Advanced Algorithms

I. Course Intro and Sorting Networks

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Easter 2021

UNIVERSITY OF CAMBRIDGE

Outline

Outline of this Course

Some Highlights

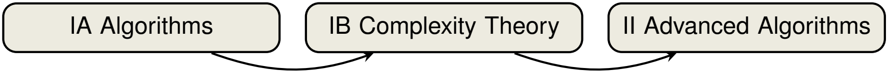
Introduction to Sorting Networks

Batcher's Sorting Network

Bonus Material: Construction of an Optimal Sorting Network (non-examinable)

Counting Networks

List of Topics



List of Topics

IA Algorithms → IB Complexity Theory → II Advanced Algorithms

- I. Sorting Networks (Sorting, Counting)
- II. Linear Programming
- III. Approximation Algorithms: Covering Problems
- IV. Approximation Algorithms via Exact Algorithms
- V. Approximation Algorithms: Travelling Salesman Problem
- VI. Approximation Algorithms: Randomisation and Rounding

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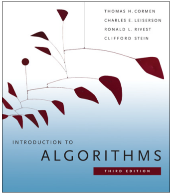

- closely follow CLRS3 and use the same numberring
- however, slides will be self-contained

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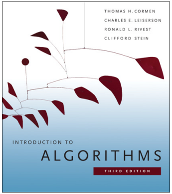

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Linear Programming and Simplex

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\begin{array}{llcccclc}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
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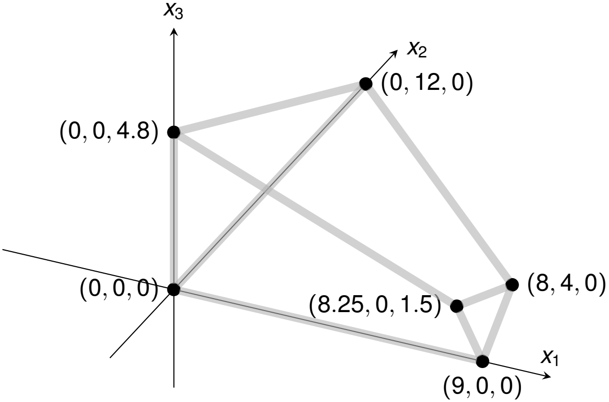
Linear Programming and Simplex



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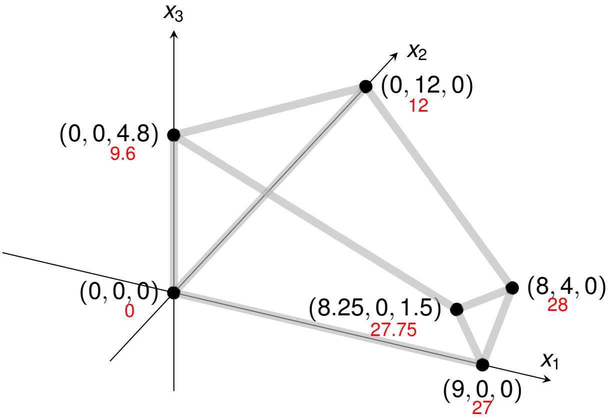

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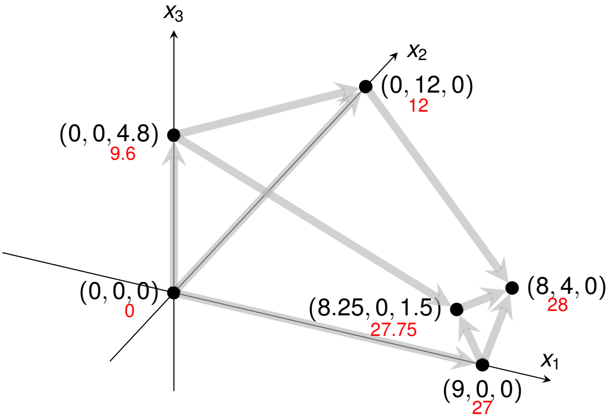

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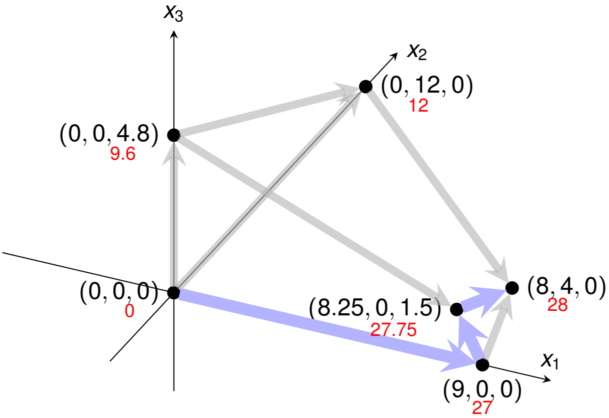

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SOLUTION OF A LARGE-SCALE TRAVELING-SALESMAN PROBLEM*

G. DANTZIG, R. FULKERSON, AND S. JOHNSON

The Rand Corporation, Santa Monica, California

(Received August 9, 1954)

It is shown that a certain tour of 49 cities, one in each of the 48 states and Washington, D. C., has the shortest road distance.

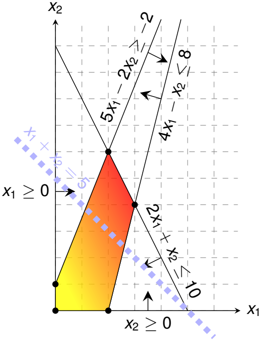
THE TRAVELING-SALESMAN PROBLEM might be described as follows: Find the shortest route (tour) for a salesman starting from a given city, visiting each of a specified group of cities, and then returning to the original point of departure. More generally, given an n by n symmetric matrix $D=(d_{IJ})$, where d_{IJ} represents the 'distance' from I to J, arrange the points in a cyclic order in such a way that the sum of the d_{IJ} between consecutive points is minimal. Since there are only a finite number of possibilities (at most $\frac{1}{2}(n-1)!$) to consider, the problem is to devise a method of picking out the optimal arrangement which is reasonably efficient for fairly large values of n. Although algorithms have been devised for problems of similar nature, e.g., the optimal assignment problem,[3,7,8] little is known about the traveling-salesman problem. We do not claim that this note alters the situation very much; what we shall do is outline a way of approaching the problem that sometimes, at least, enables one to find an optimal path and prove it so. In particular, it will be shown that a certain arrangement of 49 cities, one in each of the 48 states and Washington, D. C., is best, the d_{IJ} used representing road distances as taken from an atlas.

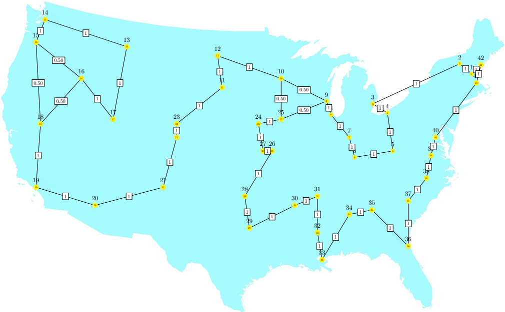
Travelling Salesman Problem: The 42 (49) Cities

1. Manchester, N. H.
2. Montpelier, Vt.
3. Detroit, Mich.
4. Cleveland, Ohio
5. Charleston, W. Va.
6. Louisville, Ky.
7. Indianapolis, Ind.
8. Chicago, Ill.
9. Milwaukee, Wis.
10. Minneapolis, Minn.
11. Pierre, S. D.
12. Bismarck, N. D.
13. Helena, Mont.
14. Seattle, Wash.
15. Portland, Ore.
16. Boise, Idaho
17. Salt Lake City, Utah
18. Carson City, Nev.
19. Los Angeles, Calif.
20. Phoenix, Ariz.
21. Santa Fe, N. M.
22. Denver, Colo.
23. Cheyenne, Wyo.
24. Omaha, Neb.
25. Des Moines, Iowa
26. Kansas City, Mo.
27. Topeka, Kans.
28. Oklahoma City, Okla.
29. Dallas, Tex.
30. Little Rock, Ark.
31. Memphis, Tenn.
32. Jackson, Miss.
33. New Orleans, La.
34. Birmingham, Ala.
35. Atlanta, Ga.
36. Jacksonville, Fla.
37. Columbia, S. C.
38. Raleigh, N. C.
39. Richmond, Va.
40. Washington, D. C.
41. Boston, Mass.
42. Portland, Me.

A. Baltimore, Md.
B. Wilmington, Del.
C. Philadelphia, Penn.
D. Newark, N. J.
E. New York, N. Y.
F. Hartford, Conn.
G. Providence, R. I.

Computing the Optimal Tour





We are going to use our own implementation of the Simplex-Algorithm along with a visulation to solve a series of linear programs in order to solve the TSP instance optimally!

There are a couple of exercises spread across the recordings to test your understanding!

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Overview: Sorting Networks

(Serial) Sorting Algorithms

- we already know several (comparison-based) sorting algorithms: Insertion sort, Bubble sort, Merge sort, Quick sort, Heap sort
- execute one operation at a time
- can handle arbitrarily large inputs
- sequence of comparisons is not set in advance

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Allows to sort n numbers in sublinear time!

Simple concept, but surprisingly deep and complex theory!

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  - comparator is a device with, on given two inputs, x and y, returns two outputs $x' = \min(x, y)$ and $y' = \max(x, y)$

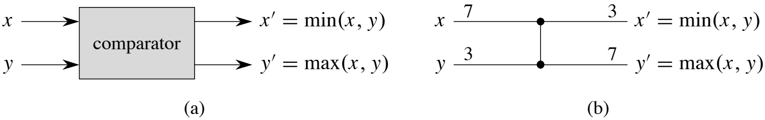

Figure 27.1 **(a)** A comparator with inputs x and y and outputs x' and y'. **(b)** The same comparator, drawn as a single vertical line. Inputs $x = 7$, $y = 3$ and outputs $x' = 3$, $y' = 7$ are shown.

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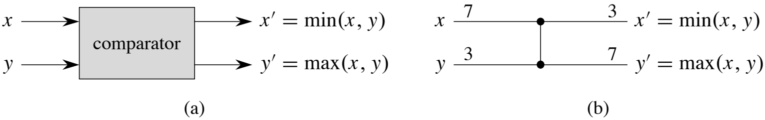
operates in $O(1)$



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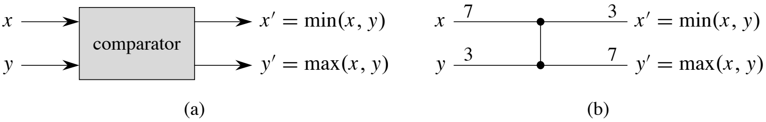

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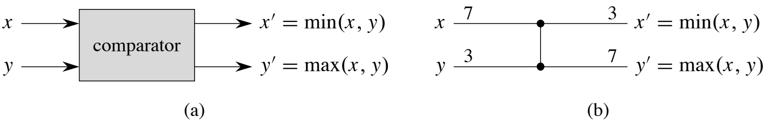

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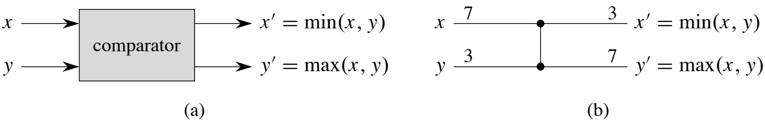
Convention: use the same name for both a wire and its value.



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A sorting network is a comparison network which works correctly (that is, it sorts every input)

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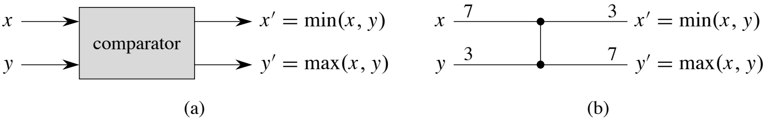

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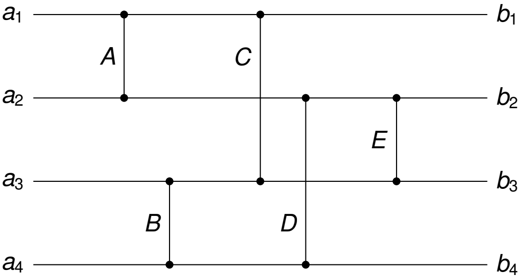

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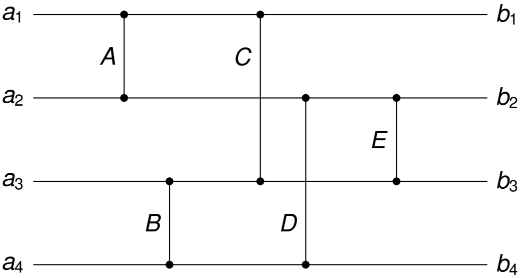
A horizontal line represents a sequence of distinct wires



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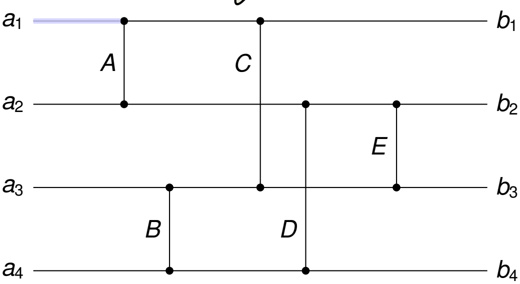

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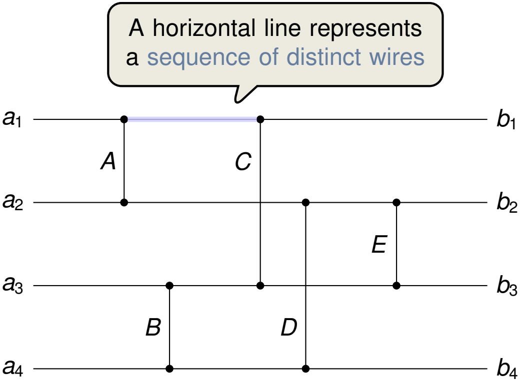

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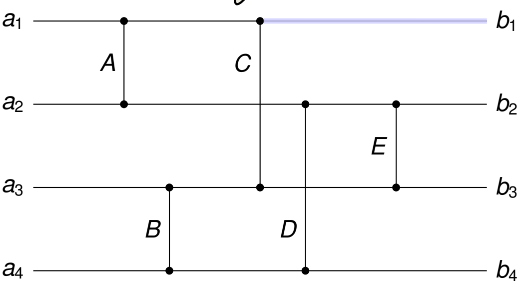

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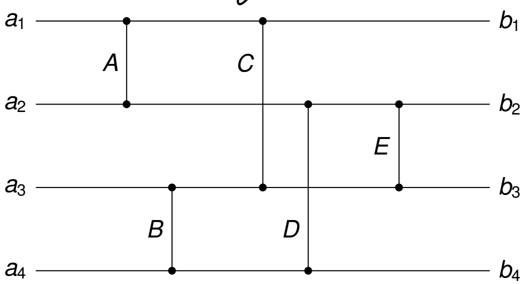
Interconnections between comparators must be acyclic



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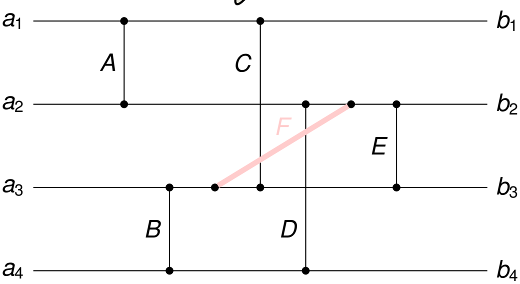

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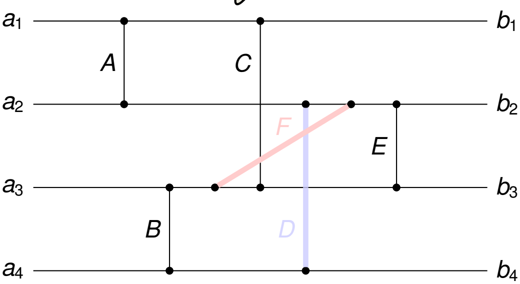

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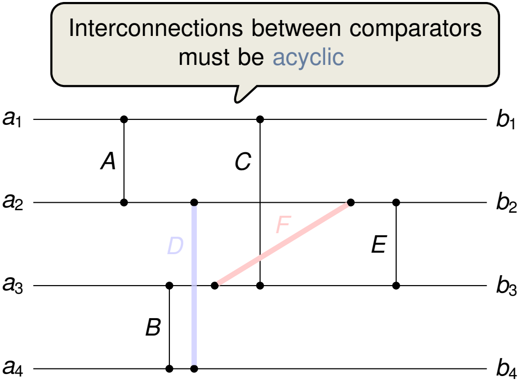

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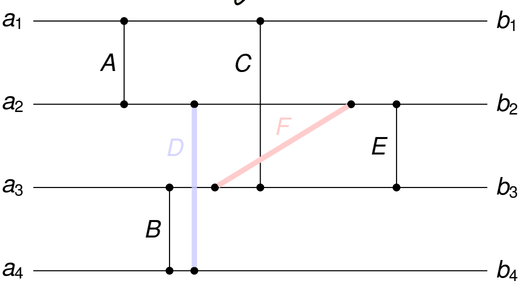

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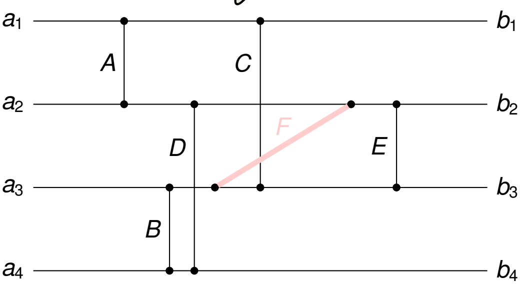

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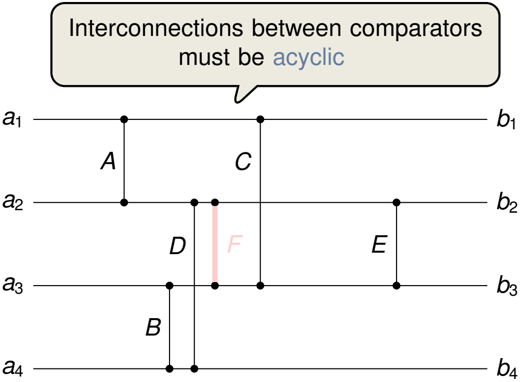

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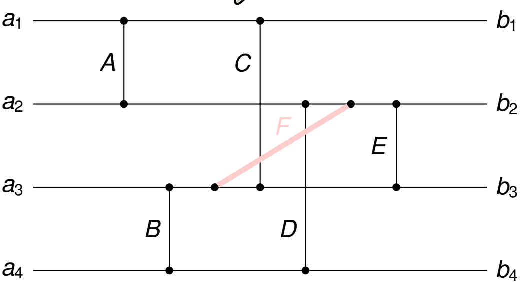

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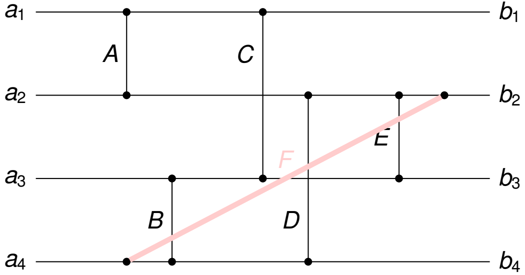

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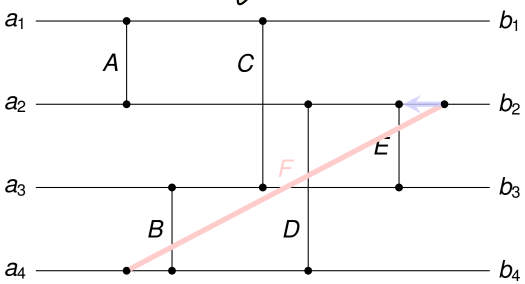

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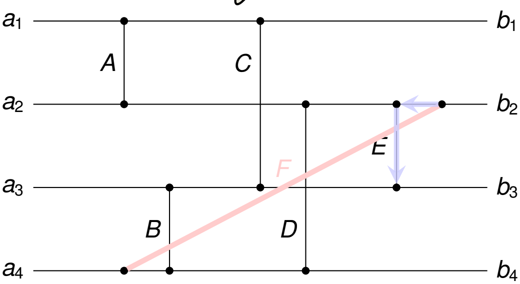

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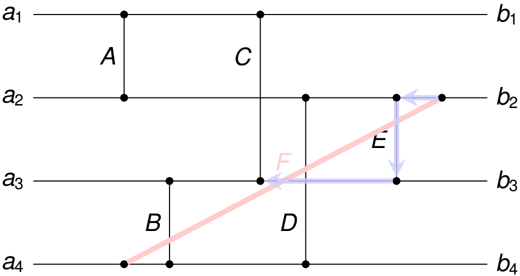

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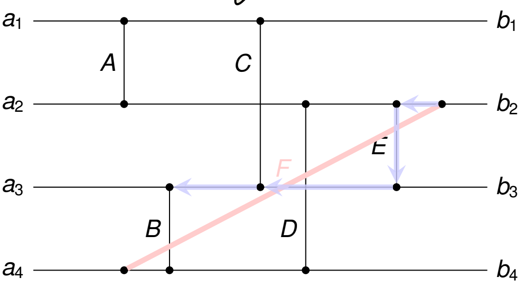

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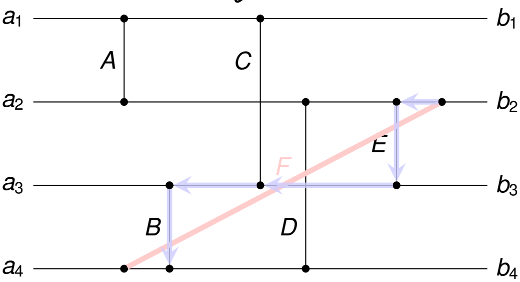

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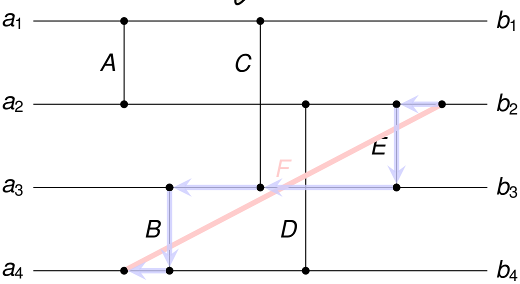

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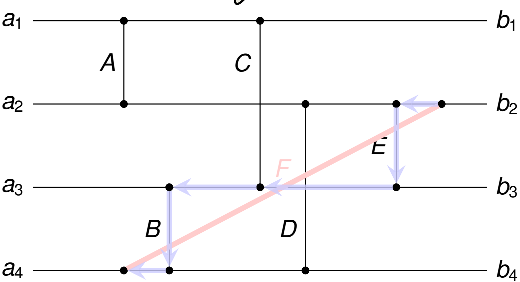
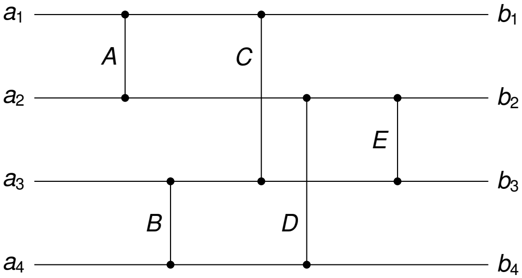
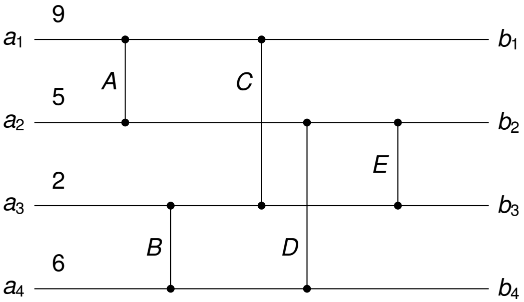


Tracing back a path must never cycle back on itself and go through the same comparator twice.

Example of a Comparison Network (Figure 27.2, CLRS2)



Example of a Comparison Network (Figure 27.2, CLRS2)



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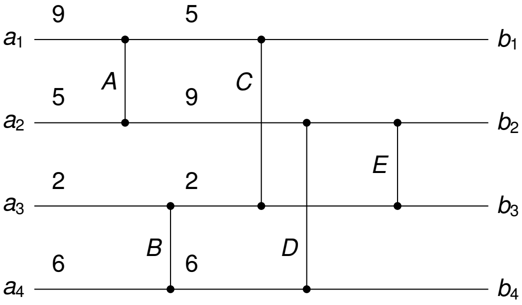
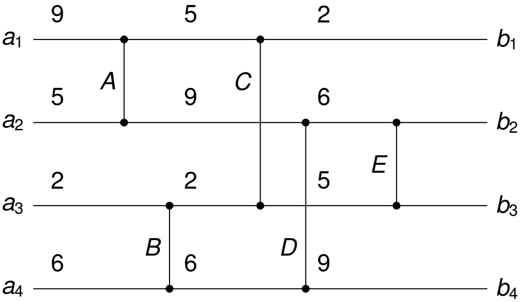
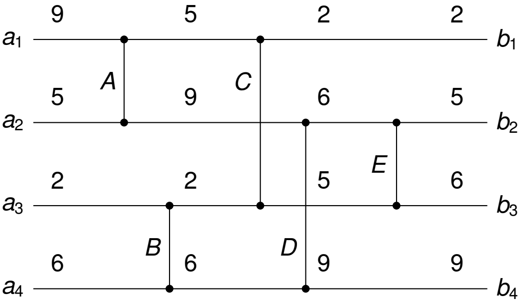


Example of a Comparison Network (Figure 27.2, CLRS2)

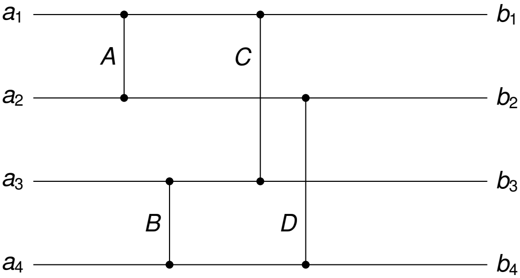


Example of a Comparison Network (Figure 27.2, CLRS2)



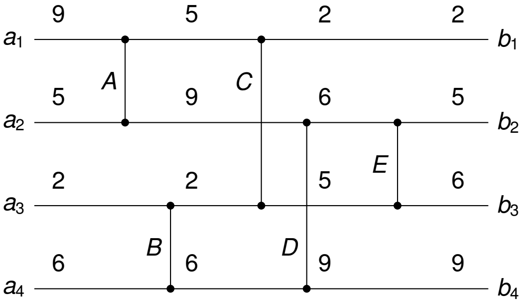
This network is in fact a sorting network (**Exercise 1**)

Example of a Comparison Network (Figure 27.2, CLRS2)



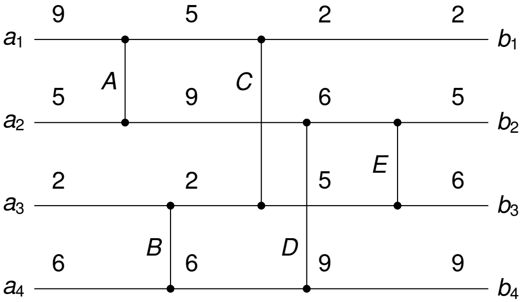
This network would not be a sorting network (**Exercise 2**)

Example of a Comparison Network (Figure 27.2, CLRS2)



Depth of a wire:

Example of a Comparison Network (Figure 27.2, CLRS2)



Depth of a wire:

- Input wire has depth 0

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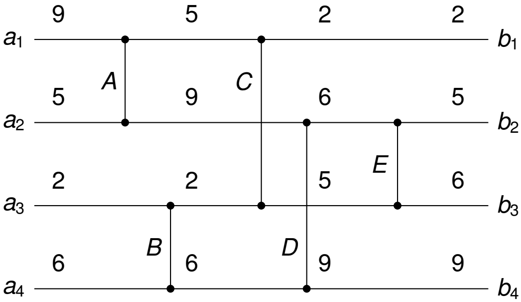

Depth of a wire:

- Input wire has depth 0
- If a comparator has two inputs of depths d_x and d_y, then outputs have depth $\max\{d_x, d_y\} + 1$

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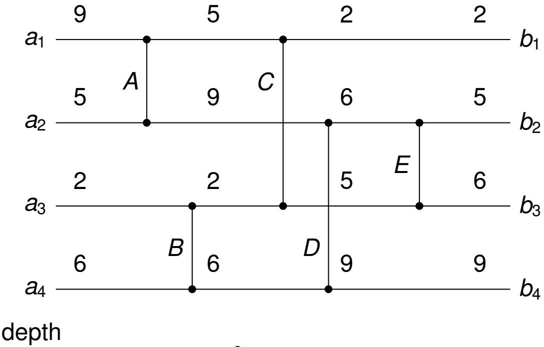

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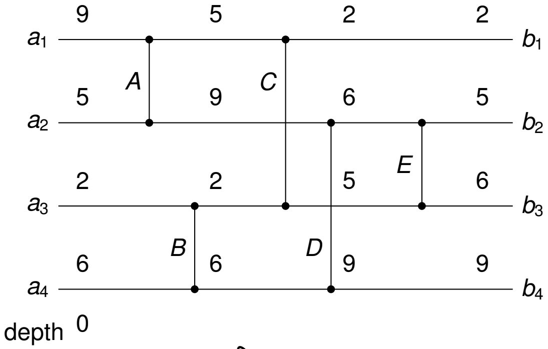

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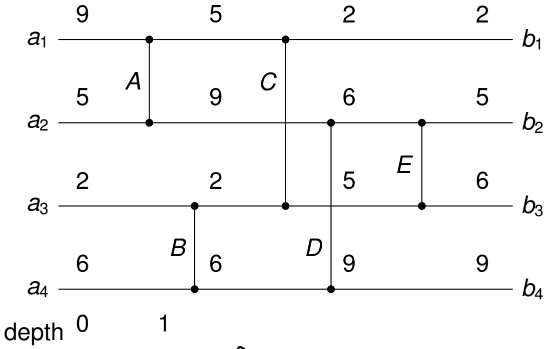

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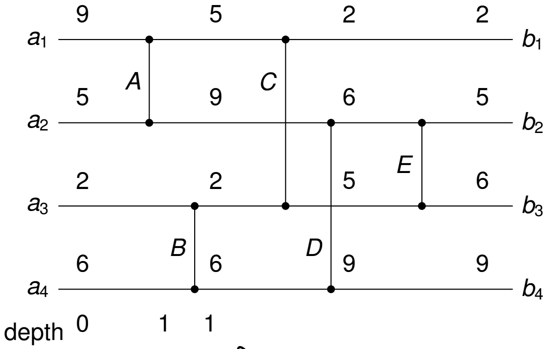

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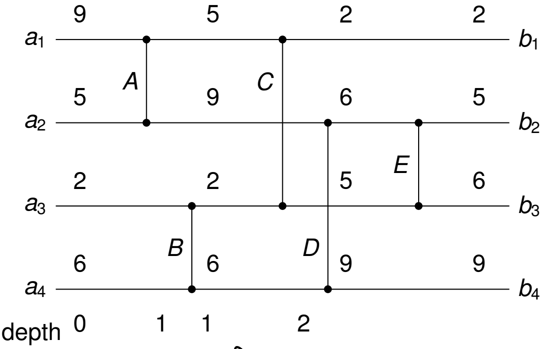

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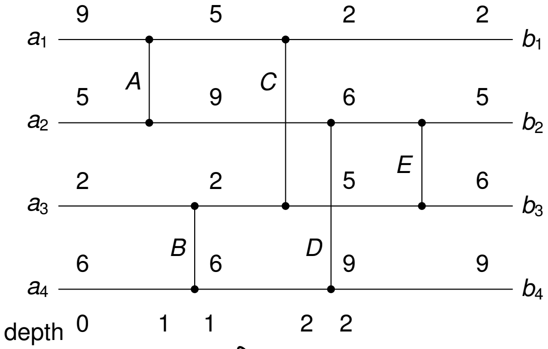

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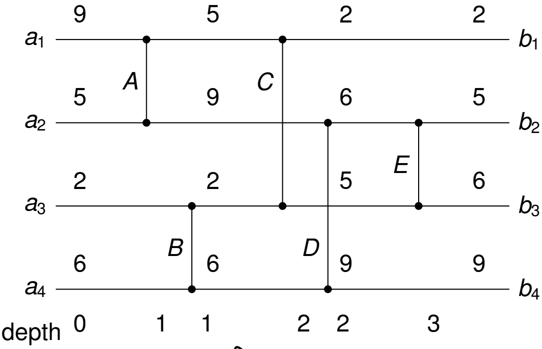

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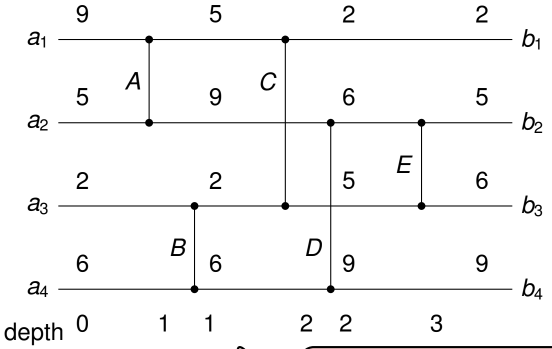

Maximum depth of an output wire equals total running time

Depth of a wire:

- Input wire has depth 0
- If a comparator has two inputs of depths d_x and d_y, then outputs have depth $\max\{d_x, d_y\} + 1$

Zero-One Principle

Zero-One Principle: A sorting networks works correctly on arbitrary inputs if it works correctly on binary inputs.

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Lemma 27.1

If a comparison network transforms the input $a = \langle a_1, a_2, \ldots, a_n \rangle$ into the output $b = \langle b_1, b_2, \ldots, b_n \rangle$, then for any monotonically increasing function f, the network transforms $f(a) = \langle f(a_1), f(a_2), \ldots, f(a_n) \rangle$ into $f(b) = \langle f(b_1), f(b_2), \ldots, f(b_n) \rangle$.

