1 Exercises Advanced Algorithms (Part II)

1.1 Sorting Networks, Counting Networks

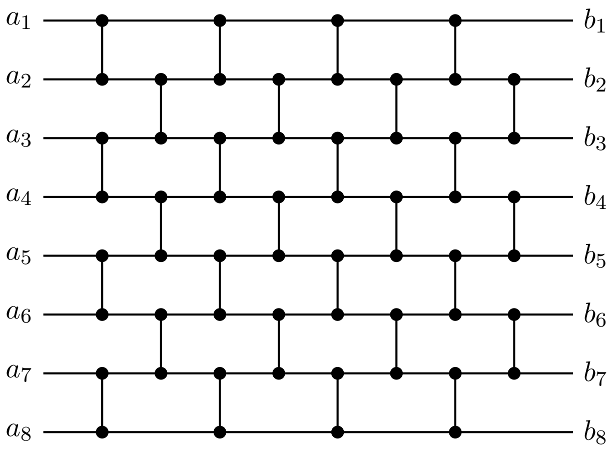
Question 1 (CLRS, Question 27.2-2)**.** Prove that a comparison network with n inputs correctly sorts the input sequence $\langle n, n-1, \ldots, 1\rangle$ if and only if it correctly sorts the $n-1$ zero-one sequences $\langle 1, 0, 0, \ldots, 0, 0\rangle$,$\langle 1, 1, 0, \ldots, 0, 0\rangle$,...,$\langle 1, 1, 1, \ldots, 1, 0\rangle$.

Question 2. [CLRS, Question 27.2-5] Prove that an n-input sorting network must contain at least one comparator between the i-th and $(i+1)$-st lines for all $i = 1, 2, \ldots, n-1$.

Question 3. [CLRS, Question 27.4-3] Show that any network that can merge 1 item with $n-1$ sorted items to produce a sorted sequence of length n must have depth at least $\log n$.

Question 4. How many binary bitonic sequences of length n are there?

Question 5. [CLRS, Problem 27-1] An **odd-even-sorting network** on n inputs $\langle a_1, a_2, \ldots, a_n\rangle$ is a transposition sorting network with n levels of comparators connected in the "brick-like" pattern illustrated below.



As can be seen in the figure, for $i = 1, 2, \ldots, n$ and $d = 1, 2, \ldots, n$, line i is connected by a depth-d comparator to line $j = i + (-1)^{i+d}$ if $1 \le j \le n$.

Prove that odd-even sorting networks actually sort.

Question 6. Prove that any sorting network must have depth $\Omega(\log n)$.

Question 7. Give a construction of a sorting network of depth $O(\log^2 n)$ that works even if n may not be a power of 2.

Question 8. Construct a network that is a sorting network but not a counting network.

Question 9. Prove that a perfect halver for n inputs must have depth $\Omega(\log n)$.

1.2 Matrix Multiplication and Multithreading

Question 10. Exam Question 2014 Paper 3 Question 1 (Algorithms II). **Warning: Parts of the solution may require Theorem 27.1 (Greedy-Scheduler-Theorem) in CLRS3, which is not covered in the lectures.**

Question 11. [CLRS, Question 4.2-7] Show how to multiply the complex numbers $a + bi$ and $c + di$ using only three multiplications of real numbers. The algorithm should take a, b, c, and d as input and produce the real component $ac - bd$ and the imaginary component $ad + bc$ separately.

Question 12. What can be said about the relation between the time complexity for multiplying two arbitrary square matrices A and B and the time complexity for multiplying a matrix C with itself?

Question 13. [CLRS, Question 4.2-2] Write pseudocode for Strassen's algorithm.

1.3 Linear Programming

Question 14. [CLRS: 29.1-5] Convert the following linear program into slack form:

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & & & - & 6x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& 3x_1 & - & x_2 & & & \geq & 8 \\
& -x_1 & + & 2x_2 & + & 2x_3 & \geq & 0 \\
& & & x_1, x_2, x_3 & & & \geq & 0
\end{array}
$$

What are the basic and non-basic variables?

