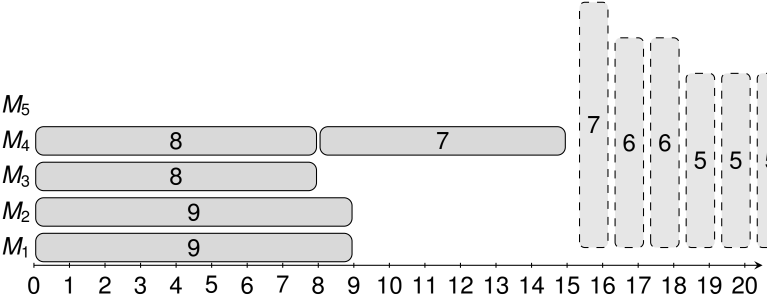

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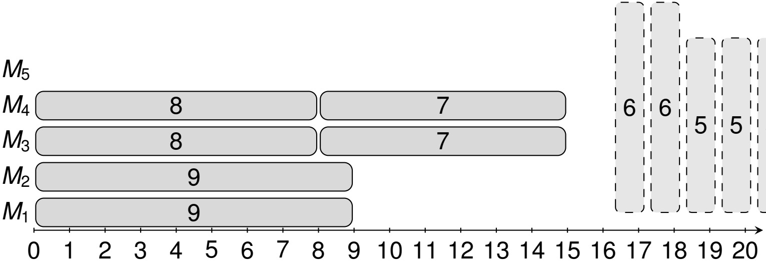

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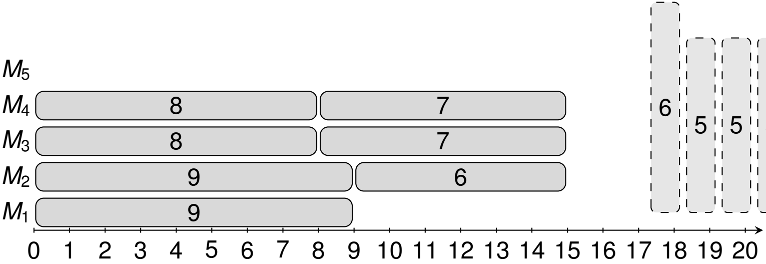
$m = 5, n = 11$ :

LPT gives $C_{\max} = 19$



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The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

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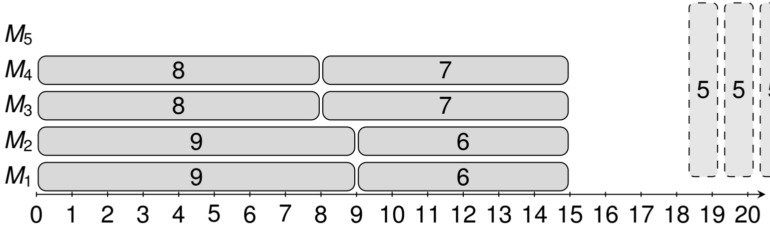

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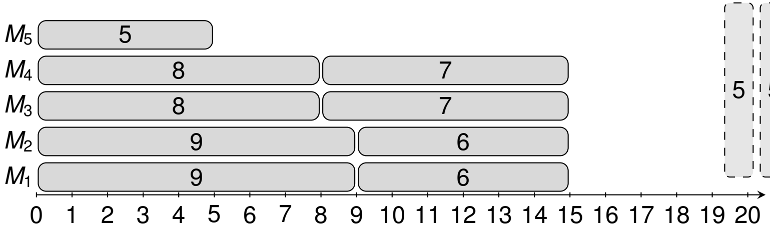

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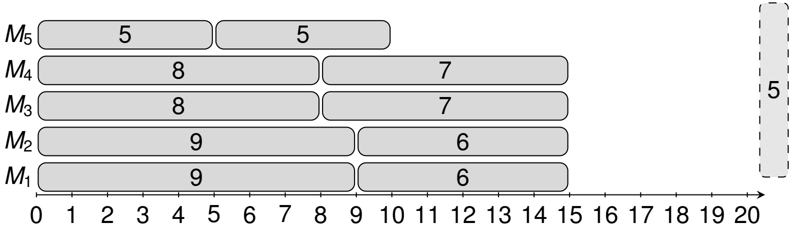

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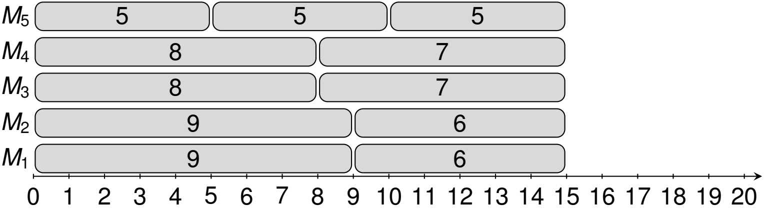