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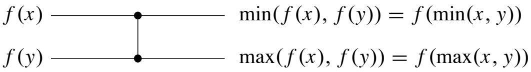

$$\begin{tikzcd} a \arrow[r, "\text{Network}"] \arrow[d, "f"'] & b \arrow[d, "f"] \\ f(a) \arrow[r, "\text{Network}"] & f(b) \end{tikzcd}$$

Figure 27.4 The operation of the comparator in the proof of Lemma 27.1. The function f is monotonically increasing.

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Theorem 27.2 (Zero-One Principle)

If a comparison network with n inputs sorts all 2^n possible sequences of 0's and 1's correctly, then it sorts all sequences of arbitrary numbers correctly.

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$$f(x) = \begin{cases} 0 & \text{if } x \leq a_i, \\ 1 & \text{if } x > a_i. \end{cases}$$

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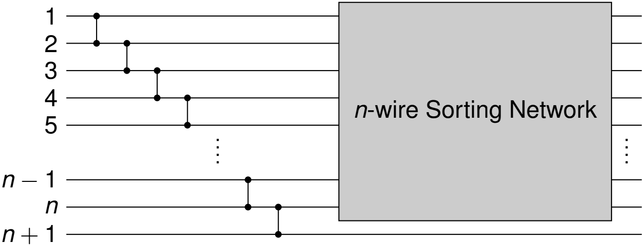
- Since the network places a_j before a_i, by the previous lemma $\Rightarrow f(a_j)$ is placed before $f(a_i)$
- But $f(a_j) = 1$ and $f(a_i) = 0$, which contradicts the assumption that the network sorts all sequences of 0's and 1's correctly $\square$

Some Basic (Recursive) Sorting Networks



???

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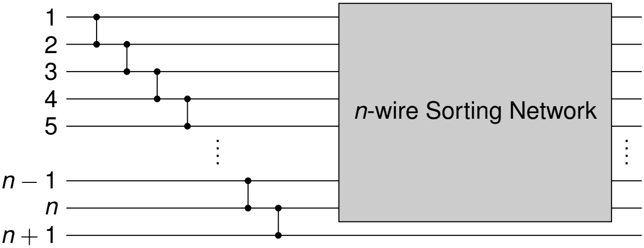

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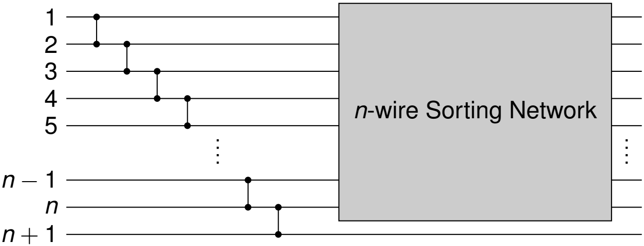

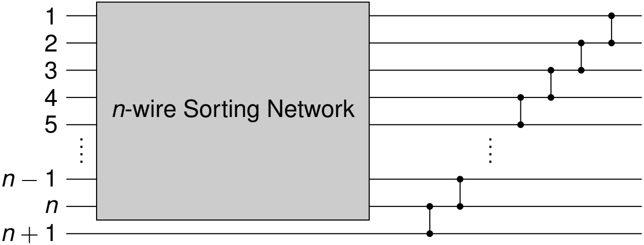

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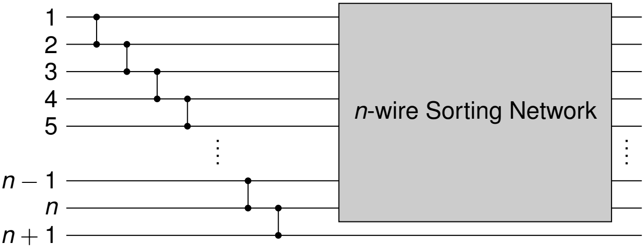

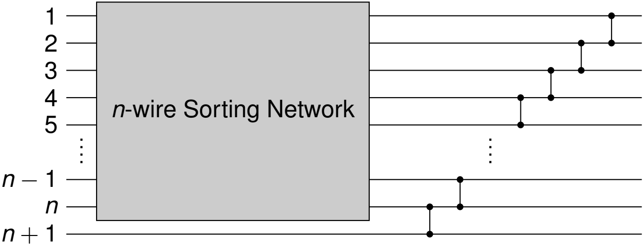

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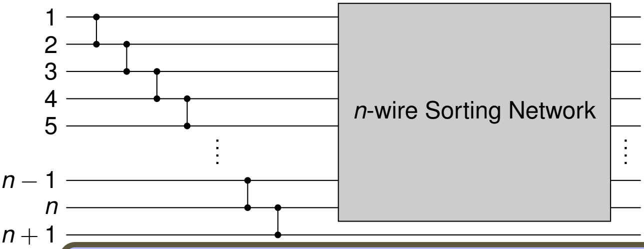
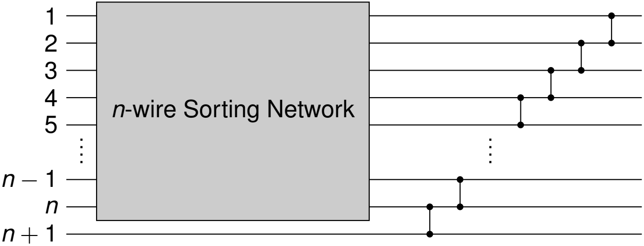


These are Sorting Networks, but with depth $\Theta(n)$.



Outline

Bitonic Sequences

Bitonic Sequence

A sequence is bitonic if it monotonically increases and then monotonically decreases, or can be circularly shifted to become monotonically increasing and then monotonically decreasing.

Sequences of one or two numbers are defined to be bitonic.

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- $\langle 9, 8, 3, 2, 4, 6 \rangle$?

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- $\cancel{\langle 4, 5, 7, 1, 2, 6 \rangle}$

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- binary sequences: ?

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- $\langle 6, 9, 4, 2, 3, 5 \rangle$ ✓
- $\langle 9, 8, 3, 2, 4, 6 \rangle$ ✓
- $\cancel{\langle 4, 5, 7, 1, 2, 6 \rangle}$
- binary sequences: $0^i 1^j 0^k$, or, $1^i 0^j 1^k$, for $i, j, k \geq 0$.

Towards Bitonic Sorting Networks

Half-Cleaner

A half-cleaner is a comparison network of depth 1 in which input wire i is compared with wire $i + n/2$ for $i = 1, 2, \ldots, n/2$.

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We always assume that n is even.

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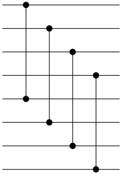

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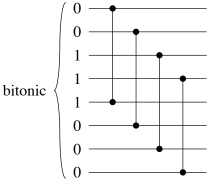

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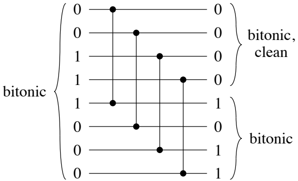

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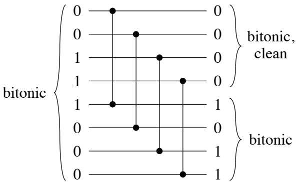

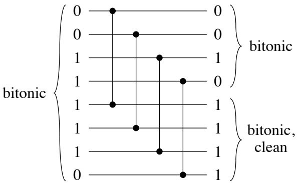

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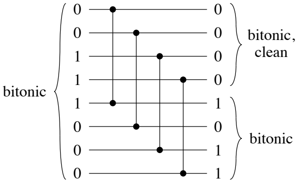
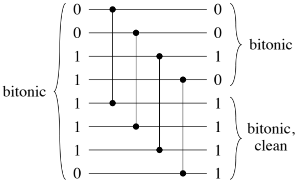
Lemma 27.3

If the input to a half-cleaner is a bitonic sequence of 0's and 1's, then the output satisfies the following properties:

- both the top half and the bottom half are bitonic,
- every element in the top is not larger than any element in the bottom,
- at least one half is clean.





Towards Bitonic Sorting Networks

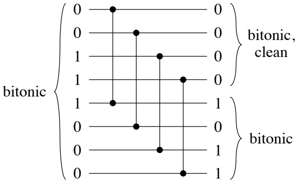
Half-Cleaner

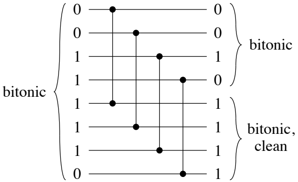
A half-cleaner is a comparison network of depth 1 in which input wire i is compared with wire $i + n/2$ for $i = 1, 2, \ldots, n/2$.

Lemma 27.3

If the input to a half-cleaner is a bitonic sequence of 0's and 1's, then the output satisfies the following properties:

- both the top half and the bottom half are bitonic,
- every element in the top is not larger than any element in the bottom,
- at least one half is clean.





Proof of Lemma 27.3

W.l.o.g. assume that the input is of the form $0^i 1^j 0^k$, for some $i, j, k \geq 0$.

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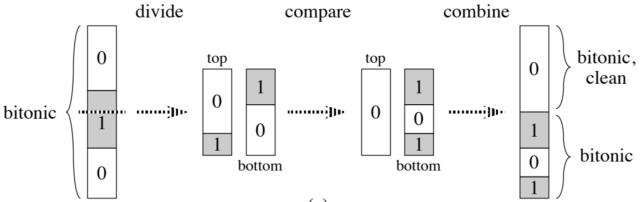

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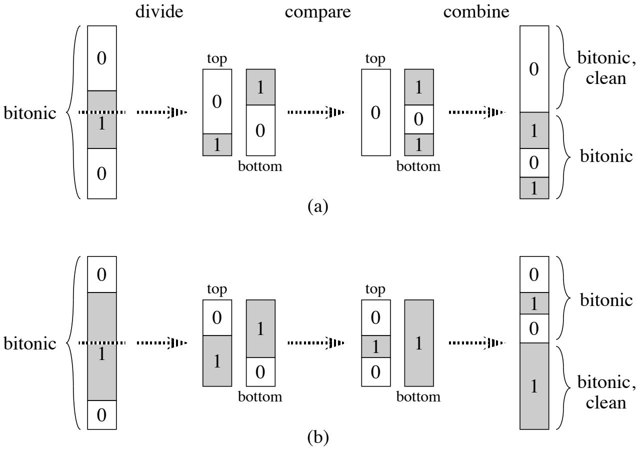

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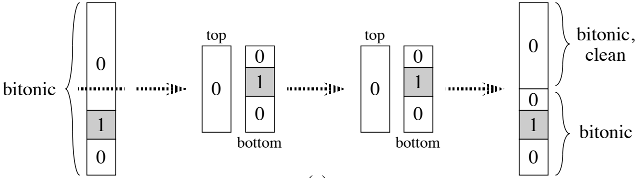
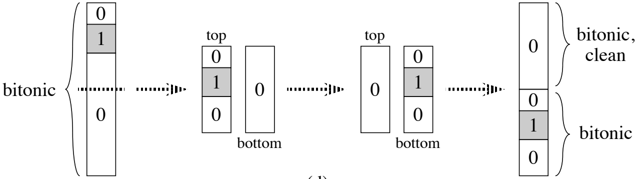


(c)



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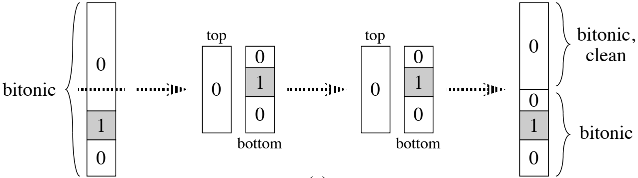

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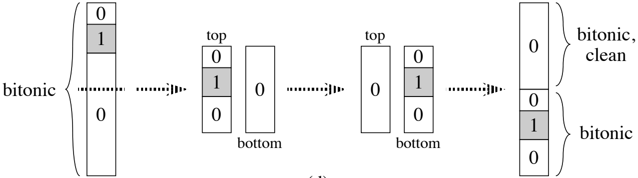

(d)

This suggests a recursive approach, since it now suffices to sort the top and bottom half separately.

The Bitonic Sorter

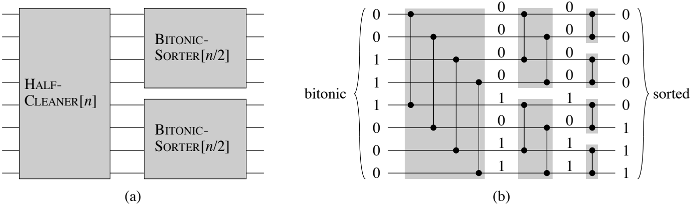

Figure 27.9 The comparison network BITONIC-SORTER[n], shown here for $n = 8$. **(a)** The recursive construction: HALF-CLEANER[n] followed by two copies of BITONIC-SORTER[$n/2$] that operate in parallel. **(b)** The network after unrolling the recursion. Each half-cleaner is shaded. Sample zero-one values are shown on the wires.

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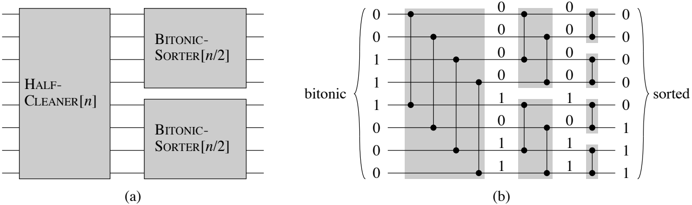

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Recursive Formula for depth $D(n)$:

$$D(n) = \begin{cases} 0 & \text{if } n = 1, \\ D(n/2) + 1 & \text{if } n = 2^k. \end{cases}$$

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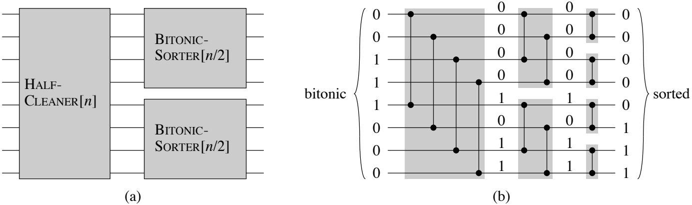

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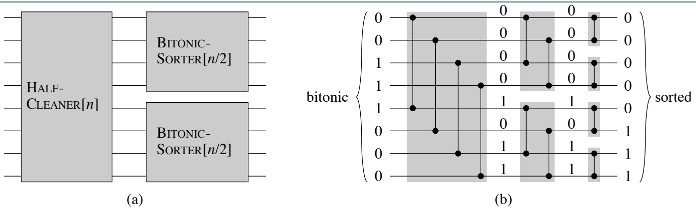

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BITONIC-SORTER[n] has depth $\log n$ and sorts any zero-one bitonic sequence.

Merging Networks

Merging Networks

- can merge two sorted input sequences into one sorted output sequence
- will be based on a modification of BITONIC-SORTER$[n]$

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This sequence is bitonic!

Hence in order to merge the sequences X and Y, it suffices to perform a bitonic sort on X concatenated with Y^R.

Construction of a Merging Network (1/2)

- Given two sorted sequences $\langle a_1, a_2, \ldots, a_{n/2} \rangle$ and $\langle a_{n/2+1}, a_{n/2+2}, \ldots, a_n \rangle$
- We know it suffices to bitonically sort $\langle a_1, a_2, \ldots, a_{n/2}, a_n, a_{n-1}, \ldots, a_{n/2+1} \rangle$
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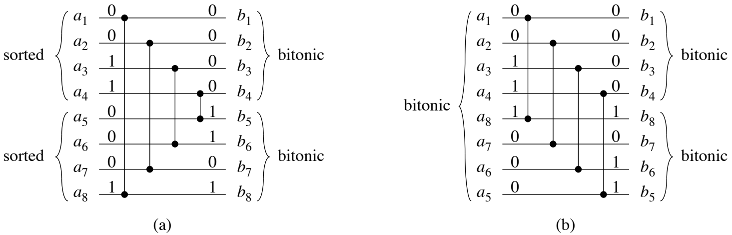

Figure 27.10 Comparing the first stage of MERGER$[n]$ with HALF-CLEANER$[n]$, for $n = 8$. **(a)** The first stage of MERGER$[n]$ transforms the two monotonic input sequences $\langle a_1, a_2, \ldots, a_{n/2} \rangle$ and $\langle a_{n/2+1}, a_{n/2+2}, \ldots, a_n \rangle$ into two bitonic sequences $\langle b_1, b_2, \ldots, b_{n/2} \rangle$ and $\langle b_{n/2+1}, b_{n/2+2}, \ldots, b_n \rangle$. **(b)** The equivalent operation for HALF-CLEANER$[n]$. The bitonic input sequence $\langle a_1, a_2, \ldots, a_{n/2-1}, a_{n/2}, a_n, a_{n-1}, \ldots, a_{n/2+2}, a_{n/2+1} \rangle$ is transformed into the two bitonic sequences $\langle b_1, b_2, \ldots, b_{n/2} \rangle$ and $\langle b_n, b_{n-1}, \ldots, b_{n/2+1} \rangle$.

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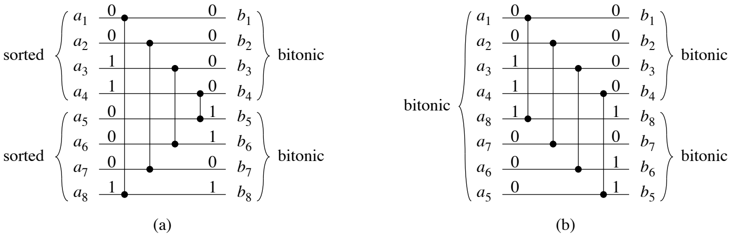

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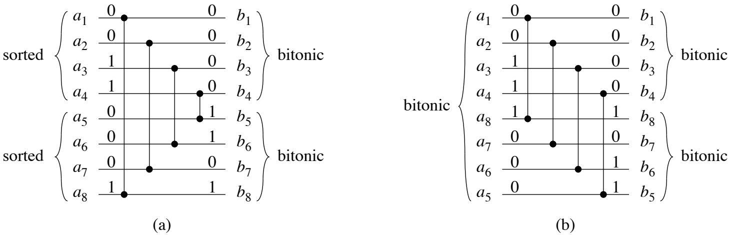
$\Rightarrow$ First part of MERGER[n] compares inputs i and $n - i + 1$ for $i = 1, 2, \ldots, n/2$

- Remaining part is identical to BITONIC-SORTER[n]



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Construction of a Merging Network (2/2)

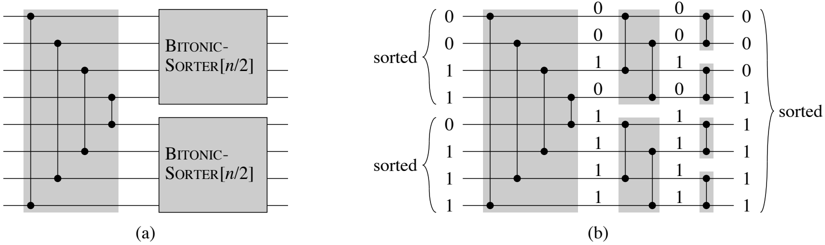


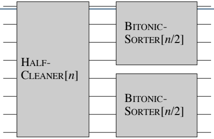
Figure 27.11 A network that merges two sorted input sequences into one sorted output sequence. The network MERGER[n] can be viewed as BITONIC-SORTER[n] with the first half-cleaner altered to compare inputs i and $n-i+1$ for $i = 1, 2, \ldots, n/2$. Here, $n = 8$. **(a)** The network decomposed into the first stage followed by two parallel copies of BITONIC-SORTER[$n/2$]. **(b)** The same network with the recursion unrolled. Sample zero-one values are shown on the wires, and the stages are shaded.

Construction of a Sorting Network

Main Components

1. BITONIC-SORTER[n]
   - sorts any bitonic sequence
   - depth $\log n$

1.



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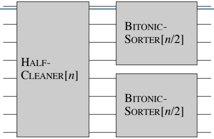
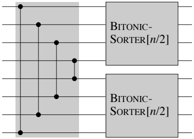


2.



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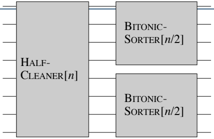

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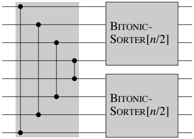
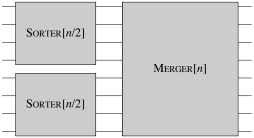


Batcher's Sorting Network

- SORTER[n] is defined recursively:
  - If $n = 2^k$, use two copies of SORTER[$n/2$] to sort two subsequences of length $n/2$ each. Then merge them using MERGER[n].
  - If $n = 1$, network consists of a single wire.



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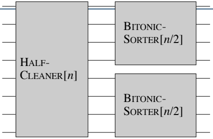

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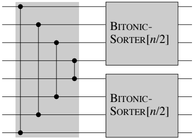

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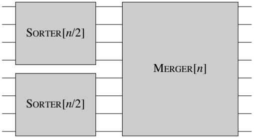
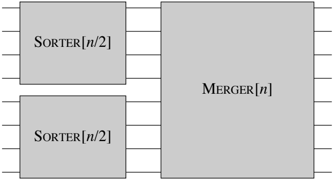
can be seen as a parallel version of merge sort



Unrolling the Recursion (Figure 27.12)



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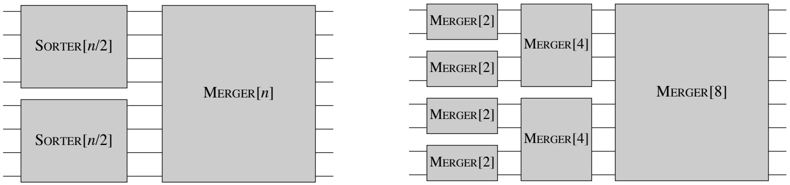

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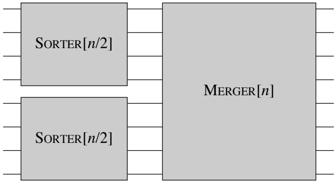

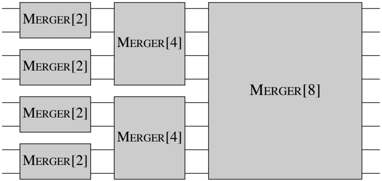

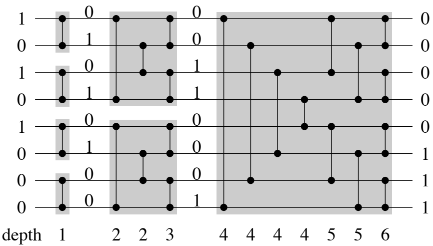

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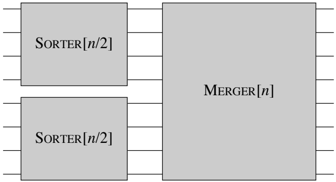

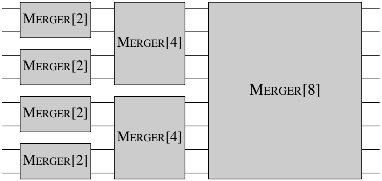

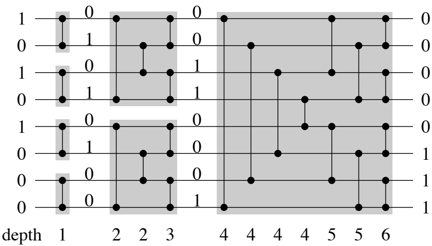

Recursion for $D(n)$:

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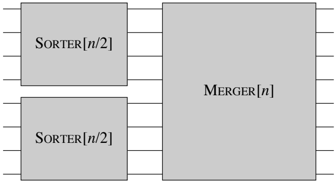

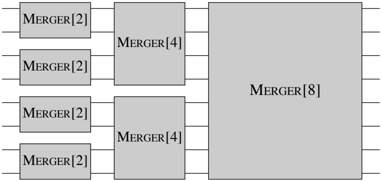

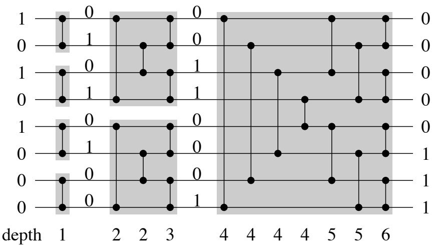

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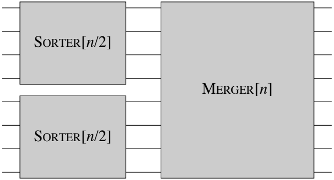
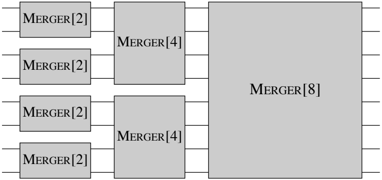
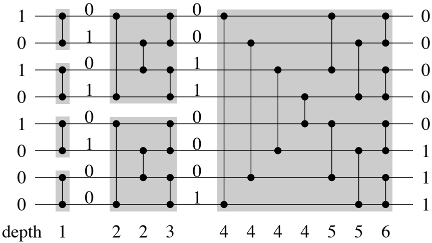
Solution: $D(n) = \Theta(\log^2 n)$.

Unrolling the Recursion (Figure 27.12)







Recursion for $D(n)$:

$$D(n) = \begin{cases} 0 & \text{if } n = 1, \\ D(n/2) + \log n & \text{if } n = 2^k. \end{cases}$$

Solution: $D(n) = \Theta(\log^2 n)$.

SORTER$[n]$ has depth $\Theta(\log^2 n)$ and sorts any input.

Outline

A Glimpse at the AKS Network

Ajtai, Komlós, Szemerédi (1983)

There exists a sorting network with depth $O(\log n)$.

A Glimpse at the AKS Network

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Quite elaborate construction, and involves huges constants.

A Glimpse at the AKS Network

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Perfect Halver

A perfect halver is a comparison network that, given any input, places the $n/2$ smaller keys in $b_1, \ldots, b_{n/2}$ and the $n/2$ larger keys in $b_{n/2+1}, \ldots, b_n$.

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Perfect halver of depth $\log n$ exist $\rightsquigarrow$ yields sorting networks of depth $\Theta((\log n)^2)$.

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Approximate Halver

An (n, ϵ)-approximate halver, $\epsilon < 1$, is a comparison network that for every $k = 1, 2, \ldots, n/2$ places at most ϵk of its k smallest keys in $b_{n/2+1}, \ldots, b_n$ and at most ϵk of its k largest keys in $b_1, \ldots, b_{n/2}$.

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We will prove that such networks can be constructed in constant depth!

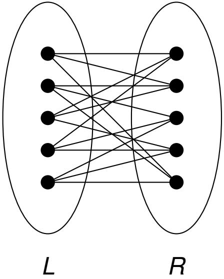
Expander Graphs

Expander Graphs

A bipartite (n, d, μ)-expander is a graph with:

- G has n vertices ($n/2$ on each side)
- the edge-set is union of d perfect matchings
- For every subset $S \subseteq V$ being in one part,

$$|N(S)| > \min\{\mu \cdot |S|, n/2 - |S|\}$$



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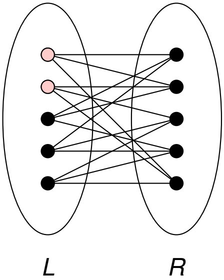

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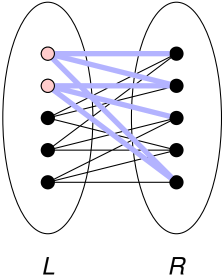

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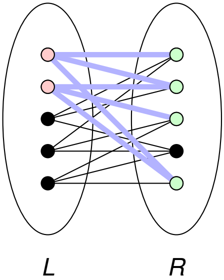

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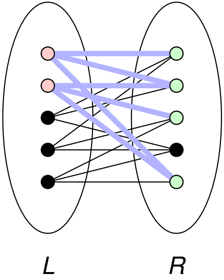
Specific definition tailored for sorting network - many other variants exist!



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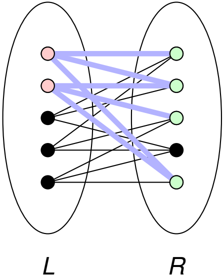
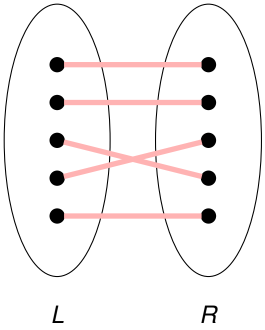


Expander Graphs:

- **probabilistic construction** “easy”: take d (disjoint) random matchings
- **explicit construction** is a deep mathematical problem with ties to number theory, group theory, combinatorics etc.
- **many applications** in networking, complexity theory and coding theory

From Expanders to Approximate Halvers



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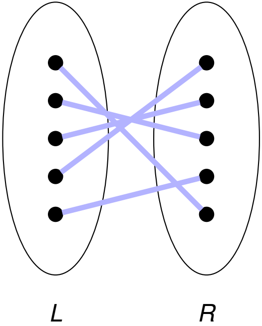

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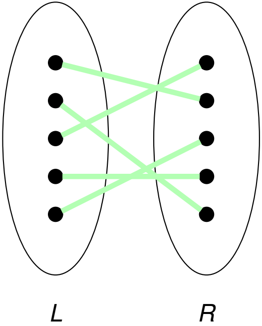

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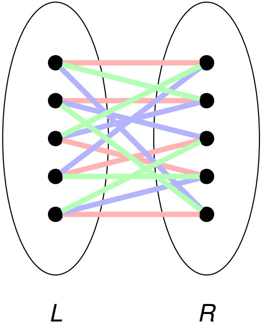

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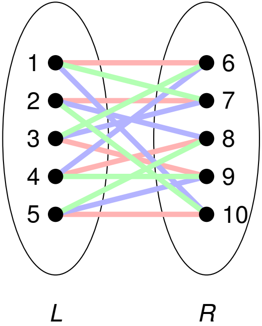

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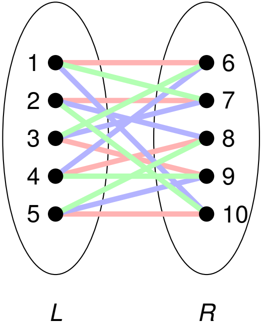

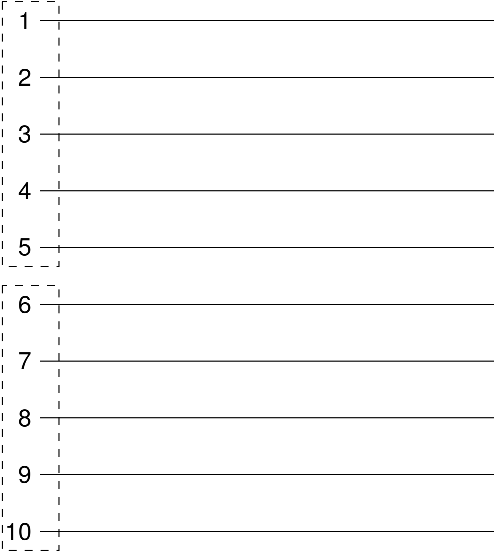

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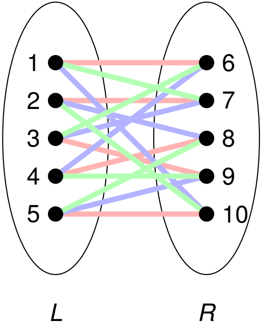

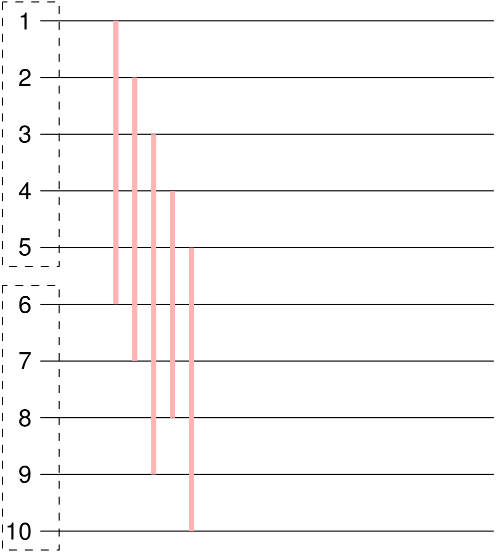

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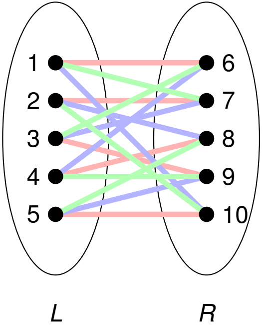

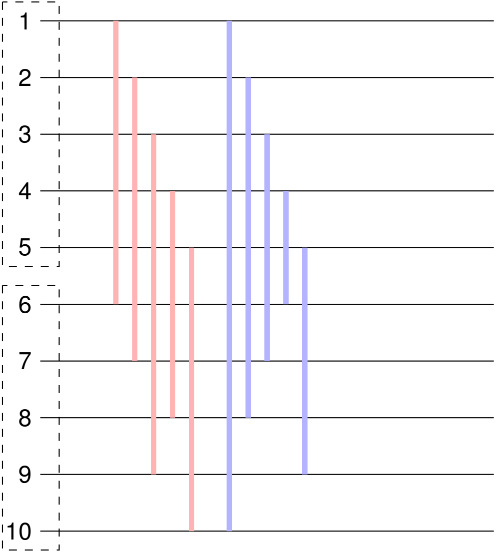

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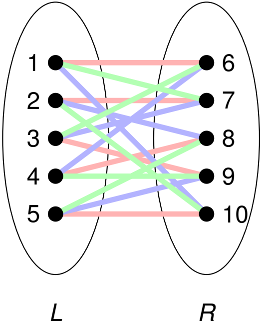
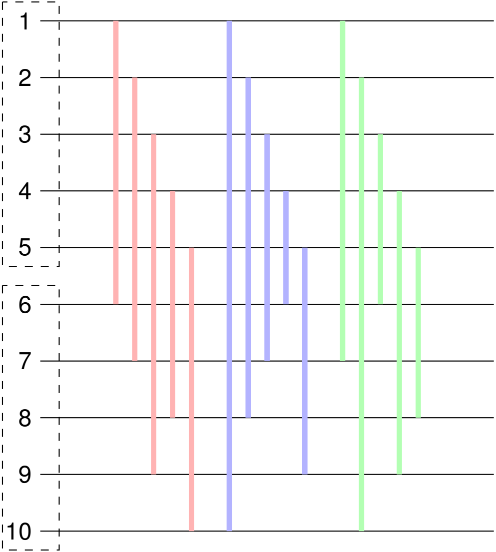
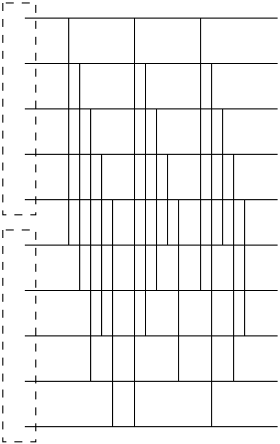