Question 15. [CLRS: 29.1-6] Show that the following linear program is infeasible:

$$
\begin{array}{lrcrcr}
\text{maximize} & 3x_1 & - & 2x_2 & & \\
\text{subject to} & & & & & \\
& x_1 & + & x_2 & \leq & 2 \\
& -2x_1 & - & 2x_2 & \leq & -10 \\
& & x_1, x_2 & & \geq & 0
\end{array}
$$

Question 16. [CLRS: 29.1-7] Show that the following linear program is unbounded:

$$
\begin{array}{lrcrcr}
\text{maximize} & x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
& -2x_1 & + & x_2 & \leq & -1 \\
& -x_1 & - & 2x_2 & \leq & -2 \\
& & x_1, x_2 & & \geq & 0
\end{array}
$$

Question 17. [Thanks to the student for mentioning this question.] Consider the linear program for the minimum-weight shortest-path from s to t from the lecture notes (Slide 23 from III Linear Programming).

1. What happens if there exists a negative-weight cycle?
2. Prove that, if there are no negative-weight cycles, the optimal solution $\overline{d_t}$ of the linear program equals the correct distance d_t.
3. Find a counter-example in which the linear program does not compute all values d_v correctly. How would you formulate the single-source-shortest path problem as a linear program?

Question 18. Prove that the set of feasible solutions of a linear program in standard form forms a convex set.

Question 19. [Thanks to the student for mentioning this question (and answer).] Find a linear program which has at least one optimal solution that is not a vertex.

Question 20. [CLRS: 29.1-8] Suppose that we have a general linear program with n variables and m constraints, and suppose we convert it into standard form. Give an upper bound on the number of variables and constraints in the resulting linear program.

Question 21. [CLRS: 29.1-9] Give an example of a linear program for which the feasible region is not bounded, but the optimal objective value is finite.

Question 22. [CLRS: 29.2-5] Rewrite the linear program for maximum flow so that it uses only $O(V + E)$ constraints.

Question 23. [CLRS: 29.3-6] Solve the following linear program using SIMPLEX:

$$\begin{array}{llcrcr}
\text{maximize} & 5x_1 & - & 3x_2 & & \\
\text{subject to} & & & & & \\
& x_1 & - & x_2 & \leq & 1 \\
& 2x_1 & + & x_2 & \leq & 2 \\
& & x_1, x_2 & & \geq & 0
\end{array}$$

Question 24. [CLRS: 29.5-5] Solve the following linear program using SIMPLEX:

$$\begin{array}{llcrcr}
\text{maximize} & x_1 & + & 3x_2 & & \\
\text{subject to} & & & & & \\
& x_1 & - & x_2 & \leq & 8 \\
& -x_1 & - & x_2 & \leq & -3 \\
& -x_1 & + & 4x_2 & \leq & 2 \\
& & x_1, x_2 & & \geq & 0
\end{array}$$

1.4 Approximation Algorithms

Question 25. Let $G = (V, E)$ be an undirected graph with maximum degree Δ. A dominating set is a subset of vertices $S \subseteq V$ so that for every vertex $u \in V$ there exists a vertex $v \in S$ with $\{u, v\} \in E(G)$. The goal is to find a dominating set as small as possible. Design an approximation algorithm based on greedy for the problem and analyse the quality of its solution.

Question 26. Given an undirected graph $G = (V, E)$, a *vertex cover* of G is a set of vertices $C \subseteq V$ so that each edge in G is incident to at least one vertex in C. A minimum vertex cover is a vertex cover with smallest possible size $|C|$. Consider a greedy approach which iteratively adds the vertex with the highest degree to C and then removes all covered edges from E. Find an example that shows that this greedy algorithm does not always find the optimum solution.

Question 27. [CLRS: 35.1-3, this one improves on the previous question and is marked with a "$\star$" in CLRS] Professor Bündchen proposes the following heuristic to solve the vertex-cover problem. Repeatedly select a vertex of highest degree, and remove all of its incident edges. Give an example to show that the professor's heuristic does not have an approximation ratio of 2. (*Hint:* Try a bipartite graph with vertices of uniform degree on the left and vertices of varying degree on the right.)

Question 28. How can you implement Approx-Vertex-Cover in time $O(V + E)$?

Question 29. [CLRS: Problem 35.3-3] Consider the analysis of Greedy-Set-Cover (Theorem 35.4). Show that the following weaker form of Theorem 35.4 is trivially true:

$$|\mathcal{C}| \leq |\mathcal{C}^*| \cdot \max\{|S| : S \in \mathcal{F}\}$$

Question 30. How would you solve an instance of the Vertex-Cover problem using the Greedy Algorithm for the Set-Cover?