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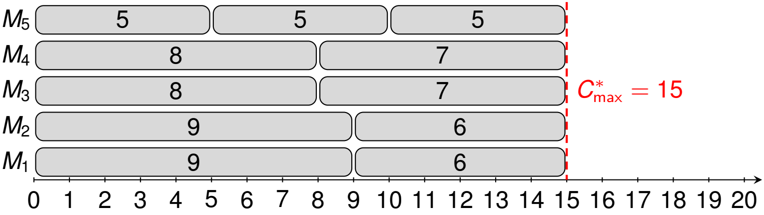

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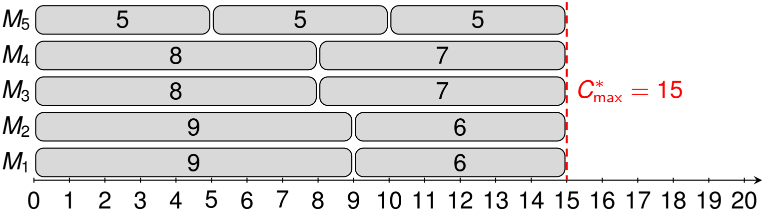

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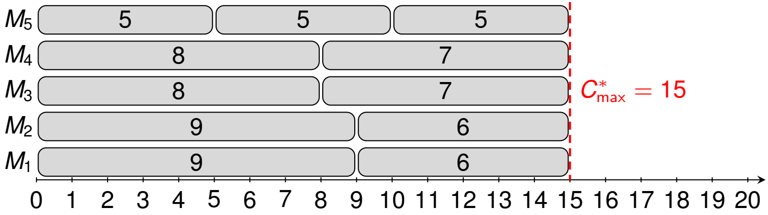

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List scheduling has an approximation ratio of 2.

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There exists a PTAS for Parallel Machine Scheduling which runs in time $O(n^{O(1/\epsilon^2)} \cdot \log P)$, where $P := \sum_{k=1}^{n} p_k$.

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Can we find a FPTAS (for polynomially bounded processing times)?
**No!**

Because for sufficiently small approximation ratio $1 + \epsilon$, the computed solution has to be optimal, and Parallel Machine Scheduling is strongly NP-hard.

**Exercise (easy):** Run the LPT algorithm on three machines and jobs having processing times $\{3, 4, 4, 3, 5, 3, 5\}$. Which allocation do you get?

1. $[3, 3, 5], [4, 5], [4, 3]$
2. $[5, 3], [5, 4], [4, 3, 3]$
3. $[3, 3, 3], [5, 4], [5, 4]$

# Outline

The Subset-Sum Problem

Parallel Machine Scheduling

Bonus Material: A PTAS for Parallel Machine Scheduling (non-examinable)

# A PTAS for Parallel Machine Scheduling

Basic Idea: For $(1+\epsilon)$-approximation, don’t have to work with exact $p_k$’s.

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$$C_j - p_i \leq \frac{1}{m} \sum_{k=1}^{n} p_k \quad \Rightarrow \quad C_j \leq p_i + \frac{1}{m} \sum_{k=1}^{n} p_k$$

the “well-known” formula

$$\leq \epsilon \cdot T + C^*_{\max}$$

$$\leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\} \quad \square$$

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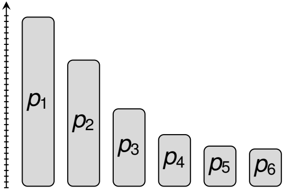

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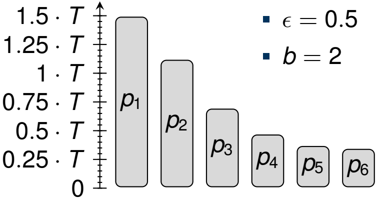

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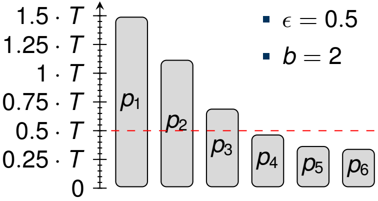

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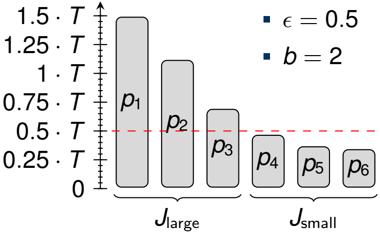

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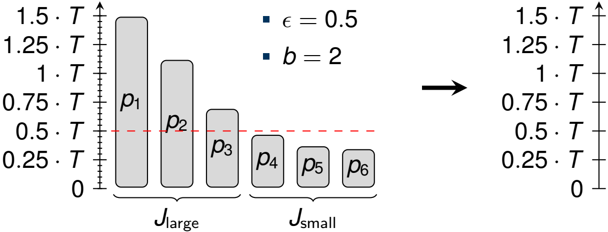

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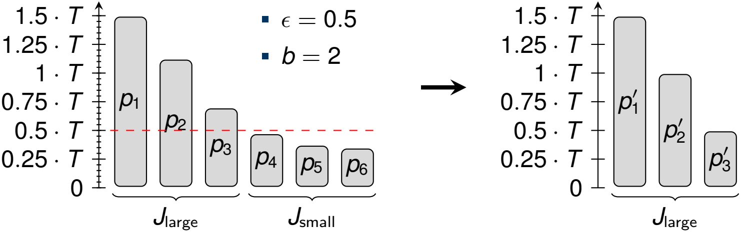