Existence of Approximate Halvers (non-examinable)

Proof:



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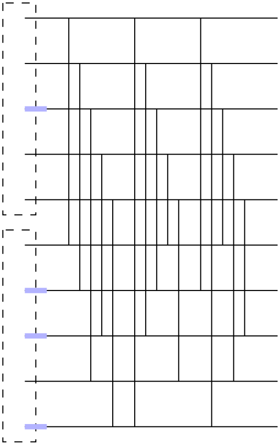
- $X :=$ keys with the k smallest inputs



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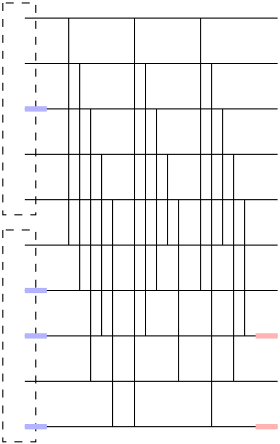

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- $X :=$ keys with the k smallest inputs
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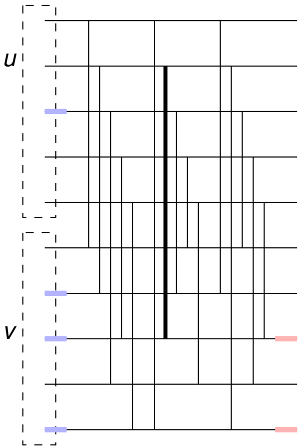

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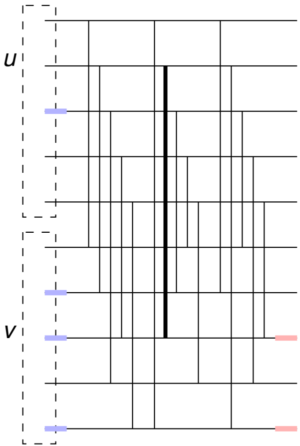

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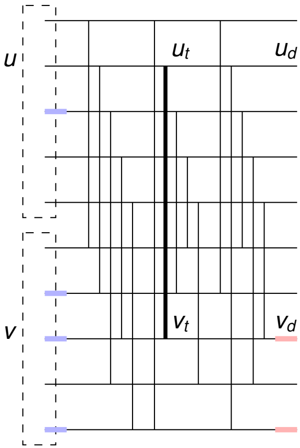

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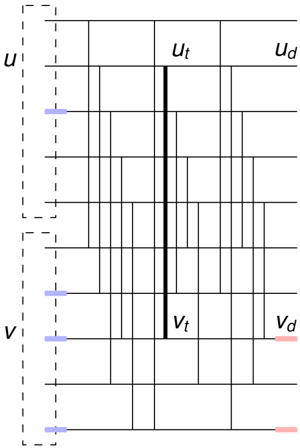

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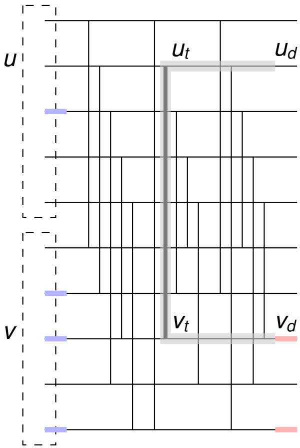

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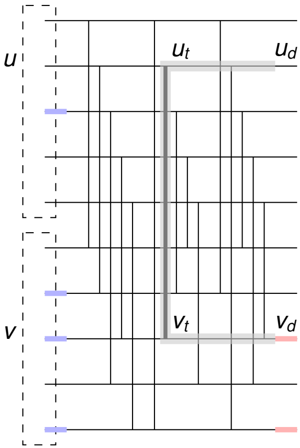
- Since u was arbitrary:

$$|Y| + |N(Y)| \leq k.$$



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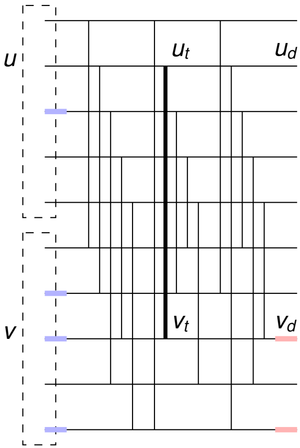

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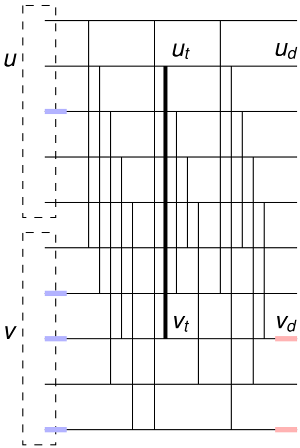

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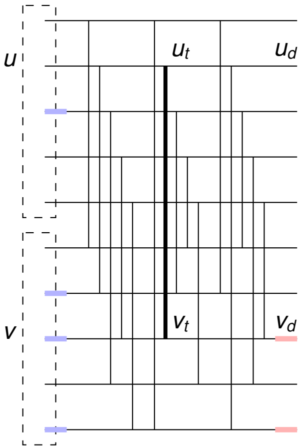

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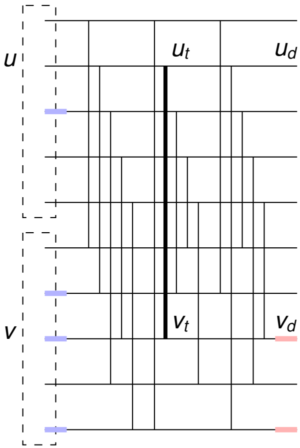

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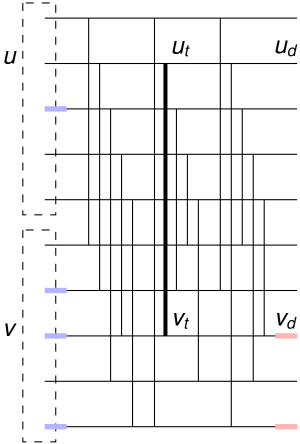

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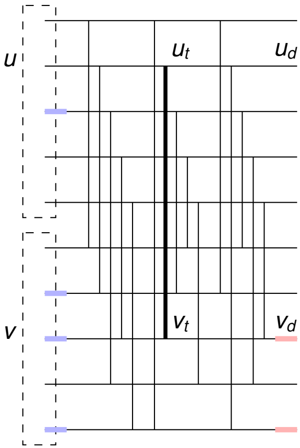
Here we used that $k \leq n/2$



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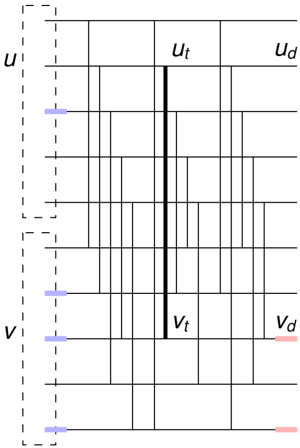
- Combining the two bounds above yields:

$$(1+\mu)|Y| \leq k.$$

- Same argument $\Rightarrow$ at most $\epsilon \cdot k$, $\epsilon := 1/(\mu+1)$, of the k largest input keys are placed in $b_1, \ldots, b_{n/2}$. □



- typical application of expander graphs in parallel algorithms
- Much more work needed to construct the AKS sorting network

AKS network vs. Batcher's network

Donald E. Knuth (Stanford)

"Batcher's method is much better, unless n exceeds the total memory capacity of all computers on earth!"

Richard J. Lipton (Georgia Tech)

*"The AKS sorting network is **galactic**: it needs that n be larger than 2^{78} or so to finally be smaller than Batcher's network for n items."*

Outline

Outline of this Course

Some Highlights

Introduction to Sorting Networks

Batcher's Sorting Network

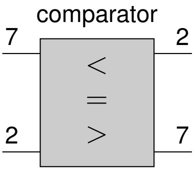
Bonus Material: Construction of an Optimal Sorting Network (non-examinable)

Counting Networks

Siblings of Sorting Network

Sorting Networks

- sorts any input of size n
- special case of Comparison Networks



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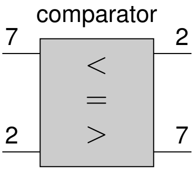
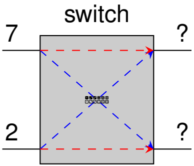


Switching (Shuffling) Networks

- creates a random permutation of *n* items
- special case of Permutation Networks



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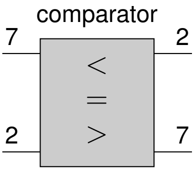

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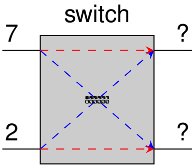
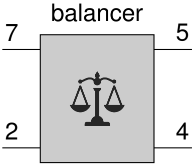


Counting Networks

- balances any stream of tokens over n wires
- special case of Balancing Networks



Counting Network

Distributed Counting

Processors collectively assign successive values from a given range.

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Values could represent addresses in memories or destinations on an interconnection network

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- constructed in a similar manner like sorting networks
- instead of comparators, consists of balancers
- balancers are asynchronous flip-flops that forward tokens from its inputs to one of its two outputs alternately (top, bottom, top,. . .)

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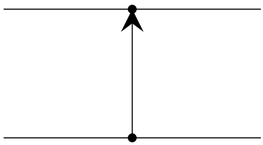

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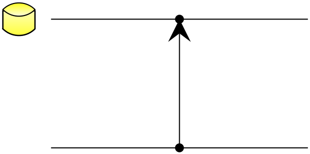

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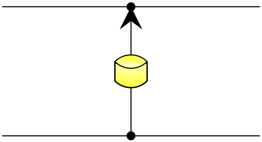

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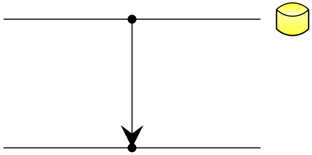

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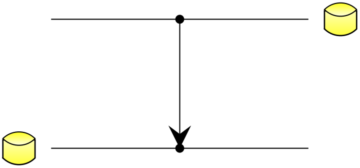

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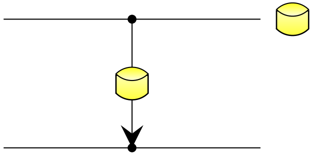

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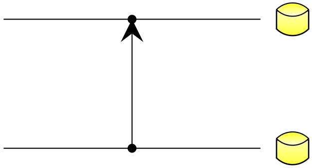

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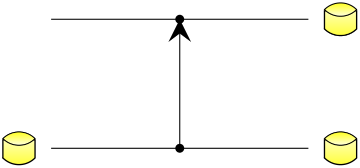

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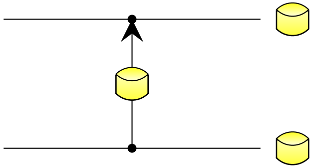

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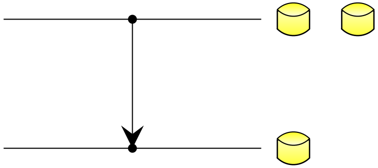

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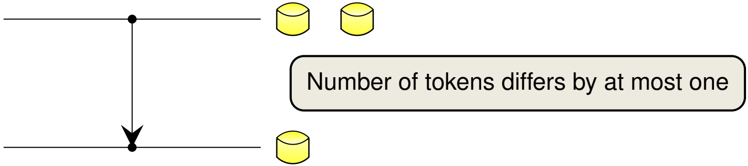

Bitonic Counting Network

Counting Network (Formal Definition)

1. Let $x_1, x_2, \ldots, x_n$ be the number of tokens (ever received) on the designated input wires
2. Let $y_1, y_2, \ldots, y_n$ be the number of tokens (ever received) on the designated output wires

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3. In a quiescent state: $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$
4. A counting network is a balancing network with the **step-property:**

$$0 \leq y_i - y_j \leq 1 \text{ for any } i < j.$$

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Bitonic Counting Network: Take Batcher's Sorting Network and replace each comparator by a balancer.

Correctness of the Bitonic Counting Network (non-examinable)

Facts

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have the step property. Then:

1. We have $\sum_{i=1}^{n/2} x_{2i-1} = \left\lceil \frac{1}{2} \sum_{i=1}^{n} x_i \right\rceil$, and $\sum_{i=1}^{n/2} x_{2i} = \left\lfloor \frac{1}{2} \sum_{i=1}^{n} x_i \right\rfloor$
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Key Lemma

Consider a MERGER$[n]$. Then if the inputs $x_1, \ldots, x_{n/2}$ and $x_{n/2+1}, \ldots, x_n$ have the step property, then so does the output $y_1, \ldots, y_n$.

Proof (by induction on n being a power of 2)

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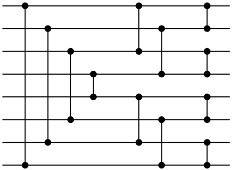

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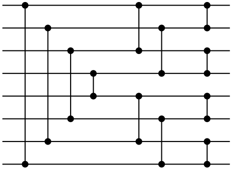

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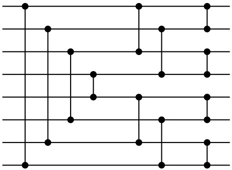

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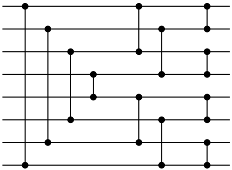

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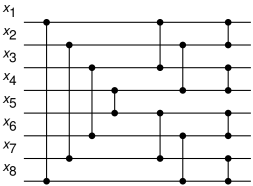

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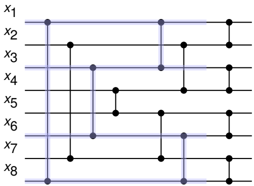

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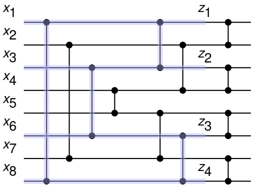

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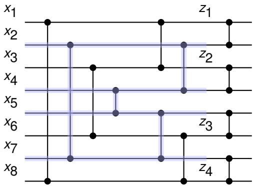

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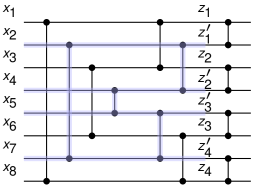
Correctness of the Bitonic Counting Network (non-examinable)

Facts

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have the step property. Then:

1. We have $\sum_{i=1}^{n/2} x_{2i-1} = \left\lceil \frac{1}{2} \sum_{i=1}^{n} x_i \right\rceil$, and $\sum_{i=1}^{n/2} x_{2i} = \left\lfloor \frac{1}{2} \sum_{i=1}^{n} x_i \right\rfloor$
2. If $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, then $x_i = y_i$ for $i = 1, \ldots, n$.
3. If $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i + 1$, then $\exists!\ j = 1, 2, \ldots, n$ with $x_j = y_j + 1$ and $x_i = y_i$ for $j \neq i$.



Proof (by induction on n being a power of 2)

- Case $n = 2$ is clear, since MERGER[2] is a single balancer
- $n > 2$: Let $z_1, \ldots, z_{n/2}$ and $z_1', \ldots, z_{n/2}'$ be the outputs of the MERGER[$n/2$] subnetworks
- IH $\Rightarrow z_1, \ldots, z_{n/2}$ and $z_1', \ldots, z_{n/2}'$ have the step property

Correctness of the Bitonic Counting Network (non-examinable)

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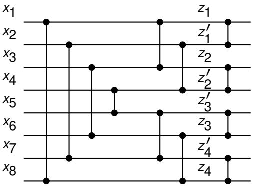

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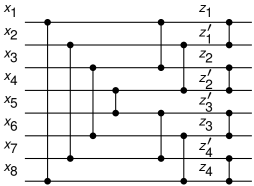

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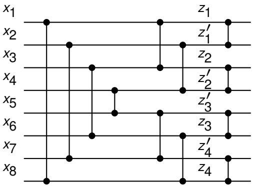

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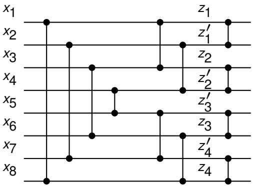

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- Case 1: If $Z = Z'$, then F2 implies the output of MERGER[n] is $y_i = z_{1+\lfloor (i-1)/2 \rfloor}$ ✓

Correctness of the Bitonic Counting Network (non-examinable)

Facts

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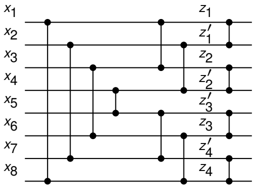

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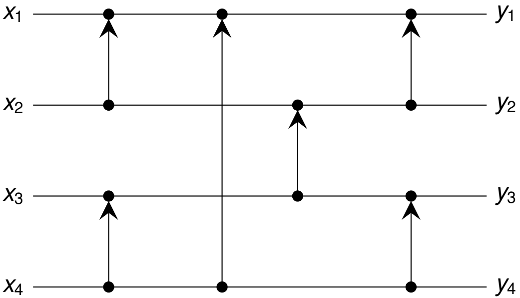
- Case 1: If $Z = Z'$, then F2 implies the output of MERGER[n] is $y_i = z_{1+\lfloor (i-1)/2 \rfloor}$ ✓
- Case 2: If $|Z - Z'| = 1$, F3 implies $z_i = z'_i$ for $i = 1, \ldots, n/2$ except a unique j with $z_j \neq z'_j$.
  Balancer between z_j and z'_j will ensure that the step property holds.

Bitonic Counting Network in Action (Asychnronous Execution)



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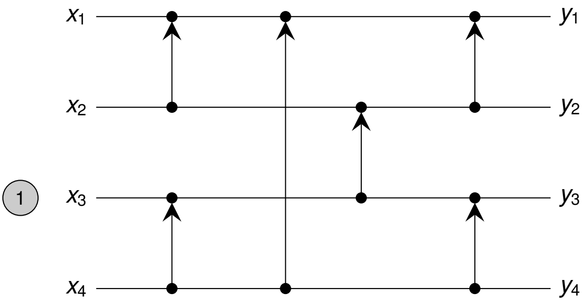

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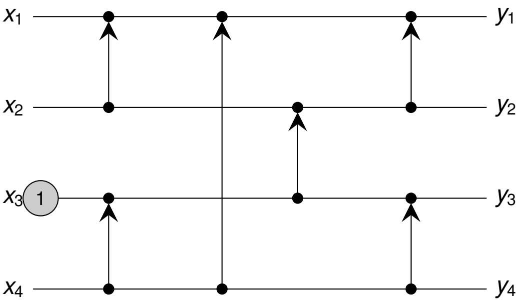

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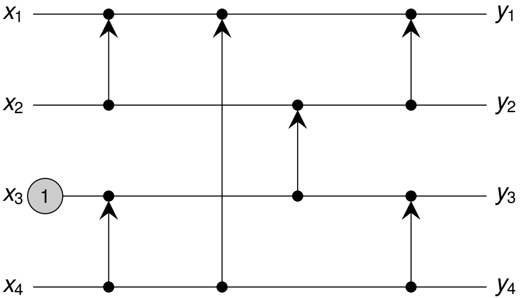

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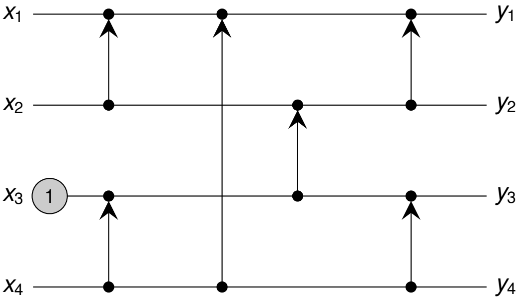

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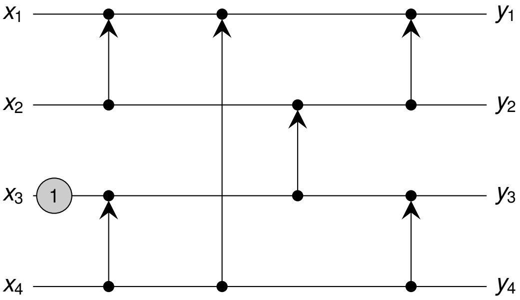

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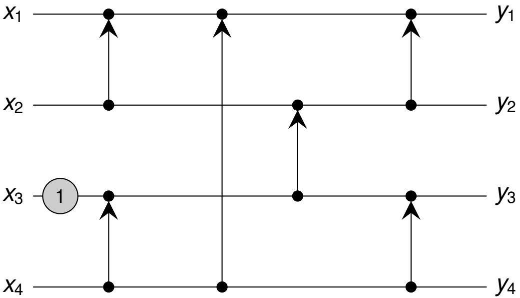

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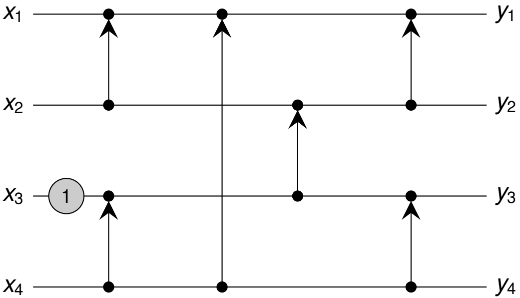

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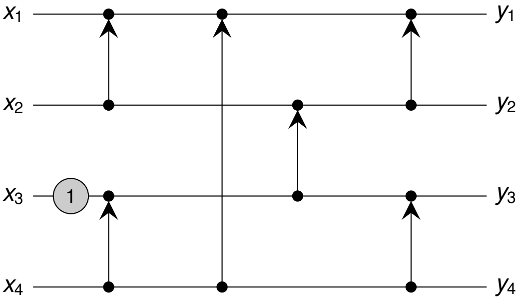

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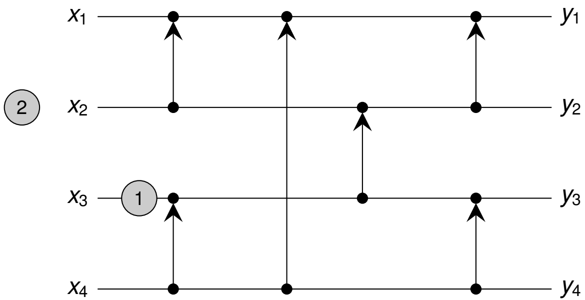

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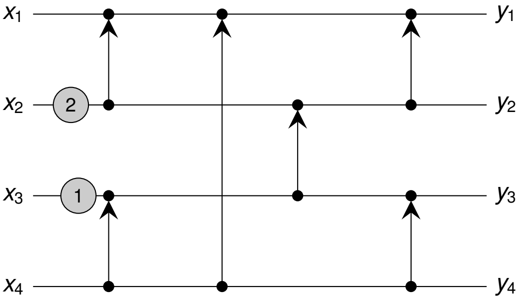

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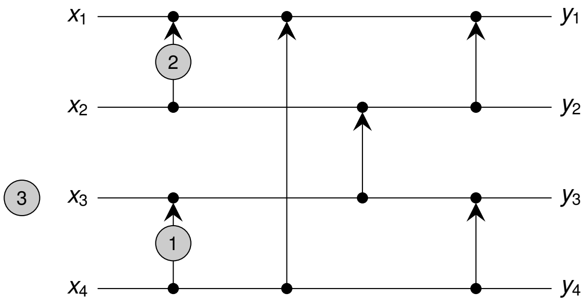

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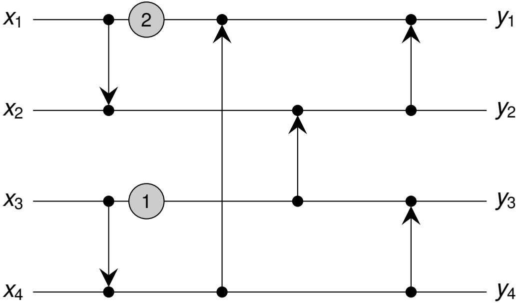

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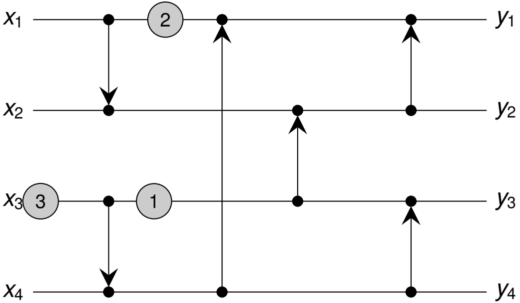

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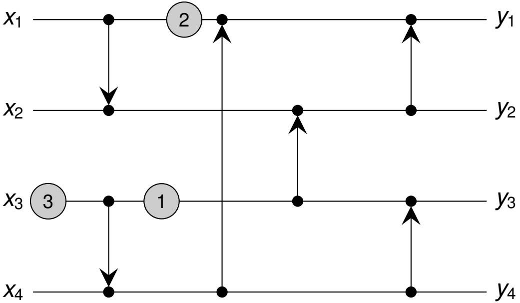

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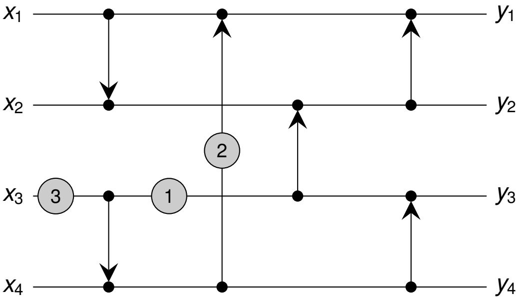

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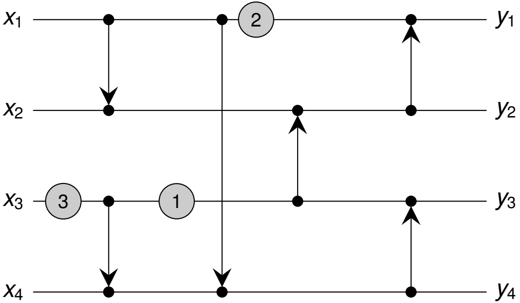

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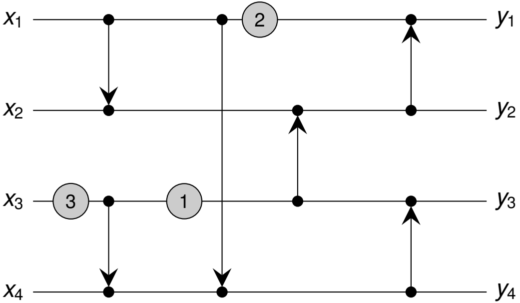

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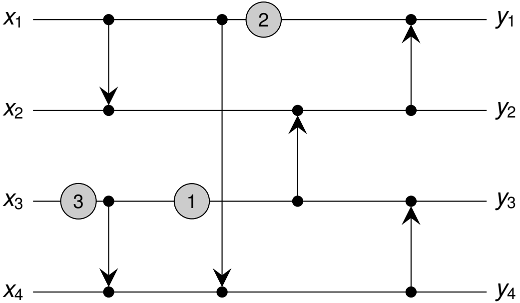

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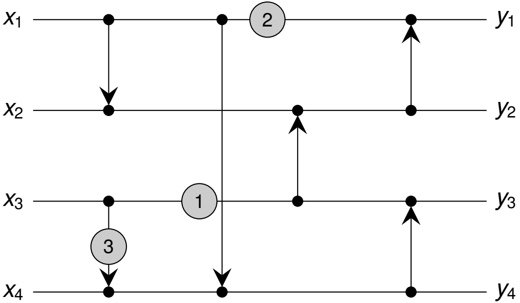

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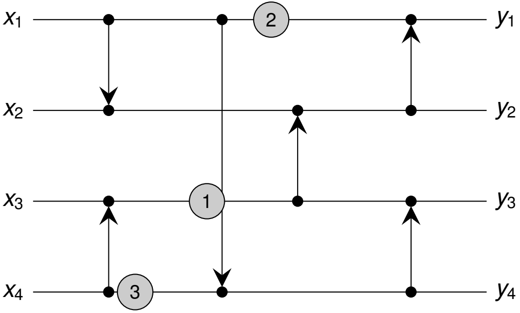

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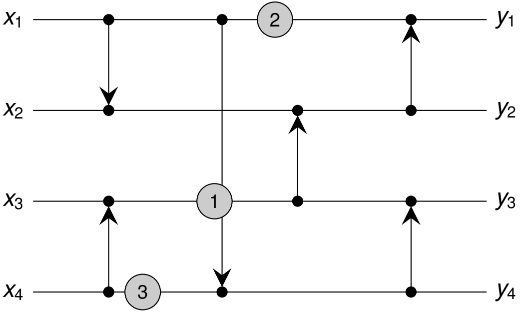

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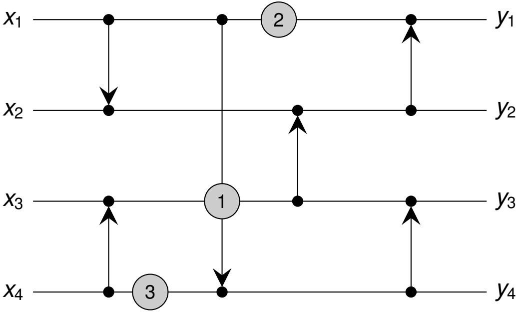

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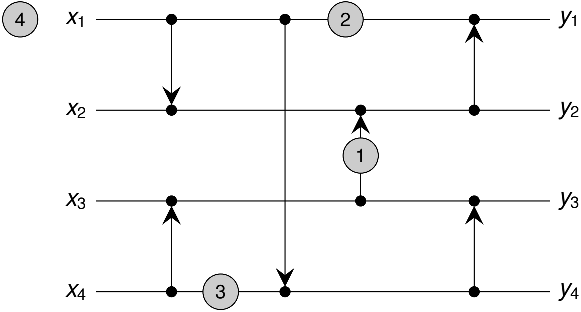

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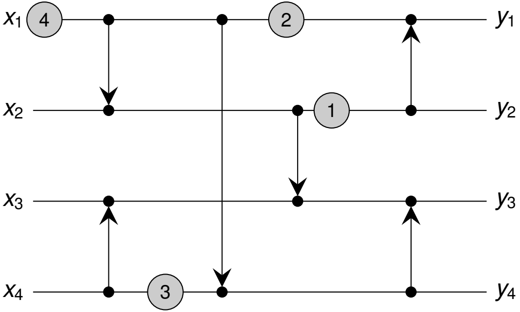

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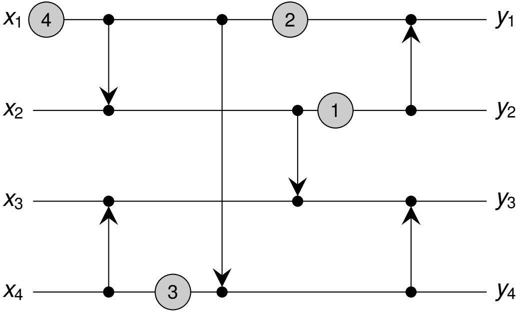

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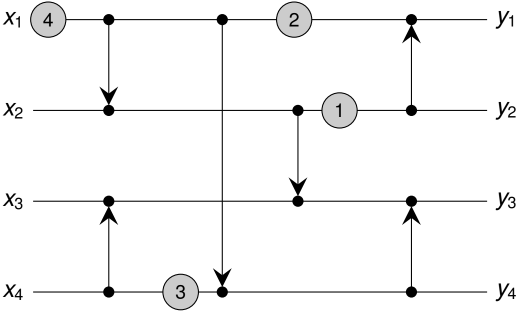

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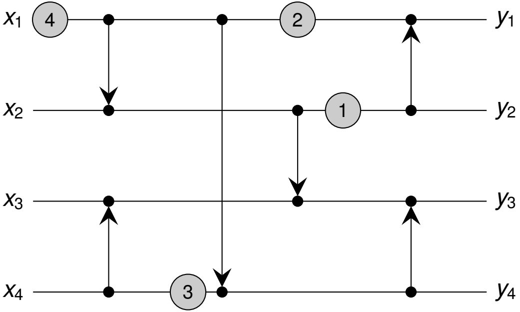

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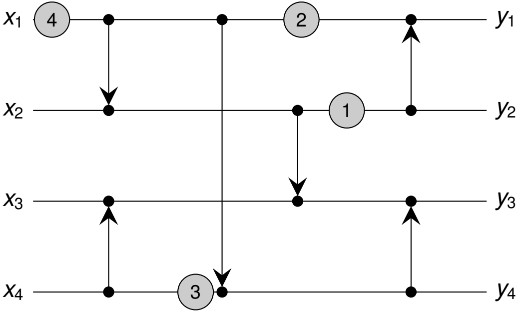

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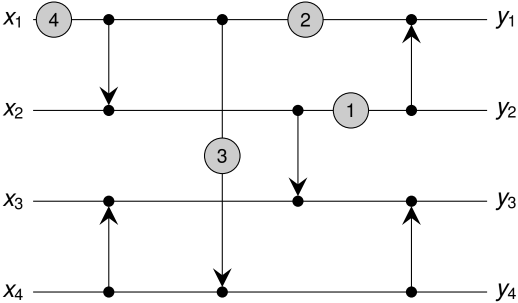

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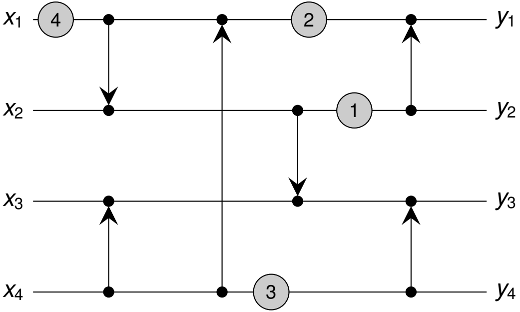