Question 31. Consider the problem Subset-Sum. Design a simple Greedy algorithm which runs in polynomial-time and achieves an approximation ratio of 2.

Question 32. Consider the algorithm Approx-Subset-Sum from the lecture. Prove formally that for every element y, at most t, which can be written as a sum of a subset of $\{x_1, x_2, \ldots, x_n\}$, there exists an element $z \in L_n$ (the list in iteration n *after* the trimming operation), such that

$$\frac{y}{(1+\delta)^n} \leq z,$$

where $0 < \delta < 1$ is the trimming parameter.

Question 33. [CLRS: 35.3-3] Show how to implement Greedy-Set-Cover in such a way that it runs in time $O(\sum_{S \in \mathcal{F}} |S|)$.

Question 34. [CLRS: 35.2-1] Suppose that a complete undirected graph $G = (V, E)$ with at least 3 vertices has a cost function that satisfies the triangle inequality. Prove that $c(u, v) \geq 0$ for all $u, v \in V$.

Question 35. [CLRS: 35.2-5] Suppose that the vertices for an instance of the travelling-salesman problem are points in the plane and that the cost $c(u, v)$ is the euclidean distance between points u and v. Show that an optimal tour never crosses itself.

Question 36. Recall the subtour elimination procedure from Lecture 10: In order to eliminate a subtour going through cities in S only, we add the following constraint:

$$\sum_{i \in S, j \notin S} x(\max(i, j), \min(i, j)) \geq 2.$$

Prove that adding this constraint to the linear program is equivalent to adding the constraint

$$\sum_{i \in S, j \in S, i < j} x(i, j) \leq |S| - 1.$$

Question 37. [CLRS: 35.2-3] Show how in polynomial time we can transform one instance of the travelling-salesman problem into another instance whose cost function satisfies the triangle inequality. The two instances must have the same set of optimal tours. Explain why such a polynomial-time transformation does not contradict the inapproximability result (Theorem 35.3), assuming that $\mathsf{P} \neq \mathsf{NP}$.

Question 38. Consider the following problem. Given an undirected, connected graph $G = (V, E)$ with non-negative, integral edge capacities $c(u, v)$ for each edge $(u, v) \in E(G)$ and $|E| \geq |V| = n$, the goal is to find a subset $E' \subseteq E$ with $|E'| = n$ so that *(i)* E' connects all vertices and *(ii)* $\sum_{e \in E'} c(e)$ is minimized. Either prove that this problem is NP-hard or design a polynomial-time algorithm.

Question 39. Find an example of a graph in the Euclidean space, with as few vertices as possible, so that the optimal TSP tour does not include a minimum spanning tree.

Question 40. [CLRS: 35.4-2] The **MAX-CNF satisfiability problem** is like the MAX-3-CNF satisfiability problem, except that it does not restrict each clause to have exactly 3 literals. Give a randomized 2-approximation algorithm for the MAX-CNF satisfiability problem.

Question 41. [CLRS: Problem 35-1] Suppose that we are given a set of n objects, where the size s_i of the ith object satisfies $0 < s_i < 1$. We wish to pack all the objects into the minimum number of unit-size bins. Each bin can hold any subset of the objects whose total size does not exceed 1.

The **first-fit** heuristic takes each object in turn and places it into the first bin that can accommodate it. Let $S := \sum_{i=1}^{n} s_i$.

1. Argue that the optimal number of bins required is at least $\lceil S \rceil$.

2. Argue that the first-fit heuristic leaves at most one bin less than half full.
3. Prove that the number of bins used by the first-fit heuristic is never more than $\lceil 2S \rceil$.
4. Prove an approximation ratio of 2 for the first-fit heuristic.
5. Give an efficient implementation of the first-fit heuristic, and analyse its running time.

Question 42. Consider the following algorithm for MAX-CUT on an unweighted, undirected graph $G = (V, E)$, which can be regarded as an iterative colouring procedure with three colours possible, grey (=unassigned), red (assigned to S) and blue (assigned to $V \setminus S$). Initially, all vertices are grey. Then the algorithm does the following in each step: If there is a grey vertex u which has more blue than red neighbours colour it **red**, if there is a grey vertex u which has more red than blue neighbours colour it **blue**. Otherwise, take a grey vertex and colour it arbitrarily. Prove that this algorithm returns a 2-approximation.