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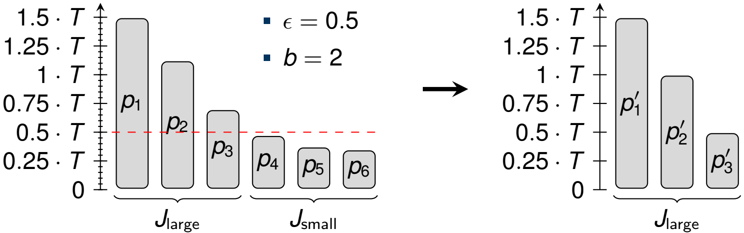
Can assume there are no jobs with $p_j \geq T$!



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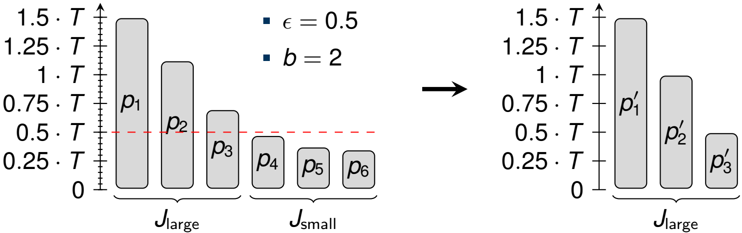

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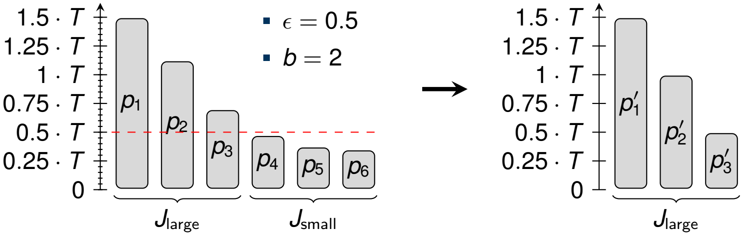
Assignments to one machine with makespan $\leq T$.



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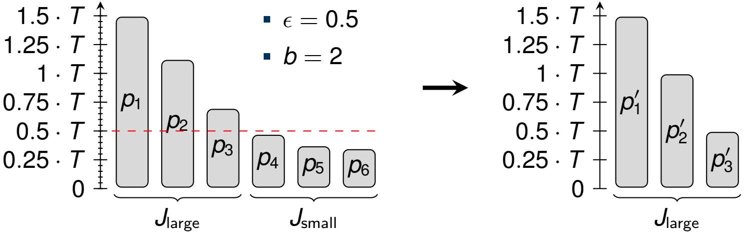

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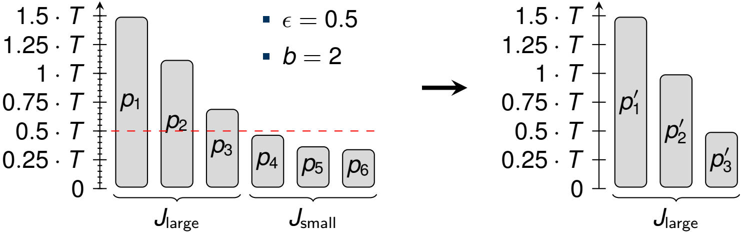

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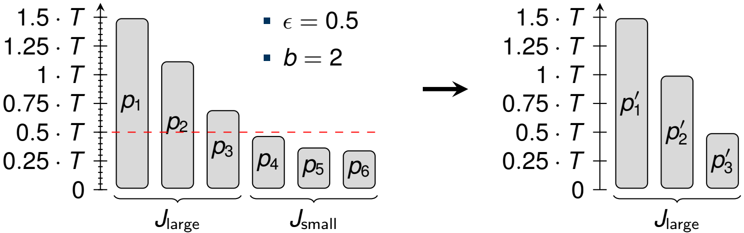

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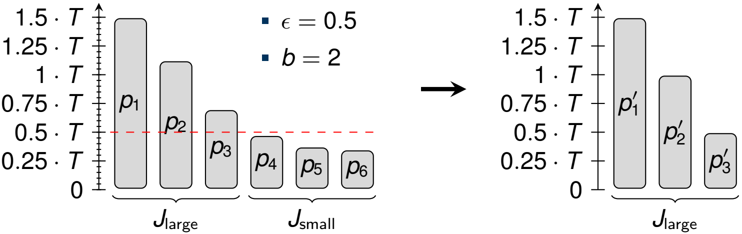
Assign some jobs to one machine, and then use as few machines as possible for the rest.



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$$C_{\max} \leq T + b \cdot \max_{i \in J_{\text{large}}} (p_i - p_i')$$

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$$\leq T + b \cdot \frac{T}{b^2} \leq (1+\epsilon) \cdot T. \qquad \square$$