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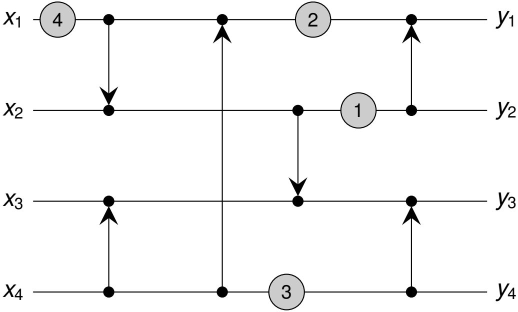

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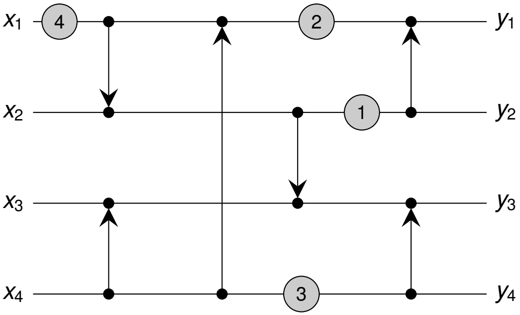

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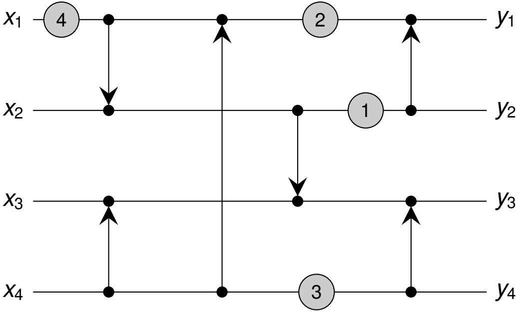

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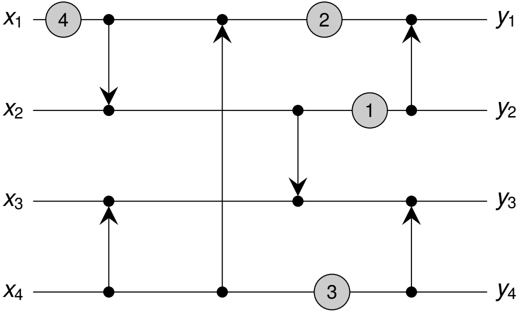

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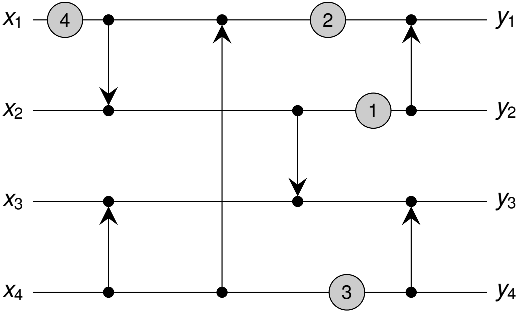

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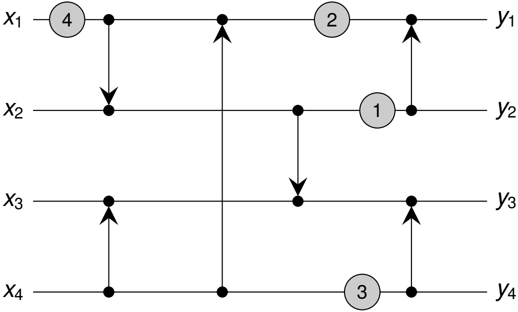

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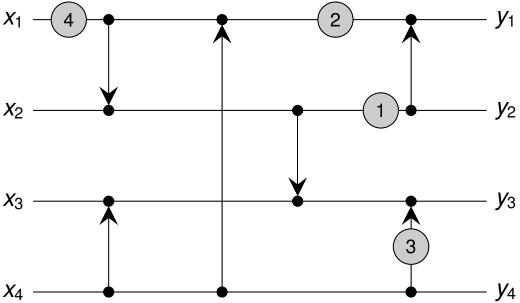

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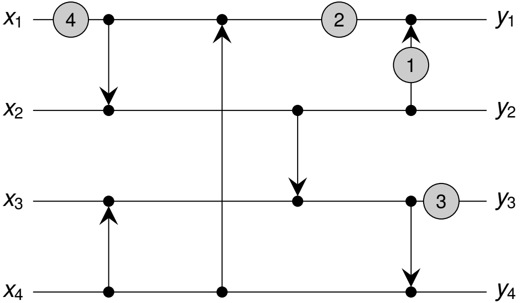

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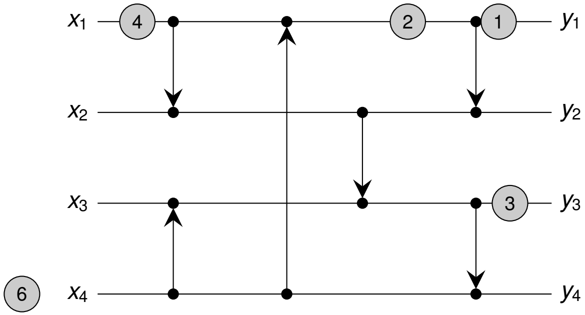

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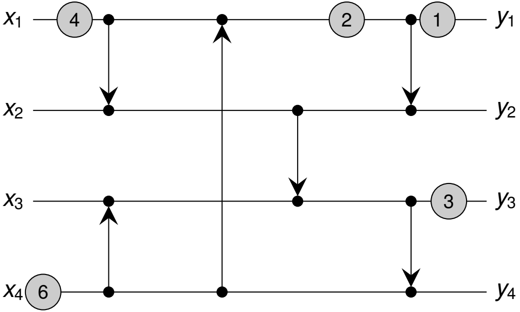

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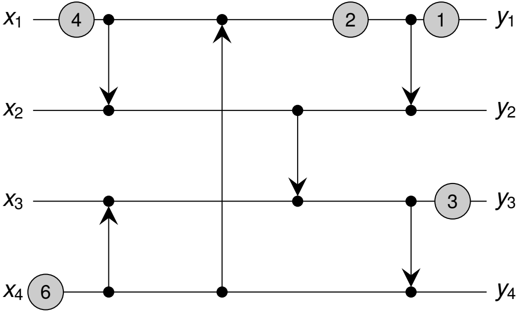

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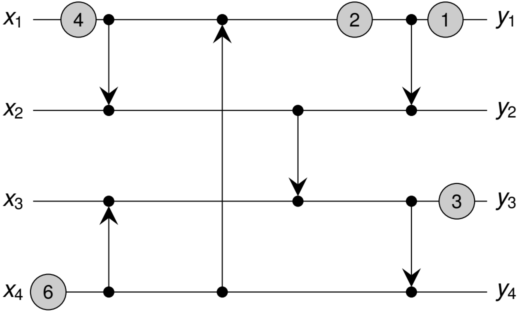

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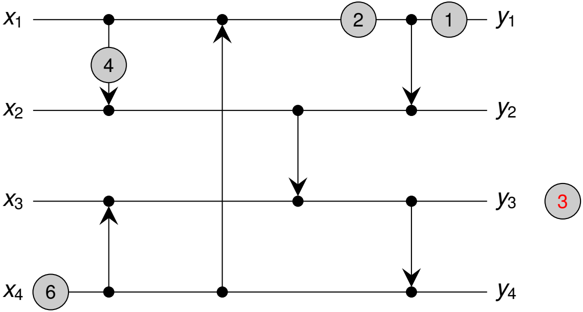

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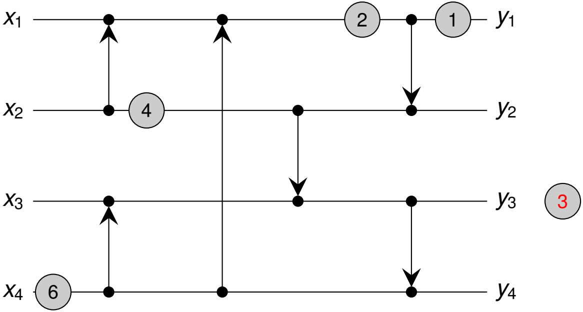

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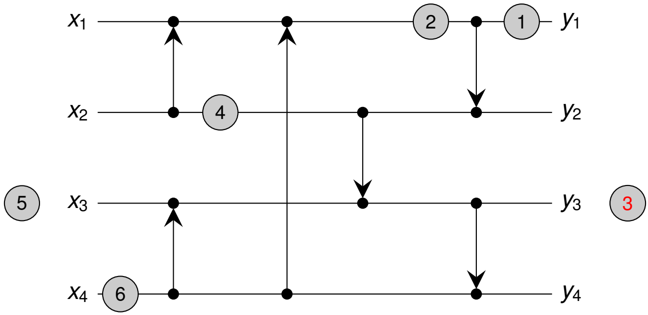

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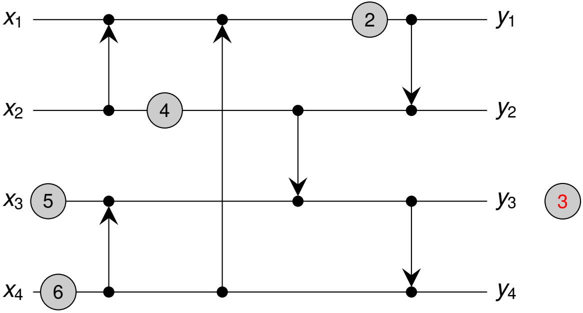

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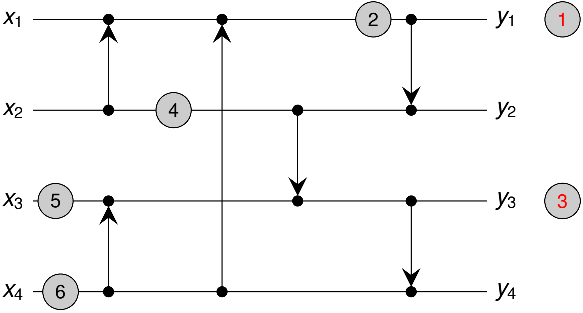

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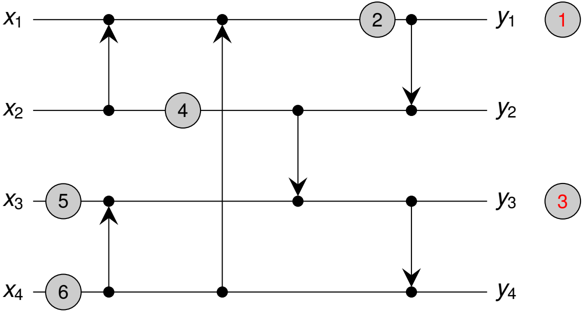

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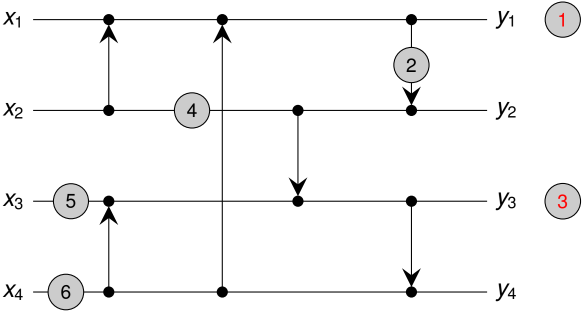

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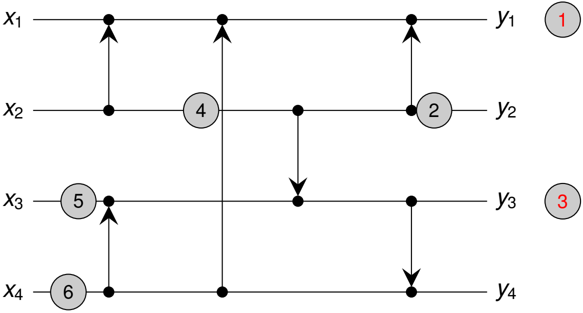

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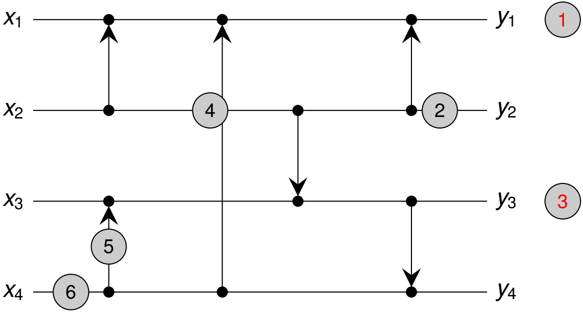

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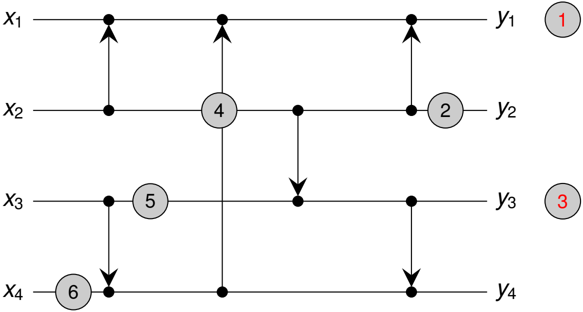

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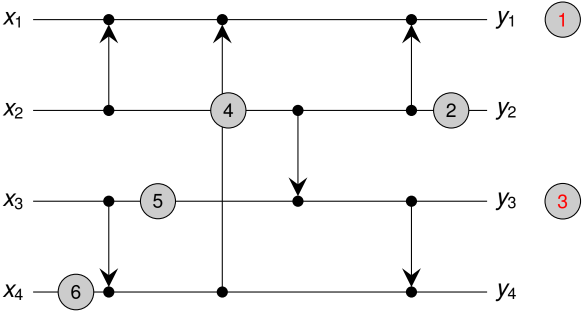

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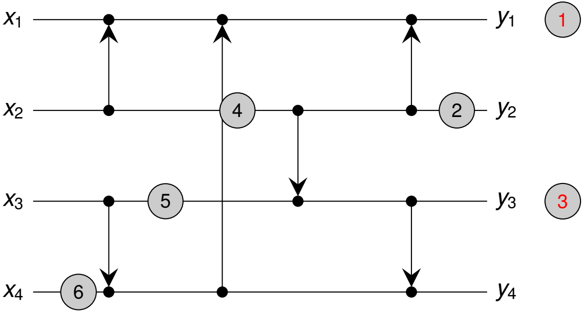

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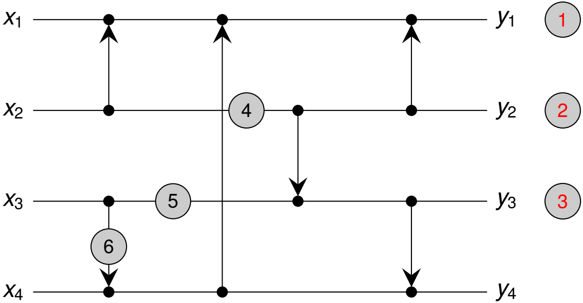

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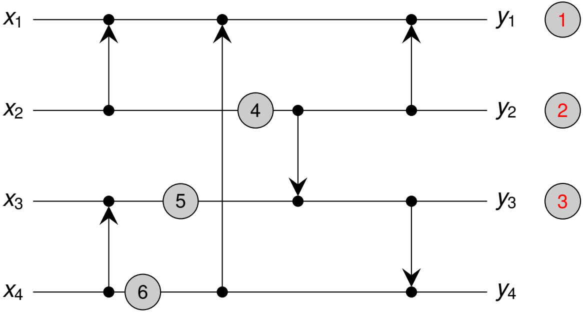

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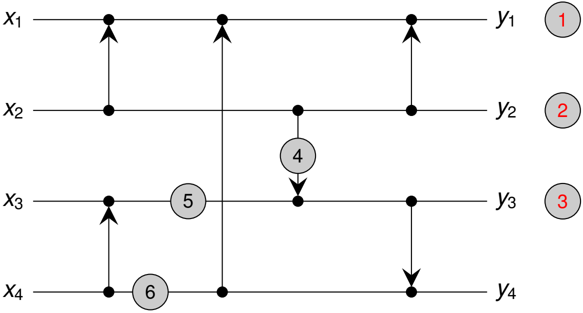

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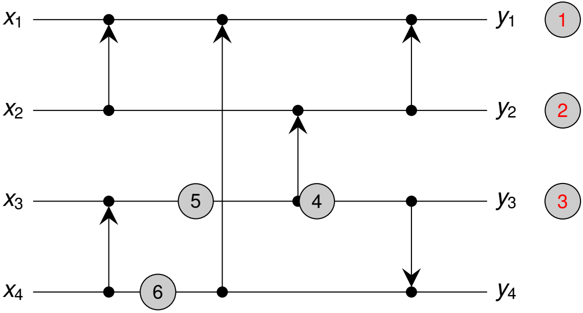

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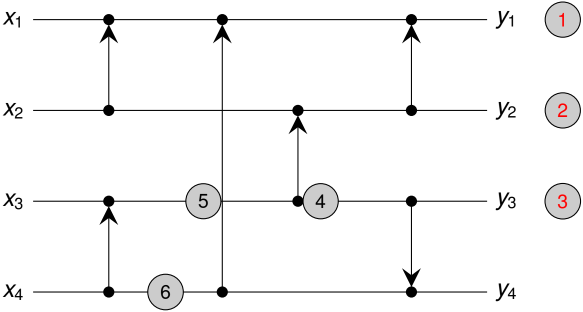

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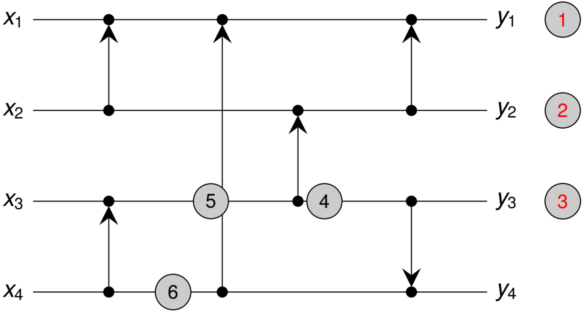

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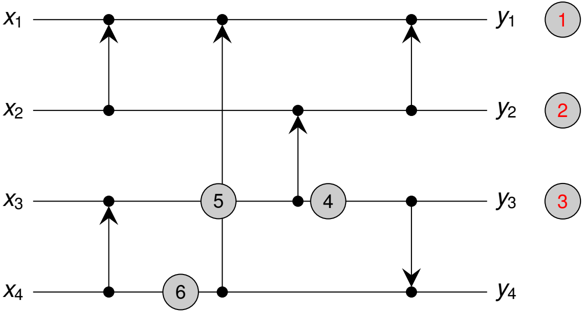

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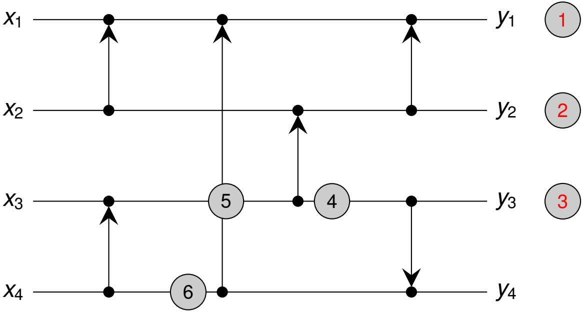

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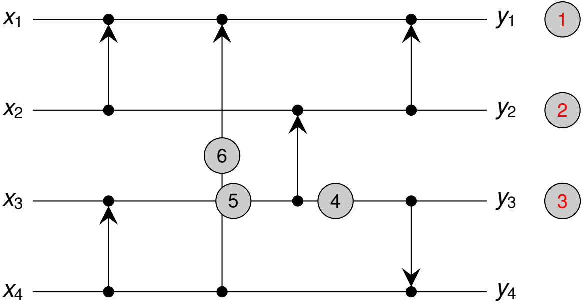

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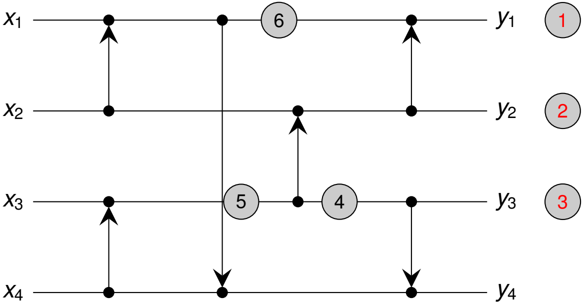

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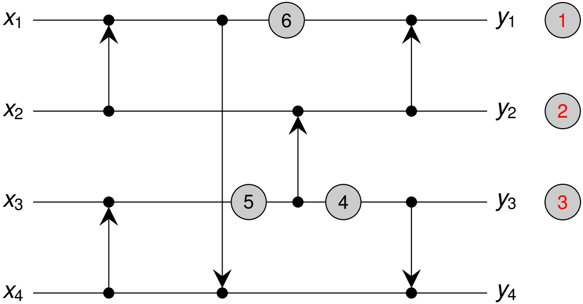

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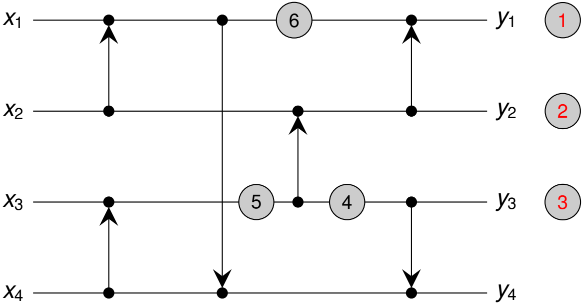

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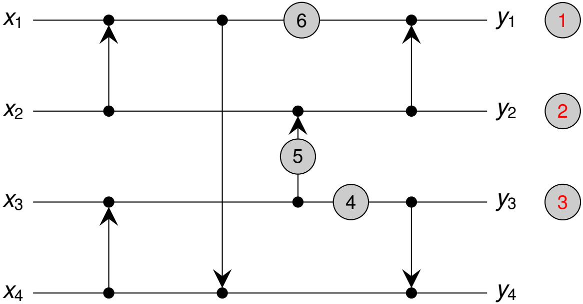

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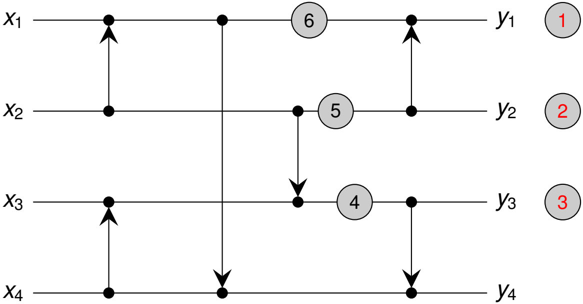

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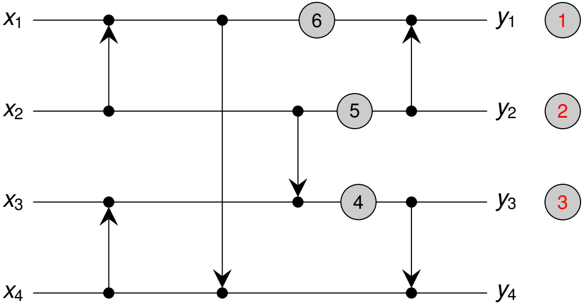

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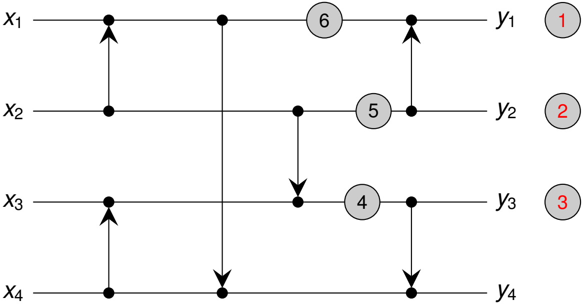

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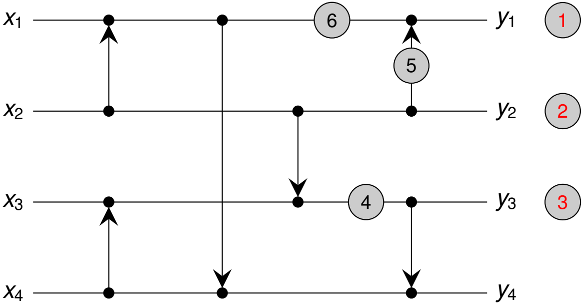

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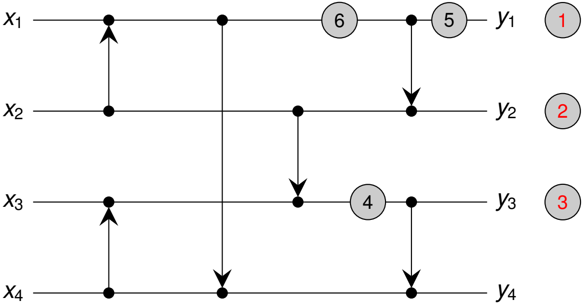

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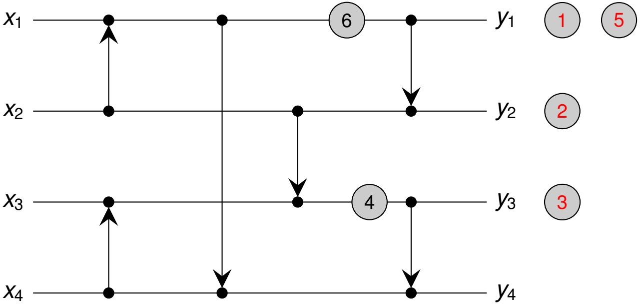

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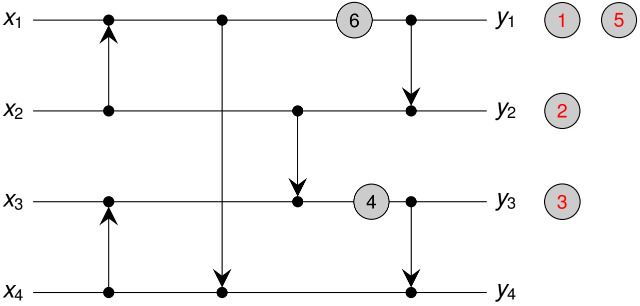

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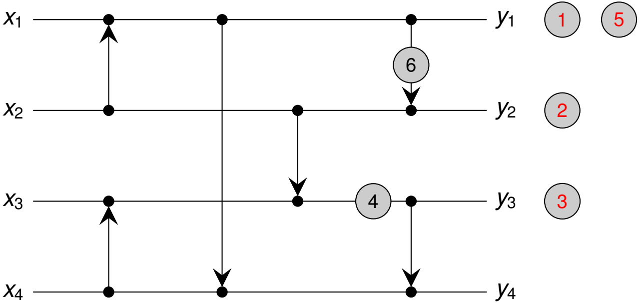

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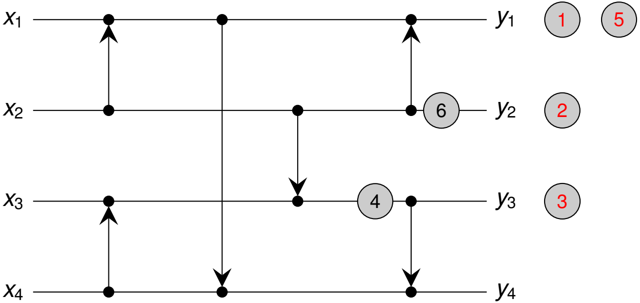

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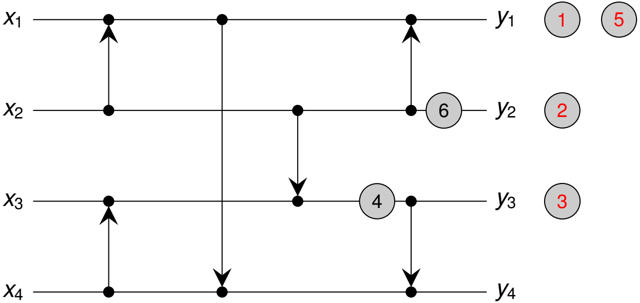

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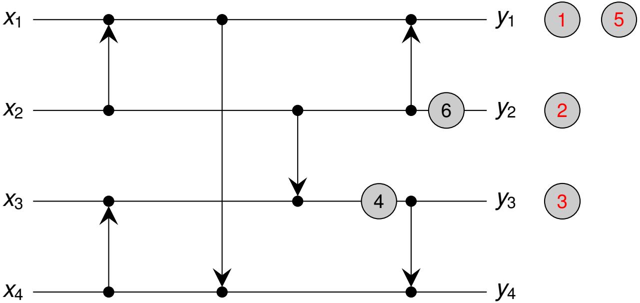

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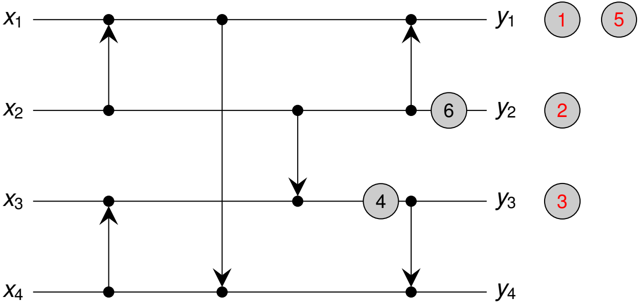

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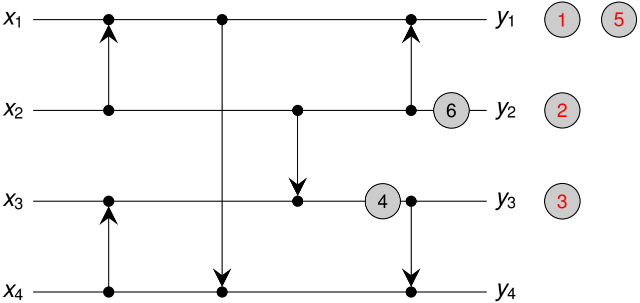

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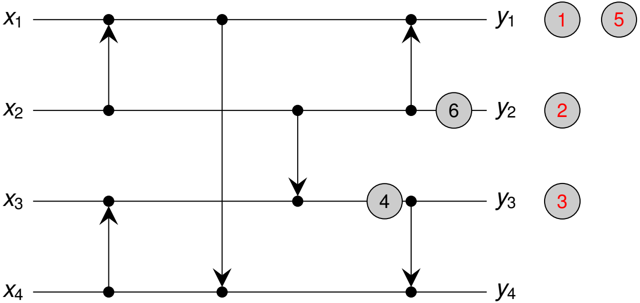

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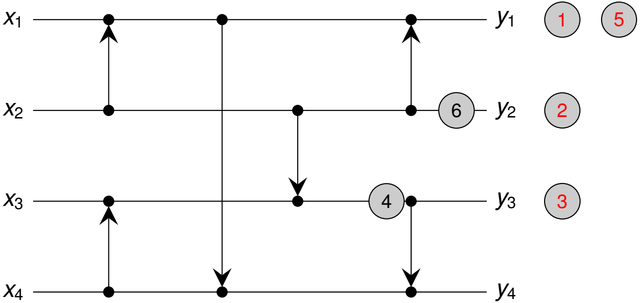

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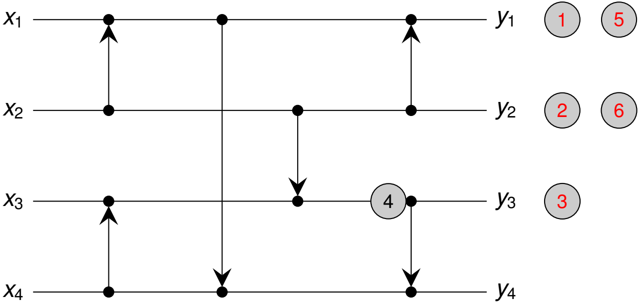

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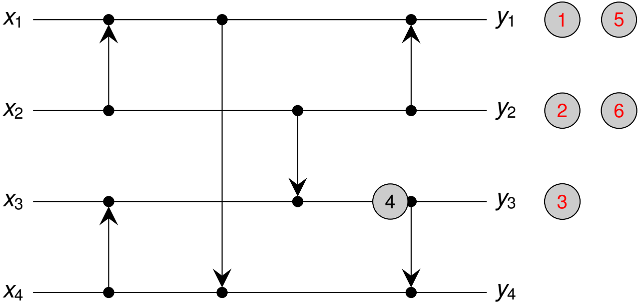

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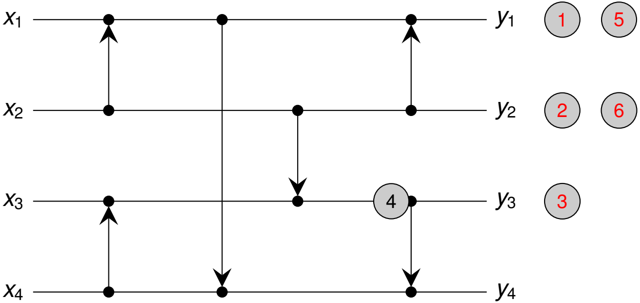

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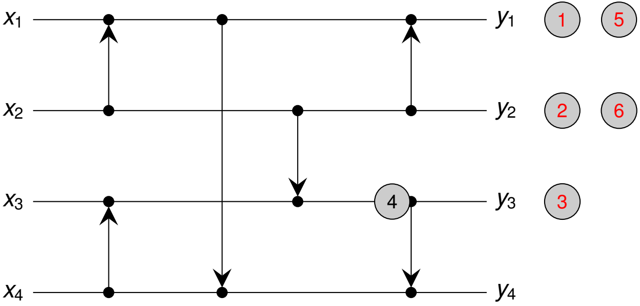

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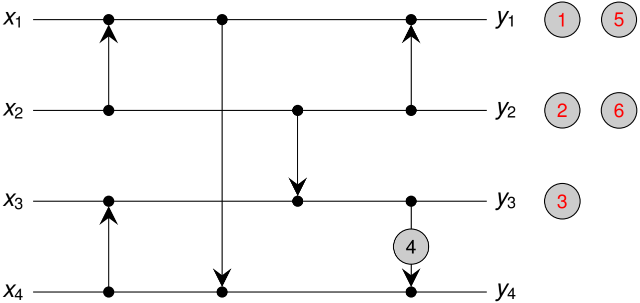

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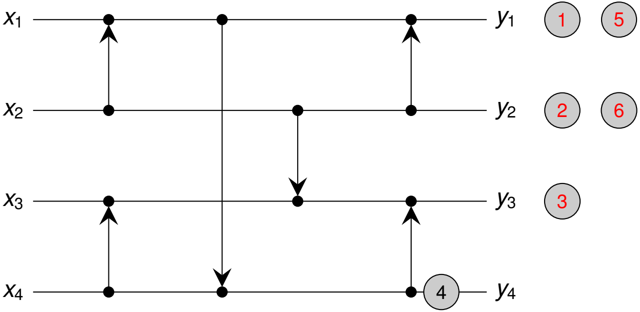

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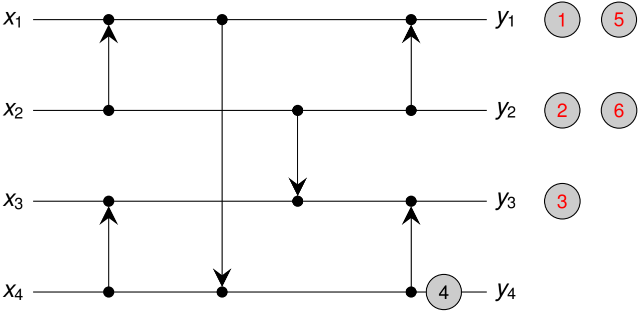

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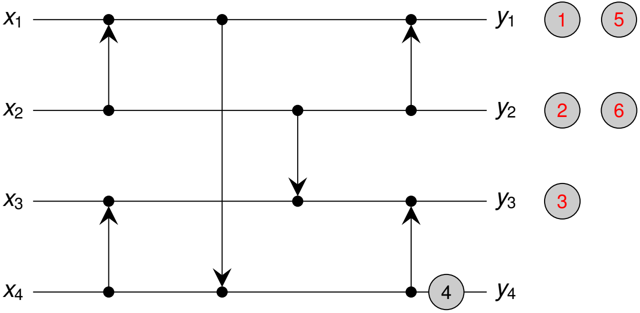

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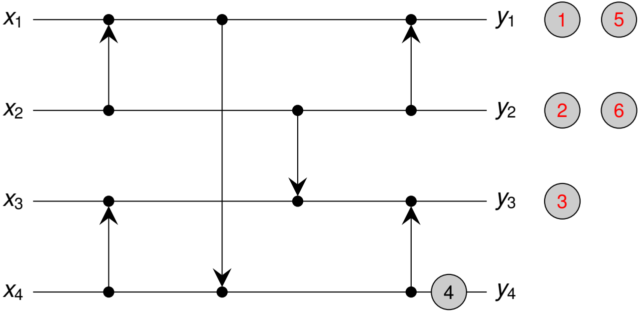

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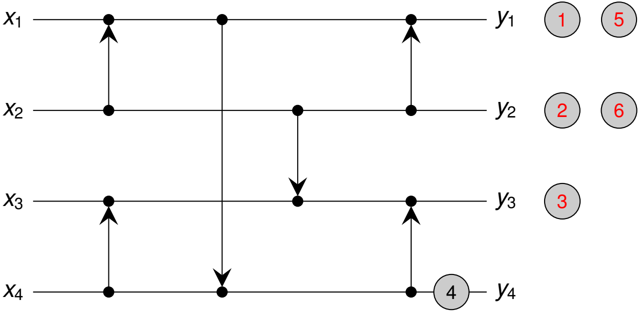

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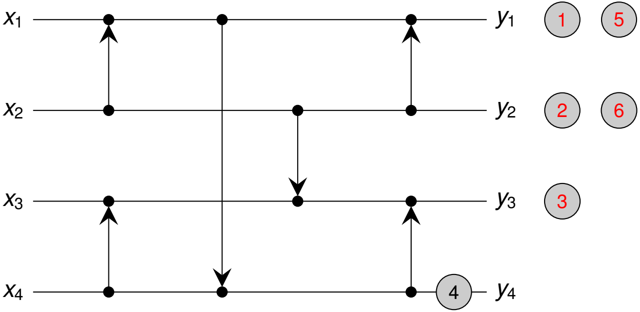

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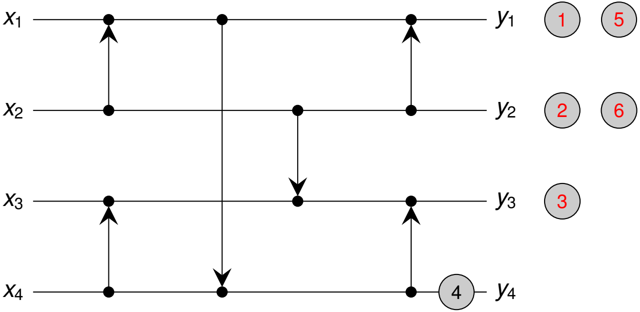

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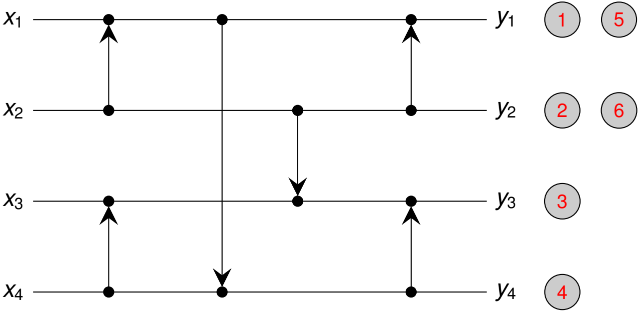

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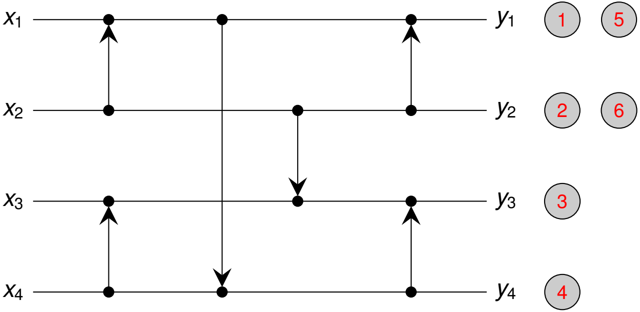
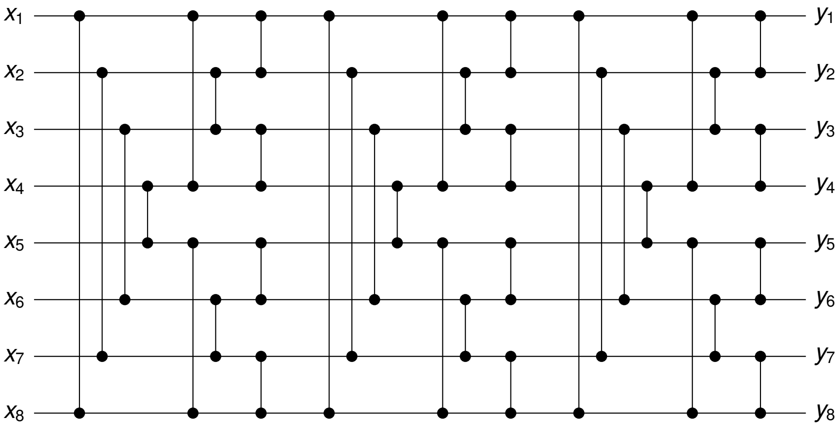


Counting can be done as follows:
Add **local counter** to each output wire i, to assign consecutive numbers $i, i+n, i+2 \cdot n, \ldots$

A Periodic Counting Network [Aspnes, Herlihy, Shavit, JACM 1994]



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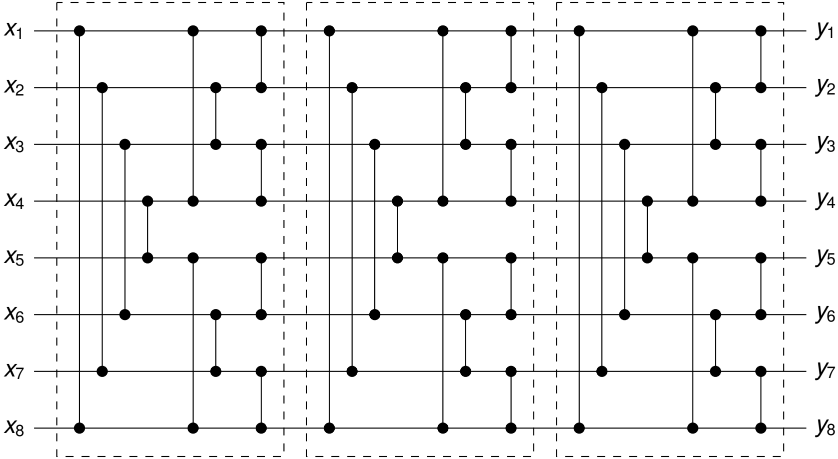


Consists of $\log n$ BLOCK[n] networks each of which has depth $\log n$

From Counting to Sorting

Counting vs. Sorting

If a network is a counting network, then it is also a sorting network.

From Counting to Sorting

The converse is not true!

Counting vs. Sorting

If a network is a counting network, then it is also a sorting network.

From Counting to Sorting

Counting vs. Sorting

If a network is a counting network, then it is also a sorting network.

Proof.

From Counting to Sorting

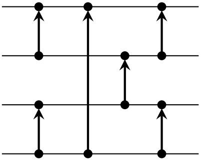
Counting vs. Sorting

If a network is a counting network, then it is also a sorting network.

Proof.

- Let C be a counting network, and S be the corresponding sorting network

C



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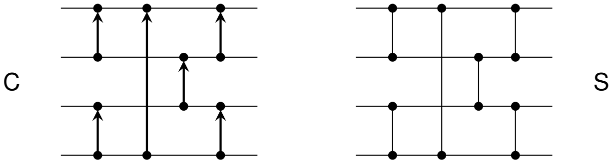

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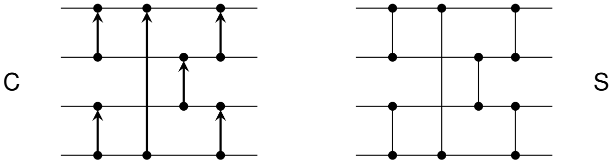

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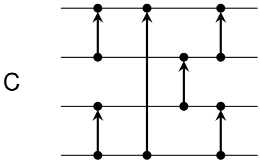

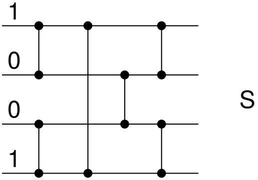

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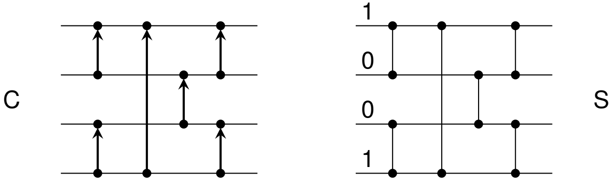

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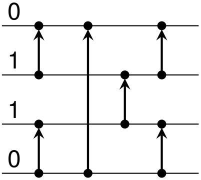

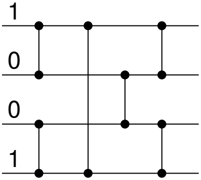

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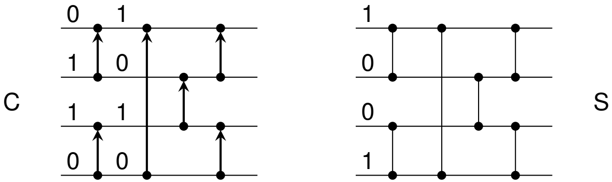

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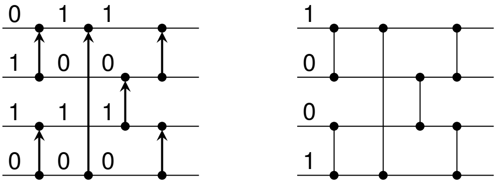

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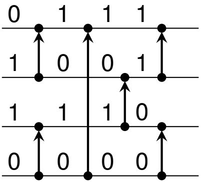

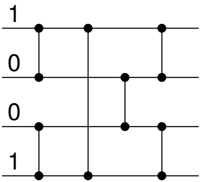

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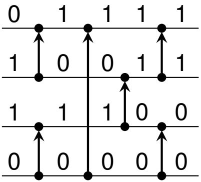

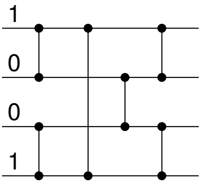

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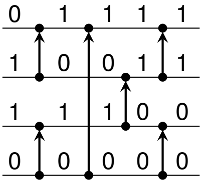

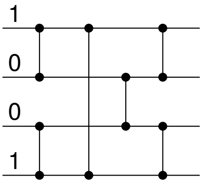

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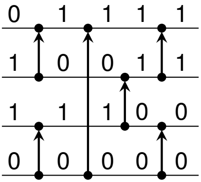

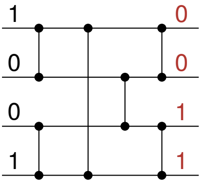

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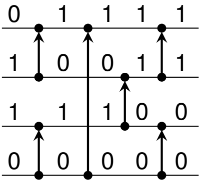
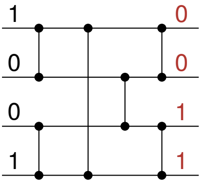
- C is a counting network $\Rightarrow$ all ones will be routed to the lower wires
- S corresponds to C $\Rightarrow$ all zeros will be routed to the lower wires
- By the Zero-One Principle, S is a sorting network. $\square$

C





S

Exercise: Consider a network which is a sorting network, but not a counting network.
Hint: Try to find a simple network with 4 wires that corresponds to a basic sequential sorting algorithm.

II. Linear Programming

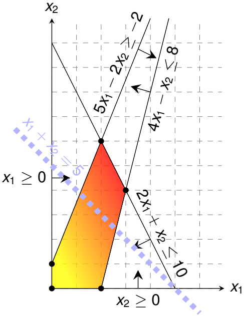
Thomas Sauerwald

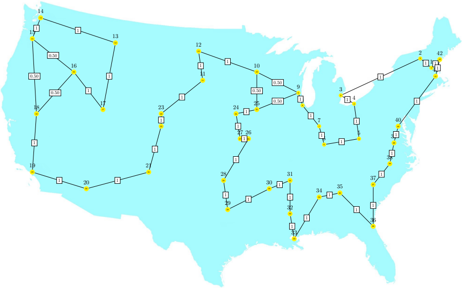
Easter 2021

UNIVERSITY OF CAMBRIDGE

Outline

Introduction





- linear programming is a powerful tool in optimisation
- inspired more sophisticated techniques such as quadratic optimisation, convex optimisation, integer programming and semi-definite programming
- we will later use the connection between linear and integer programming to tackle several problems (Vertex-Cover, Set-Cover, TSP, satisfiability)

What are Linear Programs?

Linear Programming (informal definition)

- maximize or minimize an objective, given limited resources and competing constraint
- constraints are specified as (in)equalities

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Example: Political Advertising (from CLRS3)

- Imagine you are a politician trying to win an election
- Your district has three different types of areas: Urban, suburban and rural, each with, respectively, 100,000, 200,000 and 50,000 registered voters

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- Aim: at least half of the registered voters in each of the three regions should vote for you
- Possible Actions: Advertise on one of the primary issues which are (i) building more roads, (ii) gun control, (iii) farm subsidies and (iv) a gasoline tax dedicated to improve public transit.

Political Advertising Continued

policy	urban	suburban	rural
build roads	-2	5	3
gun control	8	2	-5
farm subsidies	0	0	10
gasoline tax	10	0	-2

The effects of policies on voters. Each entry describes the number of thousands of voters who could be won (lost) over by spending $1,000 on advertising support of a policy on a particular issue.

Political Advertising Continued

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- Possible Solution:
  - $20,000 on advertising to building roads
  - $0 on advertising to gun control
  - $4,000 on advertising to farm subsidies
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What is the best possible strategy?

Towards a Linear Program

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Constraints:

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Constraints:

- $-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50$

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- $-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50$
- $5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100$
- $3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25$

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Objective: Minimize $x_1 + x_2 + x_3 + x_4$

The Linear Program

Linear Program for the Advertising Problem

$$
\begin{array}{lrcrcrcrcr}
\text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
\text{subject to} & & & & & & & & & \\
 & -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
 & 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\
 & 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\
 & & & \multicolumn{5}{l}{x_1, x_2, x_3, x_4} & \geq & 0
\end{array}
$$

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\end{array}
$$

The solution of this linear program yields the optimal advertising strategy.

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$$\begin{array}{lrcrcrcrcr}
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Formal Definition of Linear Program

- Given $a_1, a_2, \ldots, a_n$ and a set of variables $x_1, x_2, \ldots, x_n$, a linear function f is defined by

$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

The Linear Program

Linear Program for the Advertising Problem

$$
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\text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
\text{subject to} & & & & & & & & & \\
 & -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
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 & 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\
 & \multicolumn{7}{c}{x_1, x_2, x_3, x_4} & \geq & 0
\end{array}
$$

The solution of this linear program yields the optimal advertising strategy.

Formal Definition of Linear Program

- Given $a_1, a_2, \ldots, a_n$ and a set of variables $x_1, x_2, \ldots, x_n$, a linear function f is defined by

$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

- Linear Equality: $f(x_1, x_2, \ldots, x_n) = b$
- Linear Inequality: $f(x_1, x_2, \ldots, x_n) \overset{\geq}{\leq} b$

The Linear Program

Linear Program for the Advertising Problem

$$
\begin{array}{lrcrcrcrcr}
\text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
\text{subject to} & & & & & & & & & \\
 & -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
 & 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\
 & 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\
 & & & x_1, x_2, x_3, x_4 & & & & & \geq & 0
\end{array}
$$

The solution of this linear program yields the optimal advertising strategy.

Formal Definition of Linear Program

- Given $a_1, a_2, \ldots, a_n$ and a set of variables $x_1, x_2, \ldots, x_n$, a linear function f is defined by

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Linear Constraints

The Linear Program

Linear Program for the Advertising Problem

$$
\begin{array}{lrcrcrcrcr}
\text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & & \\
\text{subject to} & & & & & & & & & \\
& -2x_1 & + & 8x_2 & + & 0x_3 & + & 10x_4 & \geq & 50 \\
& 5x_1 & + & 2x_2 & + & 0x_3 & + & 0x_4 & \geq & 100 \\
& 3x_1 & - & 5x_2 & + & 10x_3 & - & 2x_4 & \geq & 25 \\
& & & x_1, x_2, x_3, x_4 & & & & & \geq & 0
\end{array}
$$

The solution of this linear program yields the optimal advertising strategy.

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- Given $a_1, a_2, \ldots, a_n$ and a set of variables $x_1, x_2, \ldots, x_n$, a linear function f is defined by

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Linear Constraints

- Linear-Progamming Problem: either minimize or maximize a linear function subject to a set of linear constraints

A Small(er) Example

$$
\begin{array}{lrcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
 & 4x_1 & - & x_2 & \leq & 8 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
 & 5x_1 & - & 2x_2 & \geq & -2 \\
 & x_1, x_2 & & & \geq & 0
\end{array}
$$

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\end{array}
$$

Any setting of x_1 and x_2 satisfying all constraints is a feasible solution

A Small(er) Example

$$\begin{array}{lrcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
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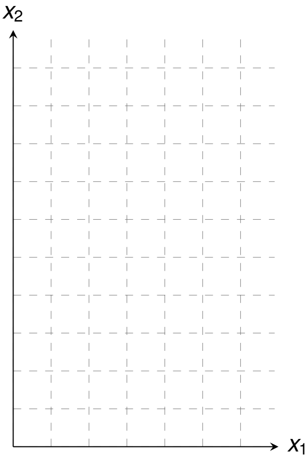

A Small(er) Example

$$\begin{array}{lrcrcr} \text{maximize} & x_1 & + & x_2 & & \\ \text{subject to} & & & & & \\ & 4x_1 & - & x_2 & \leq & 8 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & 5x_1 & - & 2x_2 & \geq & -2 \\ & x_1, x_2 & & & \geq & 0 \end{array}$$

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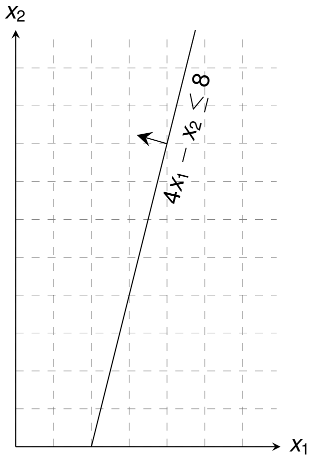

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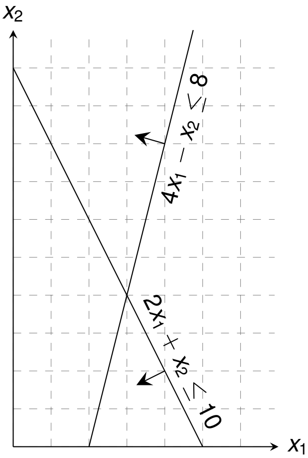

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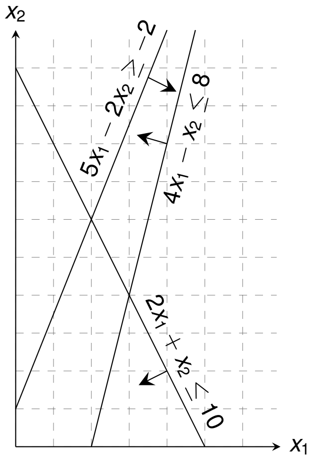

A Small(er) Example

$$
\begin{array}{llcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
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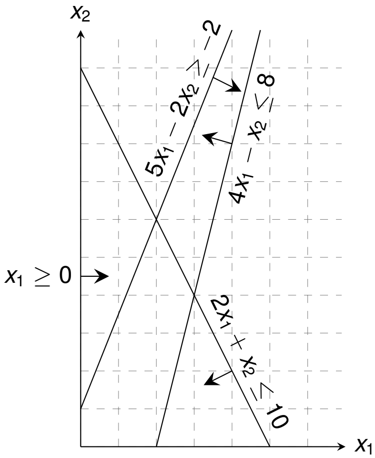

A Small(er) Example

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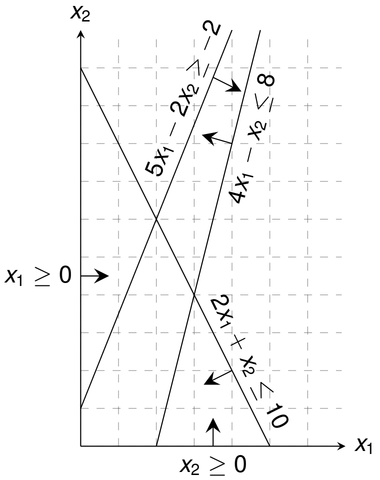

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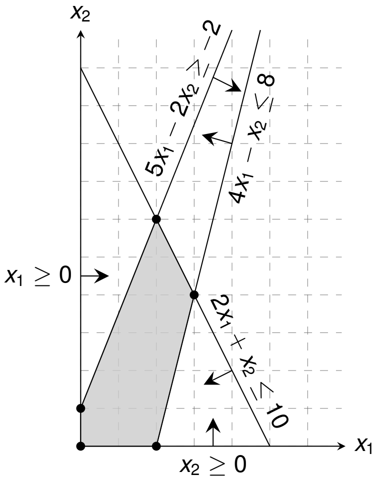
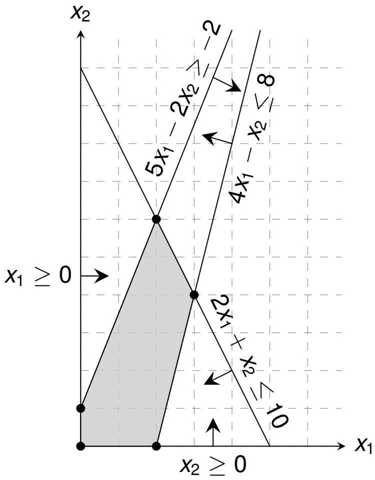


A Small(er) Example

$$\begin{array}{lrcrcr} \text{maximize} & x_1 & + & x_2 & & \\ \text{subject to} & & & & & \\ & 4x_1 & - & x_2 & \leq & 8 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & 5x_1 & - & 2x_2 & \geq & -2 \\ & x_1, x_2 & & & \geq & 0 \end{array}$$

> Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.



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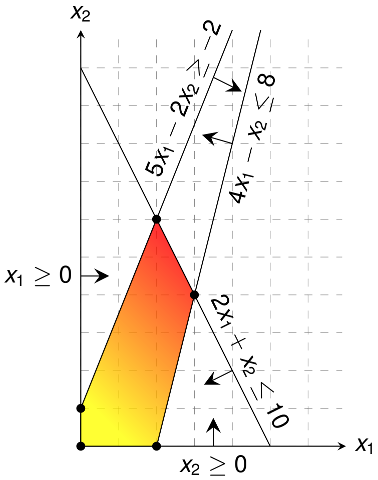
\end{array}$$

Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.



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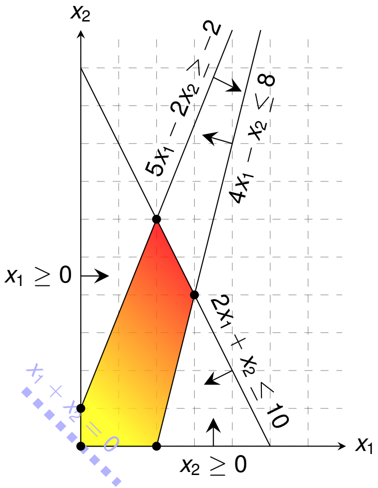

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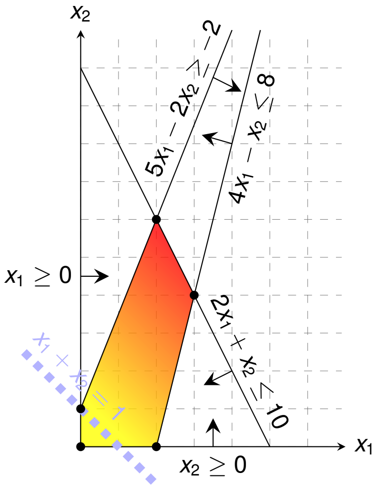

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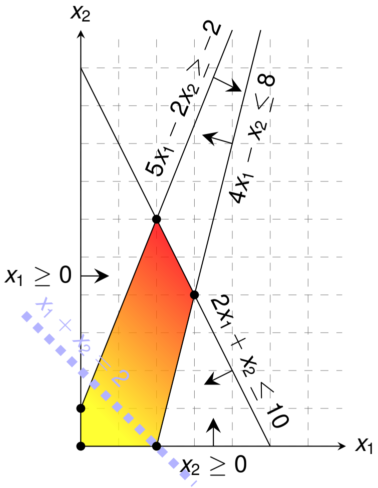

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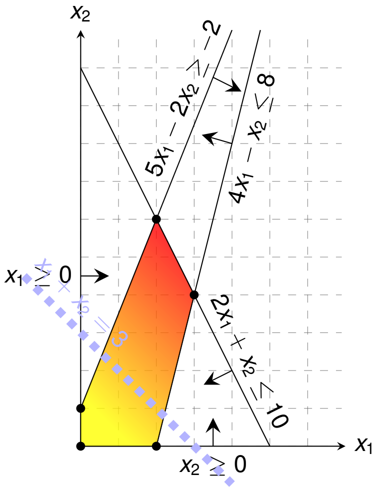

A Small(er) Example

$$
\begin{array}{llcrcr}
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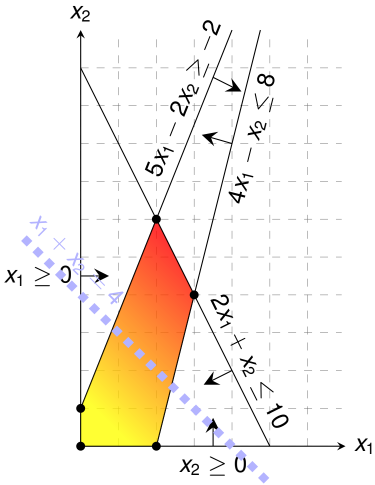

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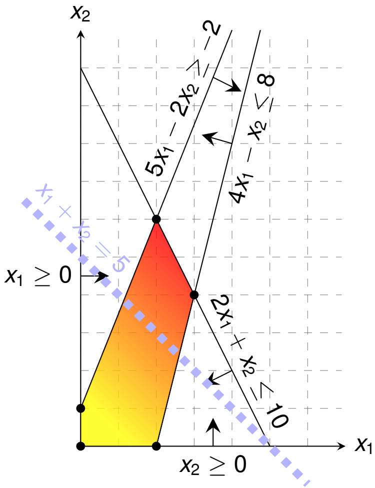

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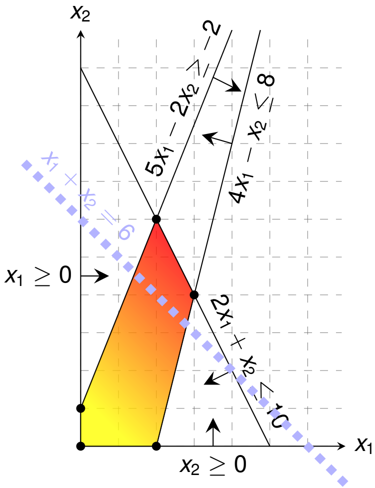

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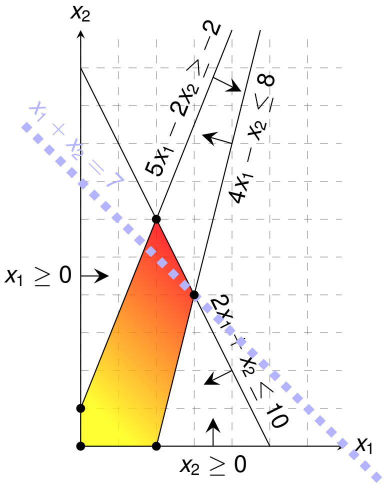

Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.

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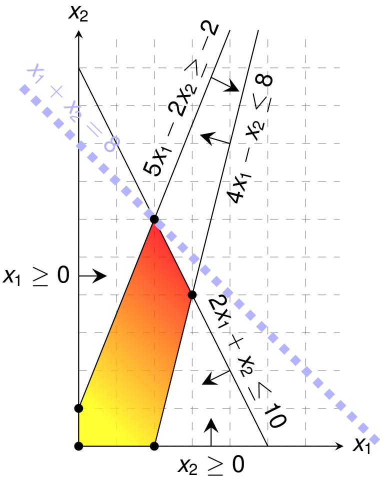

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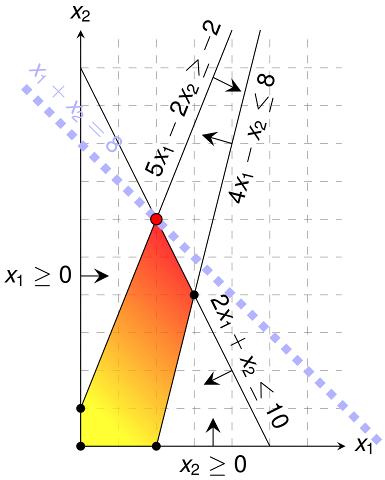

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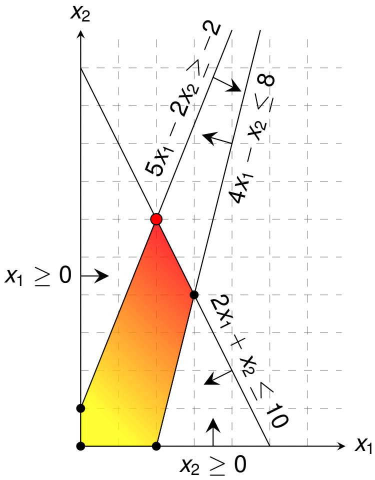

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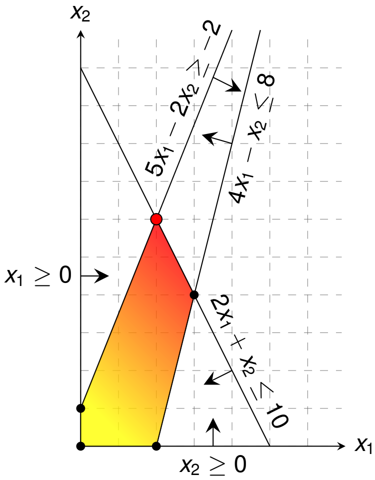


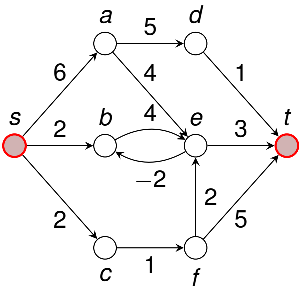
While the same approach also works for higher-dimensions, we need to take a more systematic and algebraic procedure.

Outline

Shortest Paths

Single-Pair Shortest Path Problem

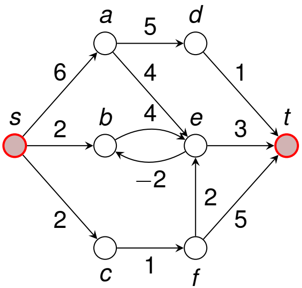
- Given: directed graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}$, pair of vertices $s, t \in V$



Shortest Paths

Single-Pair Shortest Path Problem

- Given: directed graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}$, pair of vertices $s, t \in V$
- Goal: Find a path of minimum weight from s to t in G



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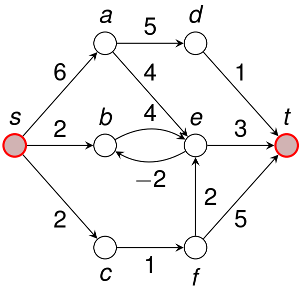

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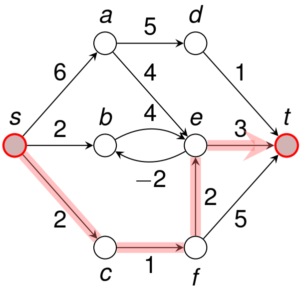

Shortest Paths

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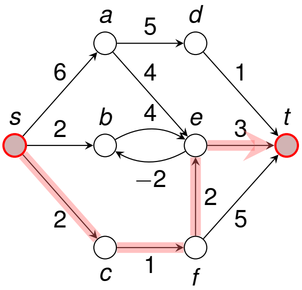

Shortest Paths as LP

subject to

Shortest Paths

Single-Pair Shortest Path Problem

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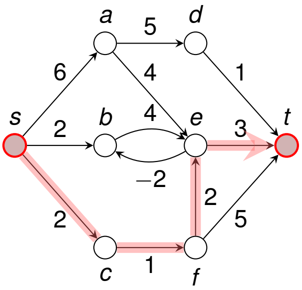


Shortest Paths as LP

subject to

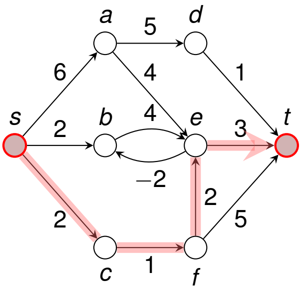
$$\begin{aligned} d_v &\leq d_u + w(u, v) \quad \text{for each edge } (u, v) \in E, \\ d_s &= 0. \end{aligned}$$

Shortest Paths

Single-Pair Shortest Path Problem

- Given: directed graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}$, pair of vertices $s, t \in V$
- Goal: Find a path of minimum weight from s to t in G

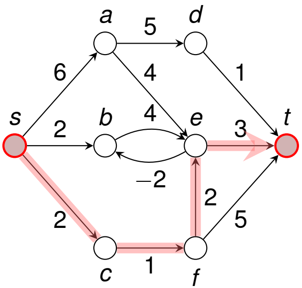
$p = (v_0 = s, v_1, \ldots, v_k = t)$ such that $w(p) = \sum_{i=1}^{k} w(v_{k-1}, v_k)$ is minimized.



Shortest Paths as LP

$$
\begin{array}{llllll}
\text{maximize} & d_t & & & & \\
\text{subject to} & & & & & \\
& d_v & \leq & d_u & + \; w(u,v) & \text{for each edge } (u,v) \in E, \\
& d_s & = & 0. & &
\end{array}
$$

Shortest Paths



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this is a maximization problem!

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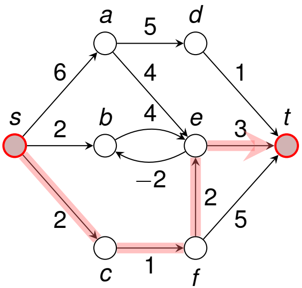

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Recall: When BELLMAN-FORD terminates, all these inequalities are satisfied.

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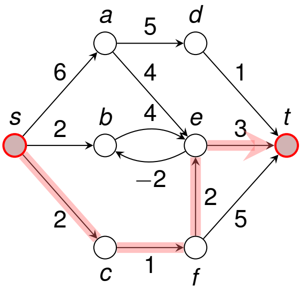

Shortest Paths as LP

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Recall: When Bellman-Ford terminates, all these inequalities are satisfied.

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Solution $\overline{d}$ satisfies $\overline{d}_v = \min_{u:\,(u,v)\in E} \left\{ \overline{d}_u + w(u,v) \right\}$

Maximum Flow

Maximum Flow Problem

- Given: directed graph $G = (V, E)$ with edge capacities $c : E \to \mathbb{R}^+$ (recall $c(u, v) = 0$ if $(u, v) \notin E$), pair of vertices $s, t \in V$

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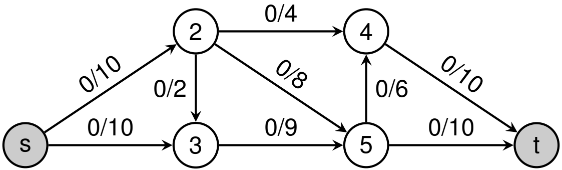

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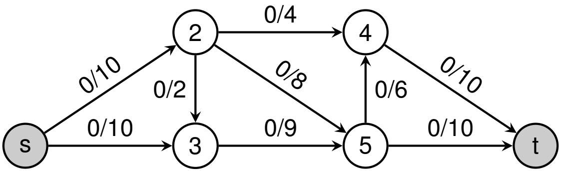

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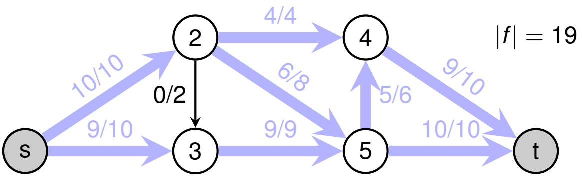

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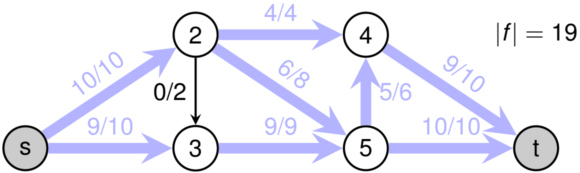

Maximum Flow as LP

$$
\begin{array}{lrcll}
\text{maximize} & \sum_{v \in V} f_{sv} & - & \sum_{v \in V} f_{vs} & \\
\text{subject to} & & & & \\
& f_{uv} & \leq & c(u, v) & \text{for each } u, v \in V, \\
& \sum_{v \in V} f_{vu} & = & \sum_{v \in V} f_{uv} & \text{for each } u \in V \setminus \{s, t\}, \\
& f_{uv} & \geq & 0 & \text{for each } u, v \in V.
\end{array}
$$

Minimum-Cost Flow

Extension of the Maximum Flow Problem

Minimum-Cost-Flow Problem

Minimum-Cost Flow

Extension of the Maximum Flow Problem

Minimum-Cost-Flow Problem

- Given: directed graph $G = (V, E)$ with capacities $c : E \to \mathbb{R}^+$, pair of vertices $s, t \in V$, cost function $a : E \to \mathbb{R}^+$, flow demand of d units

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Extension of the Maximum Flow Problem

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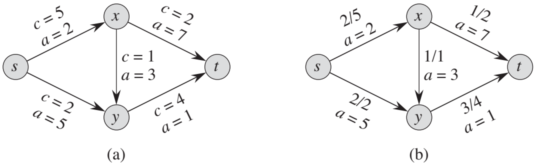

Figure 29.3 **(a)** An example of a minimum-cost-flow problem. We denote the capacities by c and the costs by a. Vertex s is the source and vertex t is the sink, and we wish to send 4 units of flow from s to t. **(b)** A solution to the minimum-cost flow problem in which 4 units of flow are sent from s to t. For each edge, the flow and capacity are written as flow/capacity.

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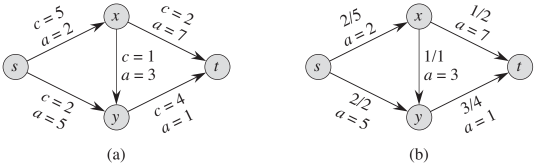
Optimal Solution with total cost:
$\sum_{(u,v)\in E} a(u,v) f_{uv} = (2\cdot2)+(5\cdot2)+(3\cdot1)+(7\cdot1)+(1\cdot3) = 27$



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Minimum-Cost Flow as a LP

Minimum Cost Flow as LP

minimize $\sum_{(u,v)\in E} a(u,v) f_{uv}$

subject to

$$
\begin{array}{rcll}
f_{uv} & \leq & c(u,v) & \text{for each } u, v \in V, \\
\sum_{v\in V} f_{vu} - \sum_{v \in V} f_{uv} & = & 0 & \text{for each } u \in V \setminus \{s,t\}, \\
\sum_{v\in V} f_{sv} - \sum_{v \in V} f_{vs} & = & d\,, & \\
f_{uv} & \geq & 0 & \text{for each } u, v \in V.
\end{array}
$$

Minimum-Cost Flow as a LP

Minimum Cost Flow as LP

$$
\begin{array}{ll}
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\end{array}
$$

$$
\begin{array}{rcll}
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\end{array}
$$

Real power of Linear Programming comes from the ability to solve **new problems**!

Outline

Standard and Slack Forms

Standard Form

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \text{for } i = 1, 2, \ldots, m$$

$$x_j \geq 0 \qquad \text{for } j = 1, 2, \ldots, n$$

Standard and Slack Forms

Standard Form

maximize $\sum_{j=1}^{n} c_j x_j$ — Objective Function

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Standard and Slack Forms

Standard Form

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

Objective Function

subject to

n + m Constraints

$$\left\{ \begin{array}{ll} \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i & \text{for } i = 1, 2, \ldots, m \\ x_j \geq 0 & \text{for } j = 1, 2, \ldots, n \end{array} \right.$$

Standard and Slack Forms

Standard Form

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subject to

$n+m$ Constraints

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Non-Negativity Constraints

Standard and Slack Forms

Standard Form

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

Objective Function

subject to

n + m Constraints

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \text{for } i = 1, 2, \ldots, m$$

$$x_j \geq 0 \qquad \text{for } j = 1, 2, \ldots, n$$

Non-Negativity Constraints

Standard Form (Matrix-Vector-Notation)

$$\text{maximize} \quad c^T x$$

Inner product of two vectors

subject to

$$Ax \leq b$$

Matrix-vector product

$$x \geq 0$$

Converting Linear Programs into Standard Form

Reasons for a LP not being in standard form:

1. The objective might be a minimization rather than maximization.
2. There might be variables without nonnegativity constraints.
3. There might be equality constraints.
4. There might be inequality constraints (with $\geq$ instead of $\leq$).

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Equivalence: a correspondence (not necessarily a bijection) between solutions.

Converting into Standard Form (1/5)

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$$\begin{array}{lrcrcl} \boxed{\text{minimize} \quad -2x_1 + 3x_2} & & & & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0 \end{array}$$

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Negate objective function

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\end{array}$$

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$$\boxed{\text{maximize} \quad 2x_1 - 3x_2}$$

$$\begin{array}{lrcrcr}
\text{subject to} & & & & & \\
& x_1 & + & x_2 & = & 7 \\
& x_1 & - & 2x_2 & \leq & 4 \\
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Replace x_2 by two non-negative variables x_2' and x_2''

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\text{subject to} & & & & & & \\
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& x_1 & - & 2x_2 & \leq & 4 \\
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\end{array}$$

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$$\begin{array}{llcllcl}
\text{maximize} & 2x_1 & - & \boxed{3x_2' + 3x_2''} & & \\
\text{subject to} & & & & & \\
& x_1 & + & \boxed{x_2' - x_2''} & = & 7 \\
& x_1 & - & \boxed{2x_2' + 2x_2''} & \leq & 4 \\
& x_1, x_2', x_2'' & & & \geq & 0
\end{array}$$

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\begin{array}{lrcrcrcr}
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\text{subject to} & & & & & & & \\
 & \boxed{x_1 & + & x_2' & - & x_2'' & = & 7} \\
 & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
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& \boxed{x_1 & + & x_2' & - & x_2'' & = & 7} \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
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\end{array}
$$

Replace each equality by two inequalities.

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\begin{array}{llcccccccc}
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\text{subject to} & & & & & & & \\
& \boxed{x_1 & + & x_2' & - & x_2'' & = & 7} \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& \multicolumn{5}{l}{x_1, x_2', x_2''} & \geq & 0
\end{array}
$$

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$$
\begin{array}{llcccccccc}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& \multicolumn{5}{l}{x_1, x_2', x_2''} & \geq & 0
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$$

Converting into Standard Form (4/5)

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$$
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\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& \boxed{x_1} & \boxed{+} & \boxed{x_2'} & \boxed{-} & \boxed{x_2''} & \boxed{\geq} & \boxed{7} \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& & x_1, x_2', x_2'' & & & & \geq & 0
\end{array}
$$

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$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
 & \boxed{x_1} & \boxed{+} & \boxed{x_2'} & \boxed{-} & \boxed{x_2''} & \boxed{\geq} & \boxed{7} \\
 & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
 & & & x_1, x_2', x_2'' & & & \geq & 0
\end{array}
$$

Negate respective inequalities.

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$$
\begin{array}{llrrrrrrrr}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& \boxed{x_1 & + & x_2' & - & x_2'' & \geq & 7} \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& & x_1, x_2', x_2'' & & & & \geq & 0
\end{array}
$$

Negate respective inequalities.

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\begin{array}{llrrrrrrrr}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& -x_1 & - & x_2' & + & x_2'' & \leq & -7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& & x_1, x_2', x_2'' & & & & \geq & 0
\end{array}
$$

Converting into Standard Form (5/5)

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & & x_1, x_2, x_3 & & & & \geq & 0
\end{array}
$$

Converting into Standard Form (5/5)

Rename variable names (for consistency).

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& \multicolumn{5}{l}{\quad x_1, x_2, x_3} & \geq & 0
\end{array}
$$

Converting into Standard Form (5/5)

Rename variable names (for consistency).

$$\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& \multicolumn{5}{l}{x_1, x_2, x_3} & \geq & 0
\end{array}$$

It is always possible to convert a linear program into standard form.

Converting Standard Form into Slack Form (1/3)

Goal: Convert standard form into slack form, where all constraints except for the non-negativity constraints are equalities.

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For the simplex algorithm, it is more convenient to work with equality constraints.

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Introducing Slack Variables

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- Let $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ be an inequality constraint

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- Introduce a slack variable s by

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$$s = b_i - \sum_{j=1}^{n} a_{ij} x_j$$

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$$s \geq 0.$$

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s measures the slack between the two sides of the inequality.

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$$s = b_i - \sum_{j=1}^{n} a_{ij} x_j$$
$$s \geq 0.$$

s measures the slack between the two sides of the inequality.

- Denote slack variable of the ith inequality by x_{n+i}

Converting Standard Form into Slack Form (2/3)

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & & x_1, x_2, x_3 & & & & \geq & 0
\end{array}
$$

Converting Standard Form into Slack Form (2/3)

$$\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & x_1, x_2, x_3 & & & & \geq & 0
\end{array}$$

$\Big\downarrow$ Introduce slack variables

Converting Standard Form into Slack Form (2/3)

$$
\begin{array}{llrcrcrcr}
\text{maximize} & & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & & \\
& & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & & x_1, x_2, x_3 & & & & \geq & 0
\end{array}
$$

$\Big\downarrow$ Introduce slack variables

subject to

$$
x_4 = 7 - x_1 - x_2 + x_3
$$

Converting Standard Form into Slack Form (2/3)

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & x_1, x_2, x_3 & & & & \geq & 0
\end{array}
$$

Introduce slack variables

subject to

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3
\end{array}
$$

Converting Standard Form into Slack Form (2/3)

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
 & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & & x_1, x_2, x_3 & & & & \geq & 0
\end{array}
$$

Introduce slack variables

$$
\begin{array}{lrcrcrcrcr}
\text{subject to} & & & & & & & & & \\
 & x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
 & x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
 & x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

Converting Standard Form into Slack Form (2/3)

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& \multicolumn{5}{l}{x_1, x_2, x_3} & \geq & 0
\end{array}
$$

Introduce slack variables

subject to

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
\multicolumn{3}{l}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & & & &
\end{array}
$$

Converting Standard Form into Slack Form (2/3)

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & & x_1, x_2, x_3 & & & \geq & 0
\end{array}
$$

Introduce slack variables

$$
\begin{array}{lrcrcrcrcr}
\text{maximize} & & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & & \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& \multicolumn{5}{r}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}
$$

Converting Standard Form into Slack Form (3/3)

$$
\begin{array}{lrcrcrcrcr}
\text{maximize} & & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & & \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& \multicolumn{5}{c}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}
$$

Converting Standard Form into Slack Form (3/3)

$$\begin{array}{lrcrcrcrcr}
\text{maximize} & & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & & \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& \multicolumn{5}{r}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}$$

Use variable z to denote objective function and omit the nonnegativity constraints.

Converting Standard Form into Slack Form (3/3)

$$
\begin{array}{lrrrrrrrr}
\text{maximize} & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & \\
 & x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
 & x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
 & x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
 & & & & x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0 & & &
\end{array}
$$

Use variable z to denote objective function and omit the nonnegativity constraints.

$$
\begin{array}{rcrcrcrcr}
\boxed{z} & \boxed{=} & & & \boxed{2x_1} & \boxed{-} & \boxed{3x_2} & \boxed{+} & \boxed{3x_3} \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

Converting Standard Form into Slack Form (3/3)

$$\begin{array}{lrrrrrrrr}
\text{maximize} & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
\end{array}$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Use variable z to denote objective function and omit the nonnegativity constraints.

$$\begin{array}{lrrrrrrrr}
\boxed{z} & \boxed{=} & & & \boxed{2x_1} & \boxed{-} & \boxed{3x_2} & \boxed{+} & \boxed{3x_3} \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
\end{array}$$

This is called slack form.

Basic and Non-Basic Variables

$$
\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

Basic and Non-Basic Variables

$$\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}$$

Basic Variables: $B = \{4, 5, 6\}$

Basic and Non-Basic Variables

$$\begin{array}{rcrcrcrcr} z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\ x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\ x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \end{array}$$

Basic Variables: $B = \{4, 5, 6\}$

Non-Basic Variables: $N = \{1, 2, 3\}$

Basic and Non-Basic Variables

$$
\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

Basic Variables: $B = \{4, 5, 6\}$

Non-Basic Variables: $N = \{1, 2, 3\}$

Slack Form (Formal Definition)

Slack form is given by a tuple (N, B, A, b, c, v) so that

$$z = v + \sum_{j \in N} c_j x_j$$

$$x_i = b_i - \sum_{j \in N} a_{ij} x_j \qquad \text{for } i \in B,$$

and all variables are non-negative.

Basic and Non-Basic Variables

$$
\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

Basic Variables: $B = \{4, 5, 6\}$

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$$z = v + \sum_{j \in N} c_j x_j$$

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and all variables are non-negative.

Variables/Coefficients on the right hand side are indexed by B and N.

Slack Form (Example)

$$\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{aligned}$$

Slack Form (Example)

$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{aligned}
$$

Slack Form Notation

Slack Form (Example)

$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{aligned}
$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$

Slack Form (Example)

$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{aligned}
$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$
- $$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$

Slack Form (Example)

$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{aligned}
$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$
- $$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$
- $$b = \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix},$$

Slack Form (Example)

$$\begin{array}{rcrcrcrcr}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$
- $$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$
- $$b = \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix}, \quad c = \begin{pmatrix} c_3 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} -1/6 \\ -1/6 \\ -2/3 \end{pmatrix}$$

Slack Form (Example)

$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{aligned}
$$

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$
- $$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$
- $$b = \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix}, \quad c = \begin{pmatrix} c_3 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} -1/6 \\ -1/6 \\ -2/3 \end{pmatrix}$$
- $v = 28$

The Structure of Optimal Solutions

Definition

A point x is a vertex if it cannot be represented as a strict convex combination of two other points in the feasible set.

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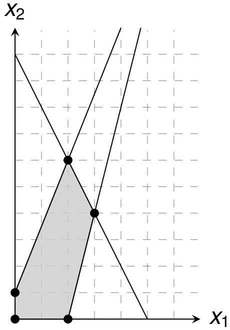
The set of feasible solutions is a convex set.

Theorem

If the slack form has an optimal solution, one of them occurs at a vertex.



The Structure of Optimal Solutions

Definition

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The set of feasible solutions is a convex set.

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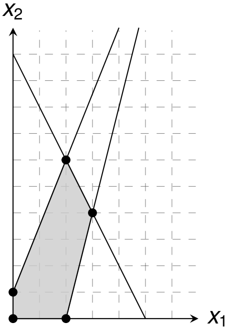
If the slack form has an optimal solution, one of them occurs at a vertex.

Proof Sketch (informal and non-examinable):

- Rewrite LP s.t. $Ax = b$. Let x be optimal but not a vertex



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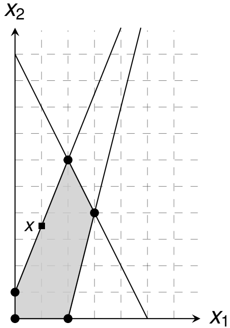

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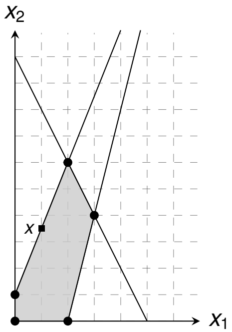

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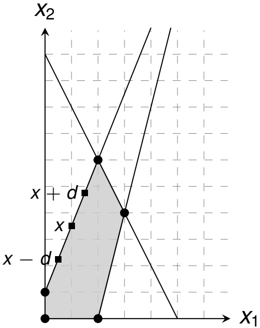

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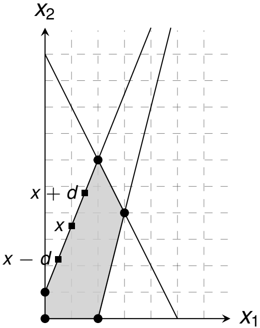

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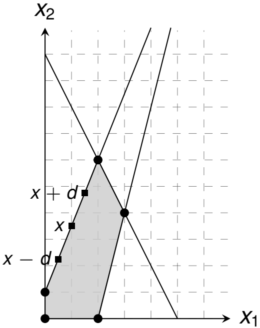

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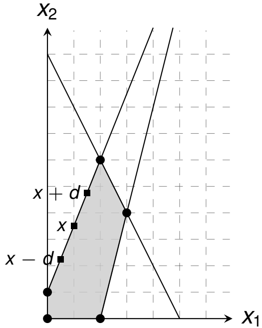

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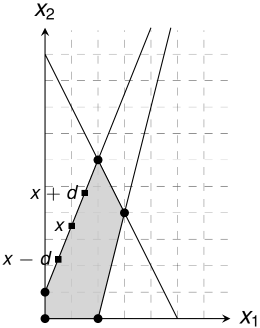

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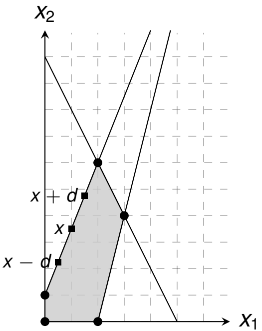

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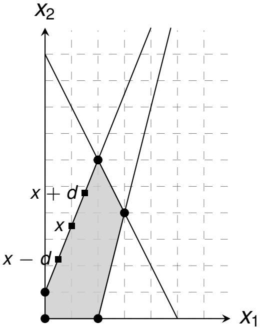

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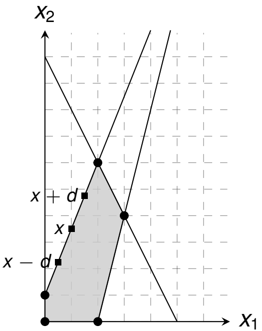

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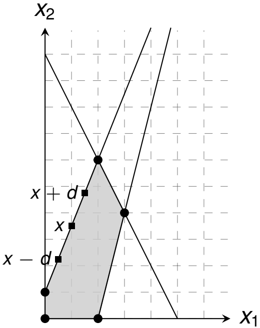

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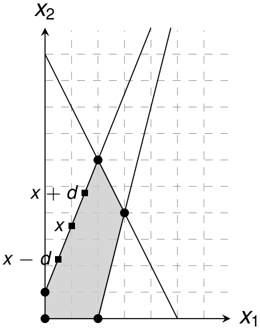

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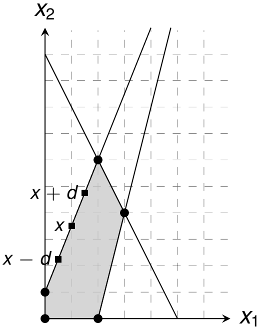

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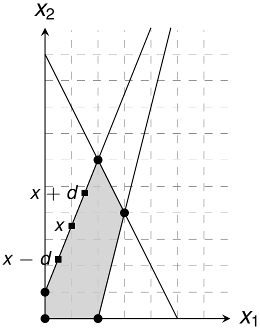

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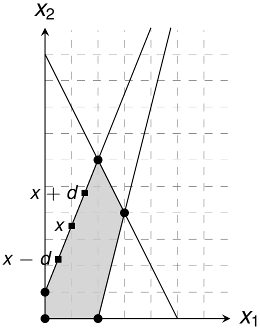

Outline

Introduction

Formulating Problems as Linear Programs

Standard and Slack Forms

Simplex Algorithm

Finding an Initial Solution

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- usually fast in practice although worst-case runtime not polynomial
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Basic Idea:

- Each iteration corresponds to a "basic solution" of the slack form
- All non-basic variables are 0, and the basic variables are determined from the equality constraints
- Each iteration converts one slack form into an equivalent one while the objective value will not decrease
- Conversion ("pivoting") is achieved by switching the roles of one basic and one non-basic variable

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  In that sense, it is a greedy algorithm.
- Conversion (“pivoting”) is achieved by switching the roles of one basic and one non-basic variable

Extended Example: Conversion into Slack Form

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
& 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
& 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
& & & x_1, x_2, x_3 & & & \geq & 0
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Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (0, 0, 0, 30, 24, 36)$

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Increasing the value of x_1 would increase the objective value.

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- Solving for x_1 yields:

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}.$$

- Substitute this into x_1 in the other three equations

Extended Example: Iteration 2

$$
\begin{array}{rclcrcrcr}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}
$$

Extended Example: Iteration 2

$$
\begin{array}{rclclclcl}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}
$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (9, 0, 0, 21, 6, 0)$ with objective value 27

Extended Example: Iteration 2

Increasing the value of x_3 would increase the objective value.

$$
\begin{array}{rcrcrcrcr}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}
$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (9, 0, 0, 21, 6, 0)$ with objective value 27

Extended Example: Iteration 2

Increasing the value of x_3 would increase the objective value.

$$
\begin{array}{lcrcccccc}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
\boxed{x_5} & \boxed{=} & \boxed{6} & \boxed{-} & \boxed{\frac{3x_2}{2}} & \boxed{-} & \boxed{4x_3} & \boxed{+} & \boxed{\frac{x_6}{2}}
\end{array}
$$

The third constraint is the tightest and limits how much we can increase x_3.

Extended Example: Iteration 2

Increasing the value of x_3 would increase the objective value.

$$
\begin{array}{lclclclcl}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}
$$

The third constraint is the tightest and limits how much we can increase x_3.

Switch roles of x_3 and x_5:

Extended Example: Iteration 2

Increasing the value of x_3 would increase the objective value.

$$\begin{array}{rclclclcl}
z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\
x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\
x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\
x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2}
\end{array}$$

The third constraint is the tightest and limits how much we can increase x_3.

Switch roles of x_3 and x_5:

- Solving for x_3 yields:

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} - \frac{x_6}{8}.$$

Extended Example: Iteration 2

Increasing the value of x_3 would increase the objective value.

$$
\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
\Aboxed{x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}}
\end{aligned}
$$

The third constraint is the tightest and limits how much we can increase x_3.

Switch roles of x_3 and x_5:

- Solving for x_3 yields:

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} - \frac{x_6}{8}.$$

- Substitute this into x_3 in the other three equations

Extended Example: Iteration 3

$$\begin{array}{rclcrcrcr} z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\ x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\ x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\ x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16} \end{array}$$

Extended Example: Iteration 3

$$
\begin{array}{rclcccccc}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}
$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (\frac{33}{4}, 0, \frac{3}{2}, \frac{69}{4}, 0, 0)$ with objective value $\frac{111}{4} = 27.75$

Extended Example: Iteration 3

Increasing the value of x_2 would increase the objective value.

$$
\begin{aligned}
z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}
\end{aligned}
$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (\frac{33}{4}, 0, \frac{3}{2}, \frac{69}{4}, 0, 0)$ with objective value $\frac{111}{4} = 27.75$

Extended Example: Iteration 3

Increasing the value of x_2 would increase the objective value.

$$
\begin{array}{rcrcrcrcr}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}
$$

The second constraint is the tightest and limits how much we can increase x_2.

Extended Example: Iteration 3

Increasing the value of x_2 would increase the objective value.

$$
\begin{array}{rcrcrcrcr}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}
$$

The second constraint is the tightest and limits how much we can increase x_2.

Switch roles of x_2 and x_3:

Extended Example: Iteration 3

Increasing the value of x_2 would increase the objective value.

$$
\begin{array}{rcrcrcrcr}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}
$$

The second constraint is the tightest and limits how much we can increase x_2.

Switch roles of x_2 and x_3:

- Solving for x_2 yields:

$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}.$$

Extended Example: Iteration 3

Increasing the value of x_2 would increase the objective value.

$$\begin{array}{rclclclcl}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}$$

The second constraint is the tightest and limits how much we can increase x_2.

Switch roles of x_2 and x_3:

- Solving for x_2 yields:

$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}.$$

- Substitute this into x_2 in the other three equations

Extended Example: Iteration 4

$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2}
\end{aligned}
$$

Extended Example: Iteration 4

$$
\begin{array}{rcrcrcrcr}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}
$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (8, 4, 0, 18, 0, 0)$ with objective value 28

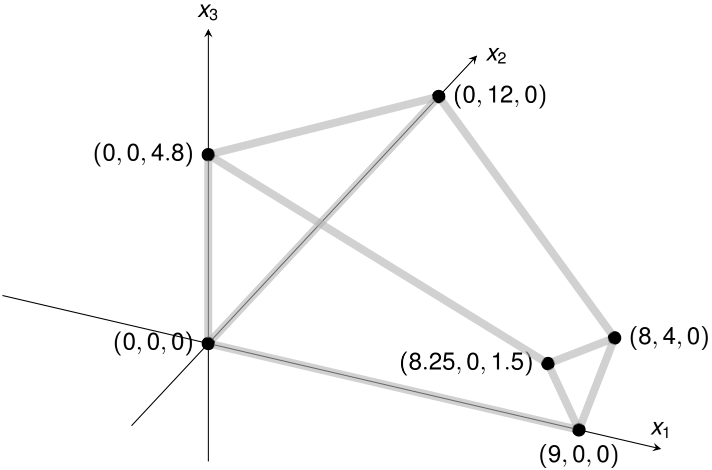
Extended Example: Iteration 4

All coefficients are negative, and hence this basic solution is **optimal**!

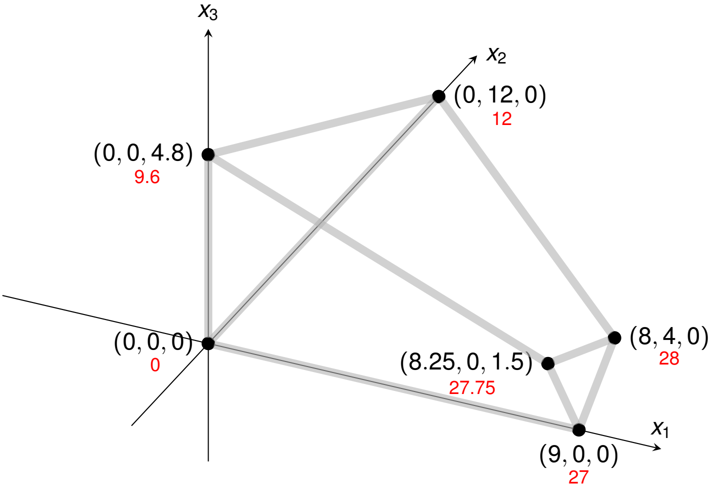
$$
\begin{array}{rclcccccc}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}
$$

Basic solution: $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_6}) = (8, 4, 0, 18, 0, 0)$ with objective value 28

Extended Example: Visualization of SIMPLEX

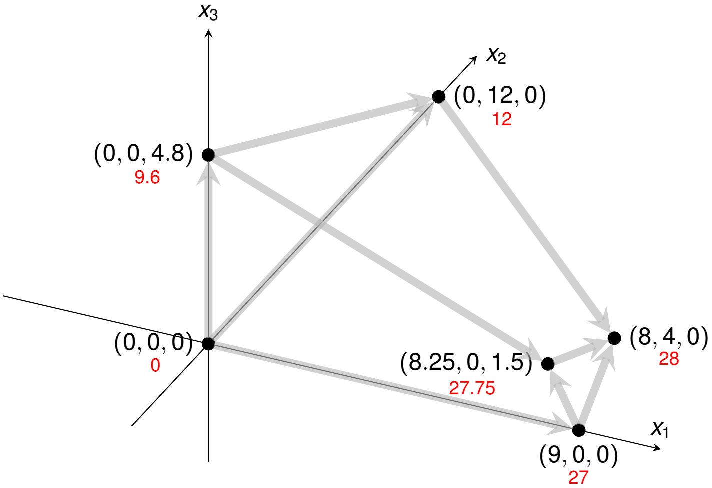


Extended Example: Visualization of SIMPLEX



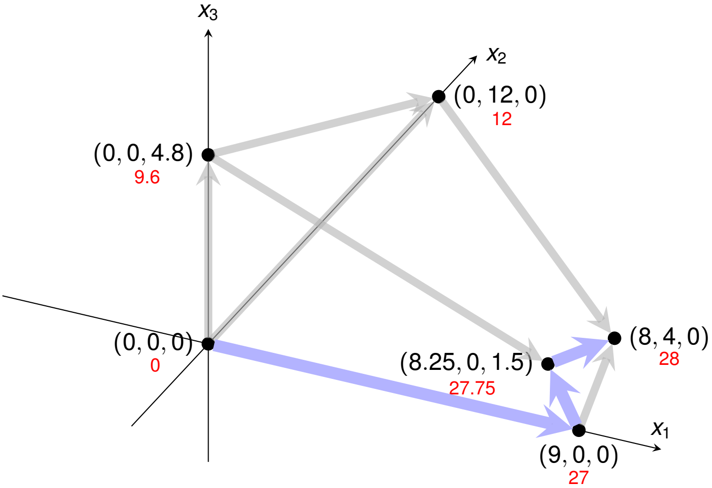
Exercise: How many basic solutions (including non-feasible ones) are there?

Extended Example: Visualization of SIMPLEX



Exercise: How many basic solutions (including non-feasible ones) are there?

Extended Example: Visualization of SIMPLEX



Exercise: How many basic solutions (including non-feasible ones) are there?

Extended Example: Alternative Runs (1/2)

$$\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}$$

Extended Example: Alternative Runs (1/2)

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

Switch roles of x_2 and x_5

Extended Example: Alternative Runs (1/2)

$$\begin{array}{lcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}$$

Switch roles of x_2 and x_5

$$\begin{array}{lcrcrcrcr}
z & = & 12 & + & 2x_1 & - & \frac{x_3}{2} & - & \frac{x_5}{2} \\
x_2 & = & 12 & - & x_1 & - & \frac{5x_3}{2} & - & \frac{x_5}{2} \\
x_4 & = & 18 & - & x_2 & - & \frac{x_3}{2} & + & \frac{x_5}{2} \\
x_6 & = & 24 & - & 3x_1 & + & \frac{x_3}{2} & + & \frac{x_5}{2}
\end{array}$$

Extended Example: Alternative Runs (1/2)

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

Switch roles of x_2 and x_5

$$
\begin{array}{rcrcrcrcr}
z & = & 12 & + & 2x_1 & - & \frac{x_3}{2} & - & \frac{x_5}{2} \\
x_2 & = & 12 & - & x_1 & - & \frac{5x_3}{2} & - & \frac{x_5}{2} \\
x_4 & = & 18 & - & x_2 & - & \frac{x_3}{2} & + & \frac{x_5}{2} \\
x_6 & = & 24 & - & 3x_1 & + & \frac{x_3}{2} & + & \frac{x_5}{2}
\end{array}
$$

Switch roles of x_1 and x_6

Extended Example: Alternative Runs (1/2)

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

$\Downarrow$ Switch roles of x_2 and x_5

$$
\begin{array}{rcrcrcrcr}
z & = & 12 & + & 2x_1 & - & \frac{x_3}{2} & - & \frac{x_5}{2} \\
x_2 & = & 12 & - & x_1 & - & \frac{5x_3}{2} & - & \frac{x_5}{2} \\
x_4 & = & 18 & - & x_2 & - & \frac{x_3}{2} & + & \frac{x_5}{2} \\
x_6 & = & 24 & - & 3x_1 & + & \frac{x_3}{2} & + & \frac{x_5}{2}
\end{array}
$$

$\Downarrow$ Switch roles of x_1 and x_6

$$
\begin{array}{rcrcrcrcr}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}
$$

Extended Example: Alternative Runs (2/2)

$$\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}$$

Extended Example: Alternative Runs (2/2)

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

Switch roles of x_3 and x_5

Extended Example: Alternative Runs (2/2)

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

Switch roles of x_3 and x_5

$$
\begin{array}{rcrcrcrcr}
z & = & \frac{48}{5} & + & \frac{11x_1}{5} & + & \frac{x_2}{5} & - & \frac{2x_5}{5} \\
x_4 & = & \frac{78}{5} & + & \frac{x_1}{5} & + & \frac{x_2}{5} & + & \frac{3x_5}{5} \\
x_3 & = & \frac{24}{5} & - & \frac{2x_1}{5} & - & \frac{2x_2}{5} & - & \frac{x_5}{5} \\
x_6 & = & \frac{132}{5} & - & \frac{16x_1}{5} & - & \frac{x_2}{5} & + & \frac{2x_3}{5}
\end{array}
$$

Extended Example: Alternative Runs (2/2)

$$
\begin{aligned}
z &= & & 3x_1 &+& x_2 &+& 2x_3 \\
x_4 &= 30 &-& x_1 &-& x_2 &-& 3x_3 \\
x_5 &= 24 &-& 2x_1 &-& 2x_2 &-& 5x_3 \\
x_6 &= 36 &-& 4x_1 &-& x_2 &-& 2x_3
\end{aligned}
$$

Switch roles of x_3 and x_5

$$
\begin{aligned}
z &= \frac{48}{5} &+& \frac{11x_1}{5} &+& \frac{x_2}{5} &-& \frac{2x_5}{5} \\
x_4 &= \frac{78}{5} &+& \frac{x_1}{5} &+& \frac{x_2}{5} &+& \frac{3x_5}{5} \\
x_3 &= \frac{24}{5} &-& \frac{2x_1}{5} &-& \frac{2x_2}{5} &-& \frac{x_5}{5} \\
x_6 &= \frac{132}{5} &-& \frac{16x_1}{5} &-& \frac{x_2}{5} &+& \frac{2x_3}{5}
\end{aligned}
$$

Switch roles of x_1 and x_6

Extended Example: Alternative Runs (2/2)

$$\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}$$

Switch roles of x_3 and x_5

$$\begin{array}{rcrcrcrcr}
z & = & \frac{48}{5} & + & \frac{11x_1}{5} & + & \frac{x_2}{5} & - & \frac{2x_5}{5} \\
x_4 & = & \frac{78}{5} & + & \frac{x_1}{5} & + & \frac{x_2}{5} & + & \frac{3x_5}{5} \\
x_3 & = & \frac{24}{5} & - & \frac{2x_1}{5} & - & \frac{2x_2}{5} & - & \frac{x_5}{5} \\
x_6 & = & \frac{132}{5} & - & \frac{16x_1}{5} & - & \frac{x_2}{5} & + & \frac{2x_3}{5}
\end{array}$$

Switch roles of x_1 and x_6

$$\begin{array}{rcrcrcrcr}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}$$

Extended Example: Alternative Runs (2/2)

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

Switch roles of x_3 and x_5

$$
\begin{array}{rcrcrcrcr}
z & = & \frac{48}{5} & + & \frac{11x_1}{5} & + & \frac{x_2}{5} & - & \frac{2x_5}{5} \\
x_4 & = & \frac{78}{5} & + & \frac{x_1}{5} & + & \frac{x_2}{5} & + & \frac{3x_5}{5} \\
x_3 & = & \frac{24}{5} & - & \frac{2x_1}{5} & - & \frac{2x_2}{5} & - & \frac{x_5}{5} \\
x_6 & = & \frac{132}{5} & - & \frac{16x_1}{5} & - & \frac{x_2}{5} & + & \frac{2x_3}{5}
\end{array}
$$

Switch roles of x_1 and x_6

Switch roles of x_2 and x_3

$$
\begin{array}{rcrcrcrcr}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}
$$

Extended Example: Alternative Runs (2/2)

$$
\begin{array}{rclllllll}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
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\begin{array}{rclllllll}
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x_6 & = & \frac{132}{5} & - & \frac{16x_1}{5} & - & \frac{x_2}{5} & + & \frac{2x_3}{5}
\end{array}
$$

Switch roles of x_1 and x_6

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\begin{array}{rclllllll}
z & = & \frac{111}{4} & + & \frac{x_2}{16} & - & \frac{x_5}{8} & - & \frac{11x_6}{16} \\
x_1 & = & \frac{33}{4} & - & \frac{x_2}{16} & + & \frac{x_5}{8} & - & \frac{5x_6}{16} \\
x_3 & = & \frac{3}{2} & - & \frac{3x_2}{8} & - & \frac{x_5}{4} & + & \frac{x_6}{8} \\
x_4 & = & \frac{69}{4} & + & \frac{3x_2}{16} & + & \frac{5x_5}{8} & - & \frac{x_6}{16}
\end{array}
$$

Switch roles of x_2 and x_3

$$
\begin{array}{rclllllll}
z & = & 28 & - & \frac{x_3}{6} & - & \frac{x_5}{6} & - & \frac{2x_6}{3} \\
x_1 & = & 8 & + & \frac{x_3}{6} & + & \frac{x_5}{6} & - & \frac{x_6}{3} \\
x_2 & = & 4 & - & \frac{8x_3}{3} & - & \frac{2x_5}{3} & + & \frac{x_6}{3} \\
x_4 & = & 18 & - & \frac{x_3}{2} & + & \frac{x_5}{2} & &
\end{array}
$$

The Pivot Step Formally

PIVOT$(N, B, A, b, c, \nu, l, e)$

// Compute the coefficients of the equation for new basic variable x_e.
let $\widehat{A}$ be a new $m \times n$ matrix
$\widehat{b}_e = b_l / a_{le}$
for each $j \in N - \{e\}$
    $\widehat{a}_{ej} = a_{lj} / a_{le}$
$\widehat{a}_{el} = 1 / a_{le}$
// Compute the coefficients of the remaining constraints.
for each $i \in B - \{l\}$
    $\widehat{b}_i = b_i - a_{ie} \widehat{b}_e$
    for each $j \in N - \{e\}$
        $\widehat{a}_{ij} = a_{ij} - a_{ie} \widehat{a}_{ej}$
    $\widehat{a}_{il} = -a_{ie} \widehat{a}_{el}$
// Compute the objective function.
$\widehat{\nu} = \nu + c_e \widehat{b}_e$
for each $j \in N - \{e\}$
    $\widehat{c}_j = c_j - c_e \widehat{a}_{ej}$
$\widehat{c}_l = -c_e \widehat{a}_{el}$
// Compute new sets of basic and nonbasic variables.
$\widehat{N} = N - \{e\} \cup \{l\}$
$\widehat{B} = B - \{l\} \cup \{e\}$
return $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{\nu})$

The Pivot Step Formally

PIVOT(N, B, A, b, c, v, l, e)

// Compute the coefficients of the equation for new basic variable x_e.
let $\widehat{A}$ be a new $m \times n$ matrix
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  $\widehat{b}_i = b_i - a_{ie}\widehat{b}_e$
  for each $j \in N - \{e\}$
    $\widehat{a}_{ij} = a_{ij} - a_{ie}\widehat{a}_{ej}$
  $\widehat{a}_{il} = -a_{ie}\widehat{a}_{el}$
// Compute the objective function.
$\widehat{v} = v + c_e\widehat{b}_e$
for each $j \in N - \{e\}$
  $\widehat{c}_j = c_j - c_e\widehat{a}_{ej}$
$\widehat{c}_l = -c_e\widehat{a}_{el}$
// Compute new sets of basic and nonbasic variables.
$\widehat{N} = N - \{e\} \cup \{l\}$
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return $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$

Rewrite “tight” equation for enterring variable x_e.

The Pivot Step Formally

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Rewrite “tight” equation for enterring variable x_e.

Substituting x_e into other equations.

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Substituting x_e into objective function.

// Compute new sets of basic and nonbasic variables.
$\widehat{N} = N - \{e\} \cup \{l\}$
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Update non-basic and basic variables

The Pivot Step Formally

PIVOT(N, B, A, b, c, v, l, e)

// Compute the coefficients of the equation for new basic variable x_e.
let $\widehat{A}$ be a new $m \times n$ matrix
$\widehat{b}_e = b_l / a_{le}$
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return $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$

Need that $a_{le} \neq 0$!

Rewrite "tight" equation for enterring variable x_e.

Substituting x_e into other equations.

Substituting x_e into objective function.

Update non-basic and basic variables

Effect of the Pivot Step (extra material, non-examinable)

Lemma 29.1

Consider a call to PIVOT(N, B, A, b, c, v, l, e) in which $a_{le} \neq 0$. Let the values returned from the call be $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$, and let $\overline{x}$ denote the basic solution after the call. Then

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1. $\overline{x}_j = 0$ for each $j \in \widehat{N}$.
2. $\overline{x}_e = b_l / a_{le}$.
3. $\overline{x}_i = b_i - a_{ie}\widehat{b}_e$ for each $i \in \widehat{B} \setminus \{e\}$.

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Proof:

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Consider a call to $\textsc{Pivot}(N, B, A, b, c, v, l, e)$ in which $a_{le} \neq 0$. Let the values returned from the call be $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$, and let $\overline{x}$ denote the basic solution after the call. Then

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Proof:

1. holds since the basic solution always sets all non-basic variables to zero.
2. When we set each non-basic variable to 0 in a constraint
$$x_i = \widehat{b}_i - \sum_{j \in \widehat{N}} \widehat{a}_{ij} x_j,$$
we have $\overline{x}_i = \widehat{b}_i$ for each $i \in \widehat{B}$. Hence $\overline{x}_e = \widehat{b}_e = b_l / a_{le}$.
3. After substituting into the other constraints, we have
$$\overline{x}_i = \widehat{b}_i = b_i - a_{ie}\widehat{b}_e.$$

Effect of the Pivot Step (extra material, non-examinable)

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Consider a call to $\textsc{Pivot}(N, B, A, b, c, v, l, e)$ in which $a_{le} \neq 0$. Let the values returned from the call be $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$, and let $\overline{x}$ denote the basic solution after the call. Then

1. $\overline{x}_j = 0$ for each $j \in \widehat{N}$.
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3. After substituting into the other constraints, we have

$$\overline{x}_i = \widehat{b}_i = b_i - a_{ie}\widehat{b}_e. \qquad \square$$

Formalizing the Simplex Algorithm: Questions

Questions:

- How do we determine whether a linear program is feasible?
- What do we do if the linear program is feasible, but the initial basic solution is not feasible?
- How do we determine whether a linear program is unbounded?
- How do we choose the entering and leaving variables?

Formalizing the Simplex Algorithm: Questions

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- How do we determine whether a linear program is feasible?
- What do we do if the linear program is feasible, but the initial basic solution is not feasible?
- How do we determine whether a linear program is unbounded?
- How do we choose the entering and leaving variables?

Example before was a particularly nice one!

The formal procedure SIMPLEX

```
SIMPLEX(A, b, c)
(N, B, A, b, c, ν) = INITIALIZE-SIMPLEX(A, b, c)
let Δ be a new vector of length m
while some index j ∈ N has c_j > 0
    choose an index e ∈ N for which c_e > 0
    for each index i ∈ B
        if a_ie > 0
            Δ_i = b_i / a_ie
        else Δ_i = ∞
    choose an index l ∈ B that minimizes Δ_i
    if Δ_l == ∞
        return "unbounded"
    else (N, B, A, b, c, ν) = PIVOT(N, B, A, b, c, ν, l, e)
for i = 1 to n
    if i ∈ B
        x̄_i = b_i
    else x̄_i = 0
return (x̄_1, x̄_2, ..., x̄_n)
```

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```
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for i = 1 to n
    if i ∈ B
        x̄_i = b_i
    else x̄_i = 0
return (x̄_1, x̄_2, ..., x̄_n)
```

Returns a slack form with a feasible basic solution (if it exists)

The formal procedure SIMPLEX

SIMPLEX(A, b, c)

$(N, B, A, b, c, \nu) = $ INITIALIZE-SIMPLEX(A, b, c)

Returns a slack form with a feasible basic solution (if it exists)

let Δ be a new vector of length m

while some index $j \in N$ has $c_j > 0$

  choose an index $e \in N$ for which $c_e > 0$

  for each index $i \in B$

    if $a_{ie} > 0$

      $\Delta_i = b_i / a_{ie}$

    else $\Delta_i = \infty$

  choose an index $l \in B$ that minimizes Δ_i

  if $\Delta_l == \infty$

    return "unbounded"

  else $(N, B, A, b, c, \nu) = $ PIVOT$(N, B, A, b, c, \nu, l, e)$

for $i = 1$ **to** n

  if $i \in B$

    $\bar{x}_i = b_i$

  else $\bar{x}_i = 0$

return $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$

The formal procedure SIMPLEX

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  else $(N, B, A, b, c, \nu) = $ PIVOT$(N, B, A, b, c, \nu, l, e)$
for $i = 1$ **to** n
  if $i \in B$
    $\bar{x}_i = b_i$
  else $\bar{x}_i = 0$
return $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$

Returns a slack form with a feasible basic solution (if it exists)

Main Loop:

The formal procedure SIMPLEX

SIMPLEX(A, b, c)

1 $(N, B, A, b, c, \nu) = $ INITIALIZE-SIMPLEX(A, b, c)
2 let Δ be a new vector of length m
3 **while** some index $j \in N$ has $c_j > 0$
4   choose an index $e \in N$ for which $c_e > 0$
5   **for** each index $i \in B$
6   **if** $a_{ie} > 0$
7             $\Delta_i = b_i / a_{ie}$
8   **else** $\Delta_i = \infty$
9   choose an index $l \in B$ that minimizes Δ_i
10   **if** $\Delta_l == \infty$
11   **return** "unbounded"
12   **else** $(N, B, A, b, c, \nu) = $ PIVOT$(N, B, A, b, c, \nu, l, e)$
13 **for** $i = 1$ **to** n
14   **if** $i \in B$
15         $\bar{x}_i = b_i$
16   **else** $\bar{x}_i = 0$
17 **return** $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$

Returns a slack form with a feasible basic solution (if it exists)

Main Loop:

- terminates if all coefficients in objective function are negative
- Line 4 picks enterring variable x_e with negative coefficient
- Lines 6 – 9 pick the tightest constraint, associated with x_l
- Line 11 returns "unbounded" if there are no constraints
- Line 12 calls PIVOT, switching roles of x_l and x_e

The formal procedure SIMPLEX

```
SIMPLEX(A, b, c)
1  (N, B, A, b, c, v) = INITIALIZE-SIMPLEX(A, b, c)
2  let Δ be a new vector of length m
3  while some index j ∈ N has c_j > 0
4      choose an index e ∈ N for which c_e > 0
5      for each index i ∈ B
6          if a_ie > 0
7              Δ_i = b_i / a_ie
8          else Δ_i = ∞
9      choose an index l ∈ B that minimizes Δ_i
10     if Δ_l == ∞
11         return "unbounded"
12     else (N, B, A, b, c, v) = PIVOT(N, B, A, b, c, v, l, e)
13 for i = 1 to n
14     if i ∈ B
15         x̄_i = b_i
16     else x̄_i = 0
17 return (x̄_1, x̄_2, ..., x̄_n)
```

Returns a slack form with a feasible basic solution (if it exists)

Main Loop:

- terminates if all coefficients in objective function are negative
- Line 4 picks enterring variable x_e with negative coefficient
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- Line 11 returns “unbounded” if there are no constraints
- Line 12 calls PIVOT, switching roles of x_l and x_e

Return corresponding solution.

The formal procedure SIMPLEX

SIMPLEX(A, b, c)

1 $(N, B, A, b, c, \nu) = $ INITIALIZE-SIMPLEX(A, b, c)
2 let Δ be a new vector of length m
3 **while** some index $j \in N$ has $c_j > 0$
4   choose an index $e \in N$ for which $c_e > 0$
5   **for** each index $i \in B$
6   **if** $a_{ie} > 0$
7             $\Delta_i = b_i / a_{ie}$
8   **else** $\Delta_i = \infty$
9   choose an index $l \in B$ that minimizes Δ_i
10   **if** $\Delta_l == \infty$
11   **return** "unbounded"
12   **else** $(N, B, A, b, c, \nu) = $ PIVOT$(N, B, A, b, c, \nu, l, e)$
13 **for** $i = 1$ **to** n
14   **if** $i \in B$
15         $\bar{x}_i = b_i$
16   **else** $\bar{x}_i = 0$
17 **return** $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$

Returns a slack form with a feasible basic solution (if it exists)

Lemma 29.2

Suppose the call to INITIALIZE-SIMPLEX in line 1 returns a slack form for which the basic solution is feasible. Then if SIMPLEX returns a solution, it is a feasible solution. If SIMPLEX returns "unbounded", the linear program is unbounded.

The formal procedure SIMPLEX

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Proof is based on the following three-part loop invariant:

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1   (N, B, A, b, c, ν) = INITIALIZE-SIMPLEX(A, b, c)
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7               Δ_i = b_i/a_ie
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9       choose an index l ∈ B that minimizes Δ_i
10      if Δ_l == ∞
11          return "unbounded"
```

Returns a slack form with a feasible basic solution (if it exists)

Proof is based on the following three-part loop invariant:

1. the slack form is always equivalent to the one returned by INITIALIZE-SIMPLEX,
2. for each $i \in B$, we have $b_i \geq 0$,
3. the basic solution associated with the (current) slack form is feasible.

Lemma 29.2

Suppose the call to INITIALIZE-SIMPLEX in line 1 returns a slack form for which the basic solution is feasible. Then if SIMPLEX returns a solution, it is a feasible solution. If SIMPLEX returns "unbounded", the linear program is unbounded.

Termination

Degeneracy: One iteration of SIMPLEX leaves the objective value unchanged.

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$$\begin{array}{rcrcrcrcr} z & = & & & x_1 & + & x_2 & + & x_3 \\ x_4 & = & 8 & - & x_1 & - & x_2 & & \\ x_5 & = & & & & & x_2 & - & x_3 \end{array}$$

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\end{array}
$$

Pivot with x_1 entering and x_4 leaving

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Pivot with x_3 entering and x_5 leaving

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$$\begin{array}{rclllllll} z & = & & & x_1 & + & x_2 & + & x_3 \\ x_4 & = & 8 & - & x_1 & - & x_2 & & \\ x_5 & = & & & & & x_2 & - & x_3 \end{array}$$

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$$\begin{array}{rclllllll} z & = & 8 & + & x_2 & - & x_4 & - & x_5 \\ x_1 & = & 8 & - & x_2 & - & x_4 & & \\ x_3 & = & & & x_2 & & & - & x_5 \end{array}$$

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x_1 & = & 8 & - & x_2 & & & - & x_4 \\
x_5 & = & & & x_2 & - & x_3 & &
\end{array}$$

Pivot with x_3 entering and x_5 leaving

Cycling: If additionally slack form at two iterations are identical, SIMPLEX fails to terminate!

$$\begin{array}{rclcccccccc}
z & = & 8 & + & x_2 & - & x_4 & - & x_5 \\
x_1 & = & 8 & - & x_2 & - & x_4 & & \\
x_3 & = & & & x_2 & & & - & x_5
\end{array}$$

Exercise: Execute one more step of the Simplex Algorithm on the tableau from the previous slide.

Termination and Running Time

Cycling: SIMPLEX may fail to terminate.

Termination and Running Time

It is theoretically possible, but very rare in practice.

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Anti-Cycling Strategies

1. Bland's rule: Choose entering variable with smallest index
2. Random rule: Choose entering variable uniformly at random
3. Perturbation: Perturb the input slightly so that it is impossible to have two solutions with the same objective value

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Replace each b_i by $\widehat{b}_i = b_i + \epsilon_i$, where $\epsilon_i \gg \epsilon_{i+1}$ are all small.

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Lemma 29.7

Assuming INITIALIZE-SIMPLEX returns a slack form for which the basic solution is feasible, SIMPLEX either reports that the program is unbounded or returns a feasible solution in at most $\binom{n+m}{m}$ iterations.

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Every set B of basic variables uniquely determines a slack form, and there are at most $\binom{n+m}{m}$ unique slack forms.

Outline

Introduction

Formulating Problems as Linear Programs

Standard and Slack Forms

Simplex Algorithm

Finding an Initial Solution

Finding an Initial Solution

$$\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
 & 2x_1 & - & x_2 & \leq & 2 \\
 & x_1 & - & 5x_2 & \leq & -4 \\
 & & x_1, x_2 & & \geq & 0
\end{array}$$

Finding an Initial Solution

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\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
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Conversion into slack form

Finding an Initial Solution

$$
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 & & x_1, x_2 & & \geq & 0
\end{array}
$$

Conversion into slack form

$$
\begin{array}{rcrcrcr}
z & = & & & 2x_1 & - & x_2 \\
x_3 & = & 2 & - & 2x_1 & + & x_2 \\
x_4 & = & -4 & - & x_1 & + & 5x_2
\end{array}
$$

Basic solution $(x_1, x_2, x_3, x_4) = (0, 0, 2, -4)$ is not feasible!

Geometric Illustration

$$\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
 & 2x_1 & - & x_2 & \leq & 2 \\
 & x_1 & - & 5x_2 & \leq & -4 \\
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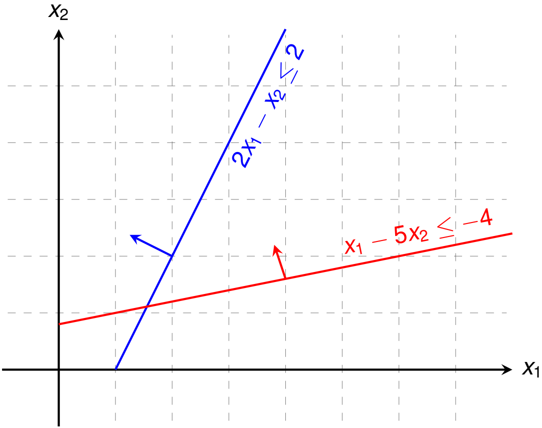
\end{array}$$



Geometric Illustration

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\begin{array}{llcrcr}
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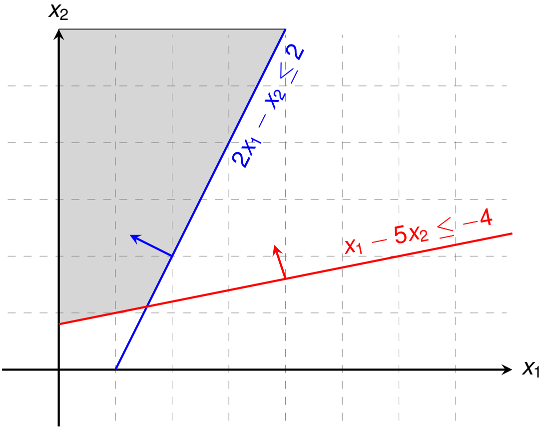

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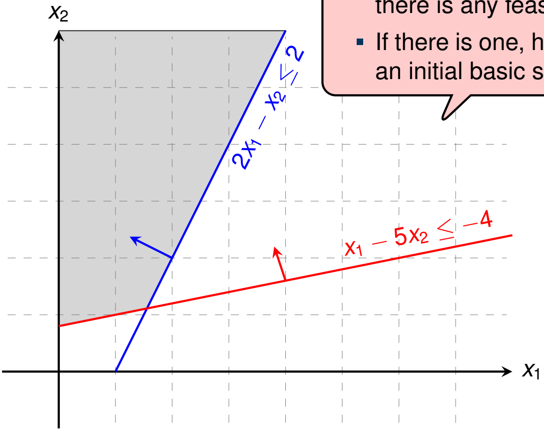

Questions:

- How to determine whether there is any feasible solution?
- If there is one, how to determine an initial basic solution?

Formulating an Auxiliary Linear Program

$$
\begin{array}{ll}
\text{maximize} & \sum_{j=1}^{n} c_j x_j \\
\text{subject to} &
\end{array}
$$

$$
\begin{array}{rcll}
\sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & \text{for } i = 1, 2, \ldots, m, \\
x_j & \geq & 0 & \text{for } j = 1, 2, \ldots, n
\end{array}
$$

Formulating an Auxiliary Linear Program

maximize $\sum_{j=1}^{n} c_j x_j$

subject to

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\leq b_i \quad \text{for } i = 1, 2, \ldots, m, \\ x_j &\geq 0 \quad \text{for } j = 1, 2, \ldots, n \end{aligned}$$

$\downarrow$ Formulating an Auxiliary Linear Program

Formulating an Auxiliary Linear Program

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$$\begin{array}{ll} \text{maximize} & -x_0 \\ \text{subject to} & \end{array}$$

$$\begin{array}{rcll} \sum_{j=1}^{n} a_{ij} x_j - x_0 & \leq & b_i & \text{for } i = 1, 2, \ldots, m, \\ x_j & \geq & 0 & \text{for } j = 0, 1, \ldots, n \end{array}$$

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Lemma 29.11

Let L_{aux} be the auxiliary LP of a linear program L in standard form. Then L is feasible if and only if the optimal objective value of L_{aux} is 0.

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Proof.

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Formulating an Auxiliary Linear Program

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  - Since $\overline{x}_0 \geq 0$ and the objective is to maximize $-x_0$, this is optimal for L_{aux}

Formulating an Auxiliary Linear Program

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\end{array}
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- "$\Leftarrow$": Suppose that the optimal objective value of L_{aux} is 0

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  - $\overline{x}_0 = 0$ combined with $\overline{x}$ is a feasible solution to L_{aux} with objective value 0.
  - Since $\overline{x}_0 \geq 0$ and the objective is to maximize $-x_0$, this is optimal for L_{aux}
- "$\Leftarrow$": Suppose that the optimal objective value of L_{aux} is 0
  - Then $\overline{x}_0 = 0$, and the remaining solution values $(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ satisfy L.

Formulating an Auxiliary Linear Program

maximize $\sum_{j=1}^{n} c_j x_j$

subject to

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maximize $-x_0$

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Proof.

- "$\Rightarrow$": Suppose L has a feasible solution $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$
  - $\overline{x}_0 = 0$ combined with $\overline{x}$ is a feasible solution to L_{aux} with objective value 0.
  - Since $\overline{x}_0 \geq 0$ and the objective is to maximize $-x_0$, this is optimal for L_{aux}
- "$\Leftarrow$": Suppose that the optimal objective value of L_{aux} is 0
  - Then $\overline{x}_0 = 0$, and the remaining solution values $(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ satisfy L. $\square$

```
INITIALIZE-SIMPLEX(A, b, c)
let k be the index of the minimum b_i
if b_k ≥ 0                          // is the initial basic solution feasible?
    return ({1, 2, ..., n}, {n + 1, n + 2, ..., n + m}, A, b, c, 0)
form L_aux by adding −x_0 to the left-hand side of each constraint
    and setting the objective function to −x_0
let (N, B, A, b, c, ν) be the resulting slack form for L_aux
l = n + k
// L_aux has n + 1 nonbasic variables and m basic variables.
(N, B, A, b, c, ν) = PIVOT(N, B, A, b, c, ν, l, 0)
// The basic solution is now feasible for L_aux.
iterate the while loop of lines 3–12 of SIMPLEX until an optimal solution
    to L_aux is found
if the optimal solution to L_aux sets x̄_0 to 0
    if x̄_0 is basic
        perform one (degenerate) pivot to make it nonbasic
    from the final slack form of L_aux, remove x_0 from the constraints and
        restore the original objective function of L, but replace each basic
        variable in this objective function by the right-hand side of its
        associated constraint
    return the modified final slack form
else return "infeasible"
```

Test solution with $N = \{1, 2, \ldots, n\}$, $B = \{n+1, n+2, \ldots, n+m\}$, $\overline{x}_i = b_i$ for $i \in B$, $\overline{x}_i = 0$ otherwise.

INITIALIZE-SIMPLEX(A, b, c)

let k be the index of the minimum b_i
if $b_k \geq 0$ // is the initial basic solution feasible?
    return $(\{1, 2, \ldots, n\}, \{n+1, n+2, \ldots, n+m\}, A, b, c, 0)$
form L_{aux} by adding $-x_0$ to the left-hand side of each constraint
    and setting the objective function to $-x_0$
let (N, B, A, b, c, ν) be the resulting slack form for L_{aux}
$l = n + k$
// L_{aux} has $n + 1$ nonbasic variables and m basic variables.
$(N, B, A, b, c, \nu) = \text{PIVOT}(N, B, A, b, c, \nu, l, 0)$
// The basic solution is now feasible for L_{aux}.
iterate the **while** loop of lines 3–12 of SIMPLEX until an optimal solution
    to L_{aux} is found
if the optimal solution to L_{aux} sets $\bar{x}_0$ to 0
    if $\bar{x}_0$ is basic
        perform one (degenerate) pivot to make it nonbasic
    from the final slack form of L_{aux}, remove x_0 from the constraints and
        restore the original objective function of L, but replace each basic
        variable in this objective function by the right-hand side of its
        associated constraint
    return the modified final slack form
else return "infeasible"

INITIALIZE-SIMPLEX

Test solution with $N = \{1, 2, \ldots, n\}$, $B = \{n+1, n+2, \ldots, n+m\}$, $\overline{x}_i = b_i$ for $i \in B$, $\overline{x}_i = 0$ otherwise.

INITIALIZE-SIMPLEX(A, b, c)

let k be the index of the minimum b_i
if $b_k \geq 0$ // is the initial basic solution feasible?
    return $(\{1, 2, \ldots, n\}, \{n+1, n+2, \ldots, n+m\}, A, b, c, 0)$
form L_{aux} by adding $-x_0$ to the left-hand side of each constraint
    and setting the objective function to $-x_0$
let (N, B, A, b, c, ν) be the resulting slack form for L_{aux}
$l = n + k$
// L_{aux} has $n + 1$ nonbasic variables and m basic variables.
$(N, B, A, b, c, \nu) = \text{PIVOT}(N, B, A, b, c, \nu, l, 0)$
// The basic solution is now feasible for L_{aux}.
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        restore the original objective function of L, but replace each basic
        variable in this objective function by the right-hand side of its
        associated constraint
    return the modified final slack form
else return "infeasible"

ℓ will be the leaving variable so that x_ℓ has the most negative value.

INITIALIZE-SIMPLEX

Test solution with $N = \{1, 2, \ldots, n\}$, $B = \{n+1, n+2, \ldots, n+m\}$, $\overline{x}_i = b_i$ for $i \in B$, $\overline{x}_i = 0$ otherwise.

INITIALIZE-SIMPLEX(A, b, c)

let k be the index of the minimum b_i
if $b_k \geq 0$ // is the initial basic solution feasible?
    return $(\{1, 2, \ldots, n\}, \{n+1, n+2, \ldots, n+m\}, A, b, c, 0)$
form L_{aux} by adding $-x_0$ to the left-hand side of each constraint
    and setting the objective function to $-x_0$
let (N, B, A, b, c, ν) be the resulting slack form for L_{aux}
$l = n + k$
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        restore the original objective function of L, but replace each basic
        variable in this objective function by the right-hand side of its
        associated constraint
    return the modified final slack form
else return "infeasible"

ℓ will be the leaving variable so that x_ℓ has the most negative value.

Pivot step with x_ℓ leaving and x_0 entering.

INITIALIZE-SIMPLEX

Test solution with $N = \{1, 2, \ldots, n\}$, $B = \{n+1, n+2, \ldots, n+m\}$, $\overline{x}_i = b_i$ for $i \in B$, $\overline{x}_i = 0$ otherwise.

```
INITIALIZE-SIMPLEX(A, b, c)
let k be the index of the minimum b_i
if b_k ≥ 0                    // is the initial basic solution feasible?
    return ({1, 2, ..., n}, {n + 1, n + 2, ..., n + m}, A, b, c, 0)
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    and setting the objective function to −x_0
let (N, B, A, b, c, ν) be the resulting slack form for L_aux
l = n + k
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    if x̄_0 is basic
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    from the final slack form of L_aux, remove x_0 from the constraints and
        restore the original objective function of L, but replace each basic
        variable in this objective function by the right-hand side of its
        associated constraint
    return the modified final slack form
else return "infeasible"
```

ℓ will be the leaving variable so that x_ℓ has the most negative value.

Pivot step with x_ℓ leaving and x_0 entering.

This pivot step does not change the value of any variable.

Example of INITIALIZE-SIMPLEX (1/3)

$$\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
& 2x_1 & - & x_2 & \leq & 2 \\
& x_1 & - & 5x_2 & \leq & -4 \\
& & x_1, x_2 & & \geq & 0
\end{array}$$

Example of Initialize-Simplex (1/3)

$$\begin{array}{lrcrcr} \text{maximize} & 2x_1 & - & x_2 & & \\ \text{subject to} & & & & & \\ & 2x_1 & - & x_2 & \leq & 2 \\ & x_1 & - & 5x_2 & \leq & -4 \\ & & x_1, x_2 & & \geq & 0 \end{array}$$

$\downarrow$ Formulating the auxiliary linear program

Example of INITIALIZE-SIMPLEX (1/3)

$$
\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
& 2x_1 & - & x_2 & \leq & 2 \\
& x_1 & - & 5x_2 & \leq & -4 \\
& & x_1, x_2 & & \geq & 0
\end{array}
$$

$\downarrow$ Formulating the auxiliary linear program

$$
\begin{array}{lrcrcrcr}
\text{maximize} & & & & - & x_0 & & \\
\text{subject to} & & & & & & & \\
& 2x_1 & - & x_2 & - & x_0 & \leq & 2 \\
& x_1 & - & 5x_2 & - & x_0 & \leq & -4 \\
& & x_1, x_2, x_0 & & & & \geq & 0
\end{array}
$$

Example of INITIALIZE-SIMPLEX (1/3)

$$\begin{array}{llcrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & & & \\
\text{subject to} & & & & & & & \\
& 2x_1 & - & x_2 & & & \leq & 2 \\
& x_1 & - & 5x_2 & & & \leq & -4 \\
& & & x_1, x_2 & & & \geq & 0
\end{array}$$

Formulating the auxiliary linear program

$$\begin{array}{llcrcrcr}
\text{maximize} & & & & - & x_0 & & \\
\text{subject to} & & & & & & & \\
& 2x_1 & - & x_2 & - & x_0 & \leq & 2 \\
& x_1 & - & 5x_2 & - & x_0 & \leq & -4 \\
& & & x_1, x_2, x_0 & & & \geq & 0
\end{array}$$

Converting into slack form

Example of INITIALIZE-SIMPLEX (1/3)

$$\begin{array}{lrcrcr} \text{maximize} & 2x_1 & - & x_2 & & \\ \text{subject to} & & & & & \\ & 2x_1 & - & x_2 & \leq & 2 \\ & x_1 & - & 5x_2 & \leq & -4 \\ & & x_1, x_2 & & \geq & 0 \end{array}$$

Formulating the auxiliary linear program

$$\begin{array}{lrcrcrcr} \text{maximize} & & & & - & x_0 & & \\ \text{subject to} & & & & & & & \\ & 2x_1 & - & x_2 & - & x_0 & \leq & 2 \\ & x_1 & - & 5x_2 & - & x_0 & \leq & -4 \\ & & x_1, x_2, x_0 & & & & \geq & 0 \end{array}$$

Converting into slack form

$$\begin{array}{rcrcrcrcr} z & = & & & & & & - & x_0 \\ x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\ x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0 \end{array}$$

Example of INITIALIZE-SIMPLEX (1/3)

$$\begin{array}{lrcrcr} \text{maximize} & 2x_1 & - & x_2 & & \\ \text{subject to} & & & & & \\ & 2x_1 & - & x_2 & \leq & 2 \\ & x_1 & - & 5x_2 & \leq & -4 \\ & & x_1, x_2 & & \geq & 0 \end{array}$$

Formulating the auxiliary linear program

$$\begin{array}{lrcrcrcr} \text{maximize} & & & & - & x_0 & & \\ \text{subject to} & & & & & & & \\ & 2x_1 & - & x_2 & - & x_0 & \leq & 2 \\ & x_1 & - & 5x_2 & - & x_0 & \leq & -4 \\ & & x_1, x_2, x_0 & & & & \geq & 0 \end{array}$$

Basic solution $(0, 0, 0, 2, -4)$ not feasible!

Converting into slack form

$$\begin{array}{rcrcrcrcr} z & = & & & & & & - & x_0 \\ x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\ x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0 \end{array}$$

Example of INITIALIZE-SIMPLEX (2/3)

$$
\begin{array}{rcrcrcrcr}
z & = & & & & & & - & x_0 \\
x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\
x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0
\end{array}
$$

Example of INITIALIZE-SIMPLEX (2/3)

$$\begin{array}{rcrcrcrcr} z & = & & & & & & - & x_0 \\ x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\ x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0 \end{array}$$

Pivot with x_0 entering and x_4 leaving

Example of INITIALIZE-SIMPLEX (2/3)

$$
\begin{array}{rcrcrcrcr}
z & = & & & & & & - & x_0 \\
x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\
x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0
\end{array}
$$

Pivot with x_0 entering and x_4 leaving

$$
\begin{array}{rcrcrcrcr}
z & = & -4 & - & x_1 & + & 5x_2 & - & x_4 \\
x_0 & = & 4 & + & x_1 & - & 5x_2 & + & x_4 \\
x_3 & = & 6 & - & x_1 & - & 4x_2 & + & x_4
\end{array}
$$

Example of INITIALIZE-SIMPLEX (2/3)

$$\begin{array}{rcrcrcrcr} z & = & & & & & & - & x_0 \\ x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\ x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0 \end{array}$$

Pivot with x_0 entering and x_4 leaving

$$\begin{array}{rcrcrcrcr} z & = & -4 & - & x_1 & + & 5x_2 & - & x_4 \\ x_0 & = & 4 & + & x_1 & - & 5x_2 & + & x_4 \\ x_3 & = & 6 & - & x_1 & - & 4x_2 & + & x_4 \end{array}$$

Basic solution $(4, 0, 0, 6, 0)$ is feasible!

Example of INITIALIZE-SIMPLEX (2/3)

$$\begin{array}{rcrcrcrcr} z & = & & & & & & - & x_0 \\ x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\ x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0 \end{array}$$

Pivot with x_0 entering and x_4 leaving

$$\begin{array}{rcrcrcrcr} z & = & -4 & - & x_1 & + & 5x_2 & - & x_4 \\ x_0 & = & 4 & + & x_1 & - & 5x_2 & + & x_4 \\ x_3 & = & 6 & - & x_1 & - & 4x_2 & + & x_4 \end{array}$$

Basic solution $(4, 0, 0, 6, 0)$ is feasible!

Pivot with x_2 entering and x_0 leaving

Example of INITIALIZE-SIMPLEX (2/3)

$$\begin{array}{rcrcrcrcr} z & = & & & & & & - & x_0 \\ x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\ x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0 \end{array}$$

Pivot with x_0 entering and x_4 leaving

$$\begin{array}{rcrcrcrcr} z & = & -4 & - & x_1 & + & 5x_2 & - & x_4 \\ x_0 & = & 4 & + & x_1 & - & 5x_2 & + & x_4 \\ x_3 & = & 6 & - & x_1 & - & 4x_2 & + & x_4 \end{array}$$

Basic solution $(4, 0, 0, 6, 0)$ is feasible!

Pivot with x_2 entering and x_0 leaving

$$\begin{array}{rcrcrcrcr} z & = & & - & x_0 & & & & \\ x_2 & = & \frac{4}{5} & - & \frac{x_0}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\ x_3 & = & \frac{14}{5} & + & \frac{4x_0}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5} \end{array}$$

Example of INITIALIZE-SIMPLEX (2/3)

$$\begin{array}{rcrcrcrcr} z & = & & & & & & - & x_0 \\ x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\ x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0 \end{array}$$

Pivot with x_0 entering and x_4 leaving

$$\begin{array}{rcrcrcrcr} z & = & -4 & - & x_1 & + & 5x_2 & - & x_4 \\ x_0 & = & 4 & + & x_1 & - & 5x_2 & + & x_4 \\ x_3 & = & 6 & - & x_1 & - & 4x_2 & + & x_4 \end{array}$$

Basic solution $(4, 0, 0, 6, 0)$ is feasible!

Pivot with x_2 entering and x_0 leaving

$$\begin{array}{rcrcrcrcr} z & = & & - & x_0 & & & & \\ x_2 & = & \frac{4}{5} & - & \frac{x_0}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\ x_3 & = & \frac{14}{5} & + & \frac{4x_0}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5} \end{array}$$

Optimal solution has $x_0 = 0$, hence the initial problem was feasible!

Example of INITIALIZE-SIMPLEX (3/3)

$$\begin{array}{rcrcrcrcr}
z & = & & - & x_0 & & & & \\
x_2 & = & \frac{4}{5} & - & \frac{x_0}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\
x_3 & = & \frac{14}{5} & + & \frac{4x_0}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5}
\end{array}$$

Example of INITIALIZE-SIMPLEX (3/3)

$$
\begin{array}{rclcrcrcr}
z & = & & - & x_0 & & & & \\
x_2 & = & \frac{4}{5} & - & \frac{x_0}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\
x_3 & = & \frac{14}{5} & + & \frac{4x_0}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5}
\end{array}
$$

Set $x_0 = 0$ and express objective function by non-basic variables

Example of INITIALIZE-SIMPLEX (3/3)

$$\begin{aligned} z &= & & - & & x_0 \\ x_2 &= & \frac{4}{5} & - & & \frac{x_0}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\ x_3 &= & \frac{14}{5} & + & & \frac{4x_0}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5} \end{aligned}$$

Set $x_0 = 0$ and express objective function by non-basic variables

$$2x_1 - x_2 = 2x_1 - \left(\tfrac{4}{5} - \tfrac{x_0}{5} + \tfrac{x_1}{5} + \tfrac{x_4}{5}\right)$$

$$\begin{aligned} z &= -\frac{4}{5} + \frac{9x_1}{5} - \frac{x_4}{5} \\ x_2 &= \frac{4}{5} + \frac{x_1}{5} + \frac{x_4}{5} \\ x_3 &= \frac{14}{5} - \frac{9x_1}{5} + \frac{x_4}{5} \end{aligned}$$

Example of INITIALIZE-SIMPLEX (3/3)

$$\begin{aligned} z &= && - && x_0 \\ x_2 &= \frac{4}{5} && - && \frac{x_0}{5} && + && \frac{x_1}{5} && + && \frac{x_4}{5} \\ x_3 &= \frac{14}{5} && + && \frac{4x_0}{5} && - && \frac{9x_1}{5} && + && \frac{x_4}{5} \end{aligned}$$

Set $x_0 = 0$ and express objective function by non-basic variables

$2x_1 - x_2 = 2x_1 - (\frac{4}{5} - \frac{x_0}{5} + \frac{x_1}{5} + \frac{x_4}{5})$

$$\begin{aligned} z &= -\frac{4}{5} && + && \frac{9x_1}{5} && - && \frac{x_4}{5} \\ x_2 &= \frac{4}{5} && + && \frac{x_1}{5} && + && \frac{x_4}{5} \\ x_3 &= \frac{14}{5} && - && \frac{9x_1}{5} && + && \frac{x_4}{5} \end{aligned}$$

Basic solution $(0, \frac{4}{5}, \frac{14}{5}, 0)$, which is feasible!

Example of INITIALIZE-SIMPLEX (3/3)

$$\begin{array}{rcrcrcrcr} z & = & & - & x_0 & & & & \\ x_2 & = & \frac{4}{5} & - & \frac{x_0}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\ x_3 & = & \frac{14}{5} & + & \frac{4x_0}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5} \end{array}$$

$2x_1 - x_2 = 2x_1 - (\frac{4}{5} - \frac{x_0}{5} + \frac{x_1}{5} + \frac{x_4}{5})$

Set $x_0 = 0$ and express objective function by non-basic variables

$$\begin{array}{rcrcrcr} z & = & -\frac{4}{5} & + & \frac{9x_1}{5} & - & \frac{x_4}{5} \\ x_2 & = & \frac{4}{5} & + & \frac{x_1}{5} & + & \frac{x_4}{5} \\ x_3 & = & \frac{14}{5} & - & \frac{9x_1}{5} & + & \frac{x_4}{5} \end{array}$$

Basic solution $(0, \frac{4}{5}, \frac{14}{5}, 0)$, which is feasible!

Lemma 29.12

If a linear program L has no feasible solution, then INITIALIZE-SIMPLEX returns "infeasible". Otherwise, it returns a valid slack form for which the basic solution is feasible.

Fundamental Theorem of Linear Programming

Theorem 29.13 (Fundamental Theorem of Linear Programming)

Any linear program L, given in standard form, either

1. has an optimal solution with a finite objective value,
2. is infeasible, or
3. is unbounded.

If L is infeasible, SIMPLEX returns "infeasible". If L is unbounded, SIMPLEX returns "unbounded". Otherwise, SIMPLEX returns an optimal solution with a finite objective value.

Fundamental Theorem of Linear Programming

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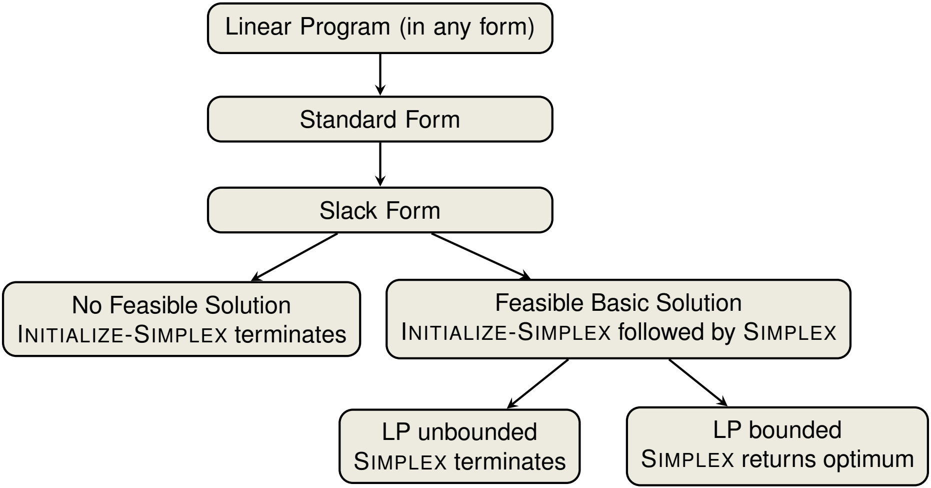
Proof requires the concept of duality, which is not covered in this course (for details see CLRS3, Chapter 29.4)

Workflow for Solving Linear Programs



Linear Programming and Simplex: Summary and Outlook

Linear Programming

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- extremely versatile tool for modelling problems of all kinds

Linear Programming and Simplex: Summary and Outlook

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- basis of Integer Programming, to be discussed in later lectures

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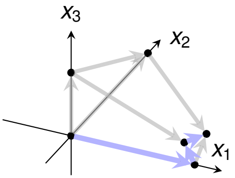
Simplex Algorithm

- In practice: usually terminates in polynomial time, i.e., $O(m+n)$



Linear Programming and Simplex: Summary and Outlook

Linear Programming

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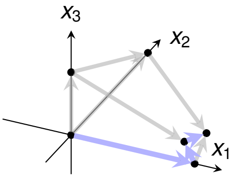
Simplex Algorithm

- In practice: usually terminates in polynomial time, i.e., $O(m+n)$
- In theory: even with anti-cycling may need exponential time



Linear Programming and Simplex: Summary and Outlook

Linear Programming

- extremely versatile tool for modelling problems of all kinds
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Simplex Algorithm

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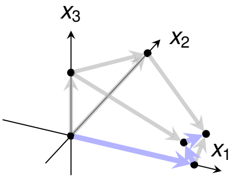

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Linear Programming and Simplex: Summary and Outlook

Linear Programming

- extremely versatile tool for modelling problems of all kinds
- basis of Integer Programming, to be discussed in later lectures

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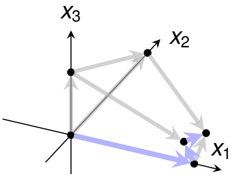

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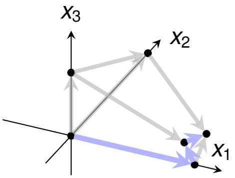

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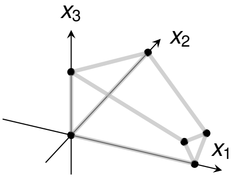

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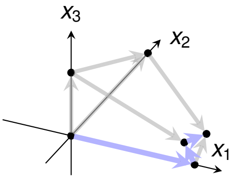

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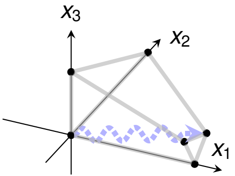

Test your Understanding

Which of the following statements are true?

1. In each iteration of the Simplex algorithm, the objective function increases.
2. There exist linear programs that have exactly two optimal solutions.
3. There exist linear programs that have infinitely many optimal solutions.
4. The Simplex algorithm always runs in worst-case polynomial time.

III. Approximation Algorithms: Covering Problems

Thomas Sauerwald

Easter 2021

UNIVERSITY OF CAMBRIDGE

Outline

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Vertex Cover

The Set-Covering Problem

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1. If inputs (or solutions) are small, an algorithm with exponential running time may be satisfactory.
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We will call these **approximation algorithms**.

Performance Ratios for Approximation Algorithms

Approximation Ratio

An algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size n, the cost C of the returned solution and optimal cost C^* satisfy:

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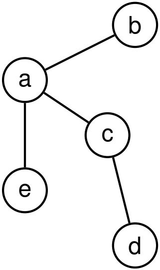

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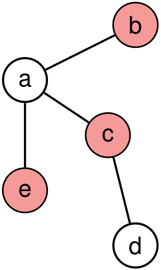

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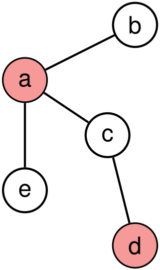

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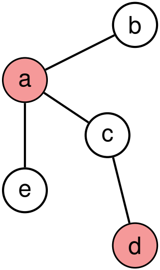

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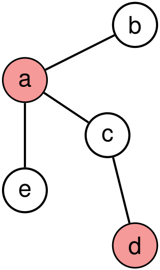

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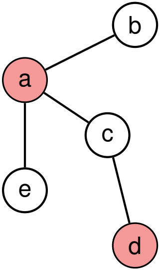

Applications:

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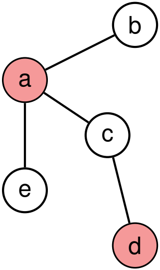

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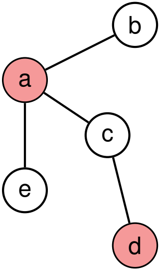

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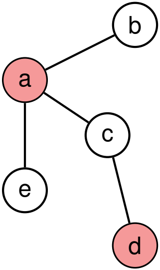

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- Extensions: weighted vertices or hypergraphs ($\rightsquigarrow$ Set-Covering Problem)

Exercise: Be creative and design your own algorithm for VERTEX-COVER!

An Approximation Algorithm based on Greedy

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APPROX-VERTEX-COVER(G)
C = ∅
E' = G.E
while E' ≠ ∅
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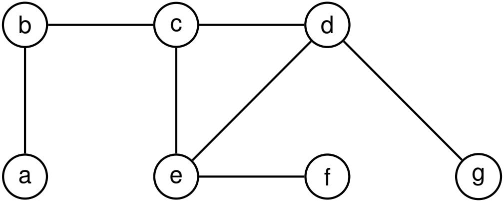

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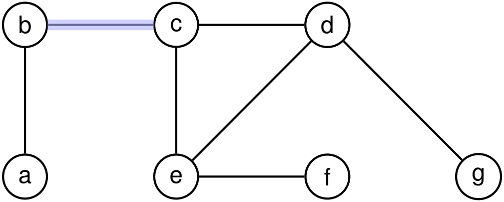

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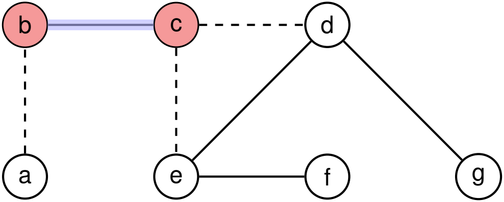

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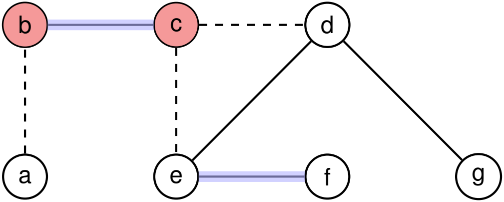

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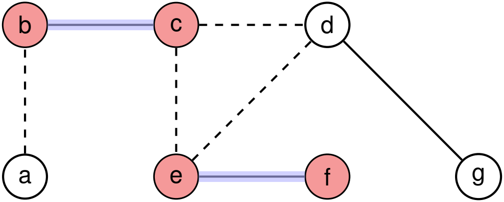

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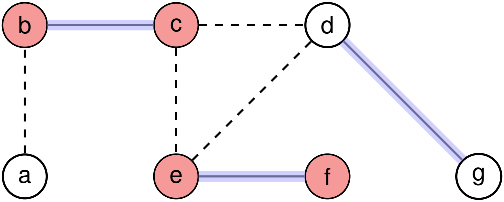

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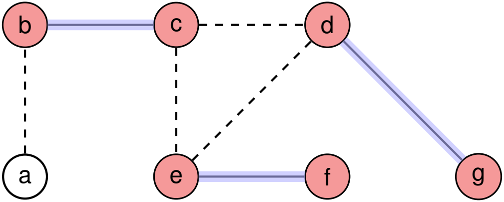

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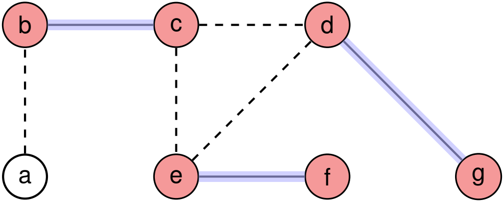

APPROX-VERTEX-COVER produces a set of size 6.

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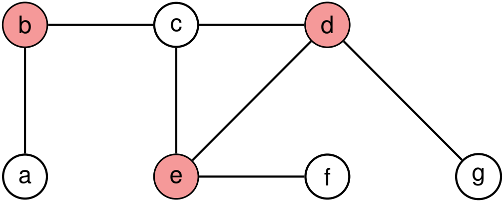

The optimal solution has size 3.

Analysis of Greedy for Vertex Cover

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APPROX-VERTEX-COVER is a poly-time 2-approximation algorithm.

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- Let $A \subseteq E$ denote the set of edges picked in line 4

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Theorem 35.1

APPROX-VERTEX-COVER is a poly-time 2-approximation algorithm.

Proof:

- Running time is $O(V + E)$ (using adjacency lists to represent E')
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A "vertex-based" Greedy that adds **one** vertex at each iteration fails to achieve an approximation ratio of 2 (Supervision Exercise)!

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Strategies to cope with NP-complete problems

1. If inputs are small, an algorithm with exponential running time may be satisfactory.
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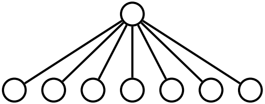

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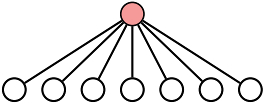

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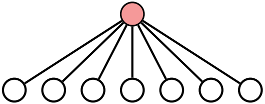

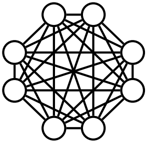

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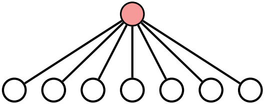

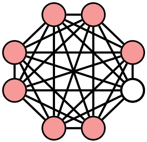

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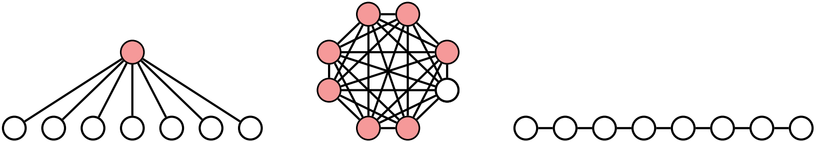

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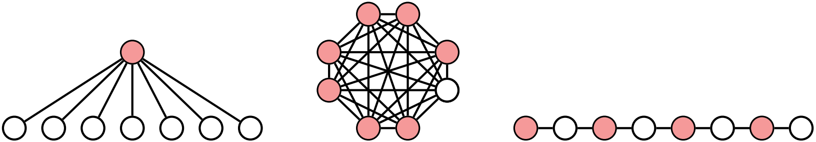
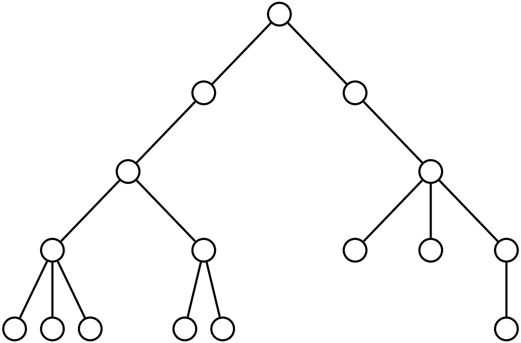


Vertex Cover on Trees



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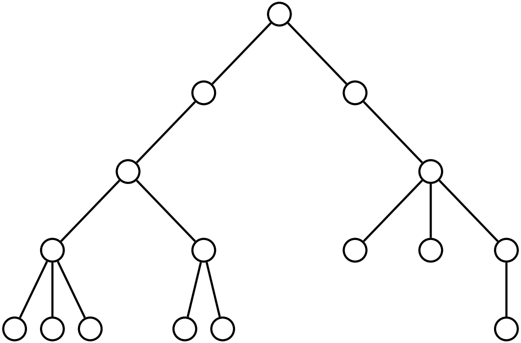

There exists an optimal vertex cover which does not include any leaves.

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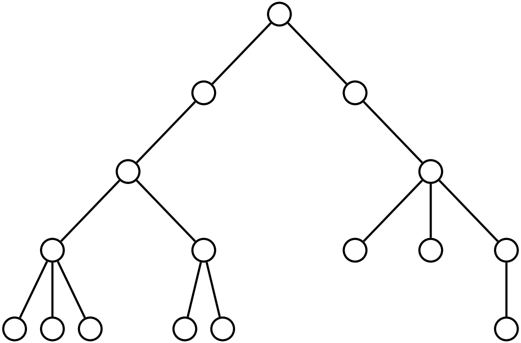

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Exchange-Argument: Replace any leaf in the cover by its parent.

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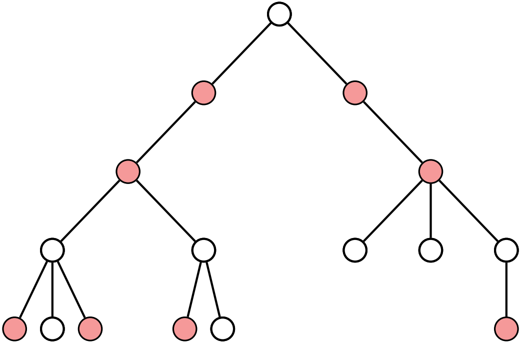

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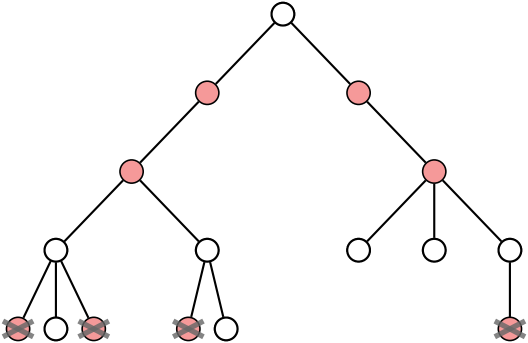

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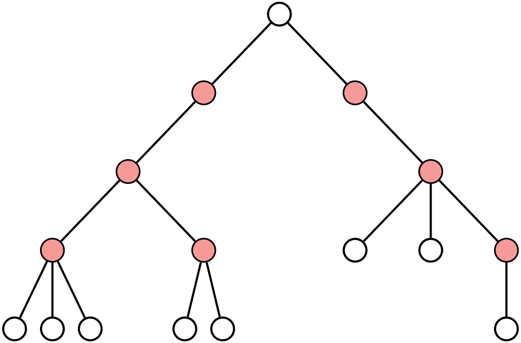

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Clear: Running time is $O(V)$, and the returned solution is a vertex cover.

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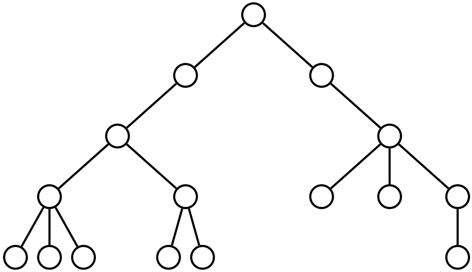
Solution is also optimal. (Use inductively the existence of an optimal vertex cover without leaves)

Execution on a Small Example



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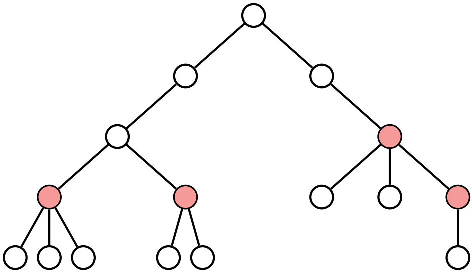

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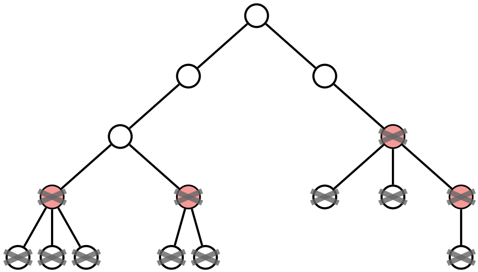

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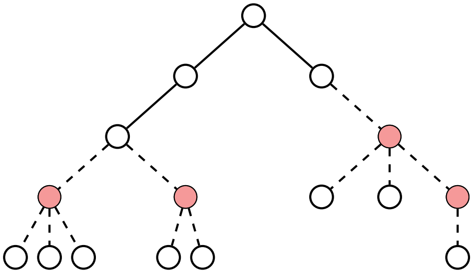

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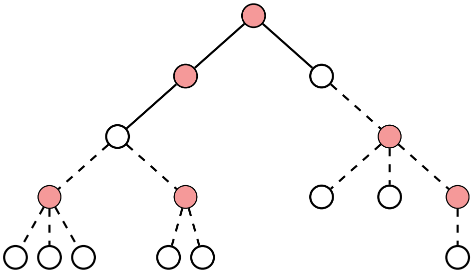

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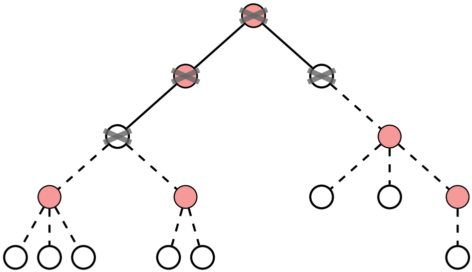

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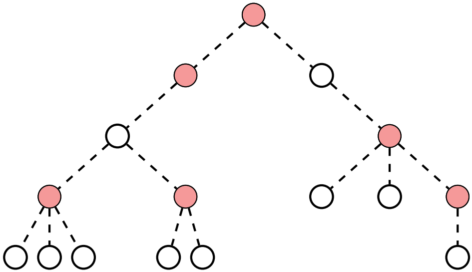

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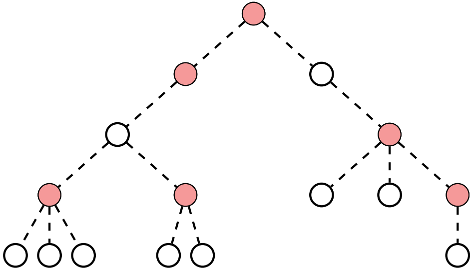

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Problem can be also solved on bipartite graphs, using Max-Flows and Min-Cuts.

Exact Algorithms

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Simple Brute-Force Search would take $\approx \binom{n}{k} = \Theta(n^k)$ time.

Towards a more efficient Search

Substructure Lemma

Consider a graph $G = (V, E)$, edge $\{u, v\} \in E(G)$ and integer $k \geq 1$. Let G_u be the graph obtained by deleting u and its incident edges (G_v is defined similarly). Then G has a vertex cover of size k if and only if G_u or G_v (or both) have a vertex cover of size $k - 1$.

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Reminiscent of Dynamic Programming.

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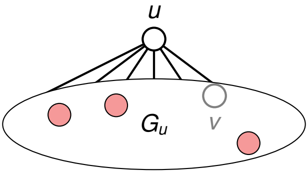

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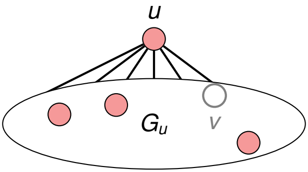

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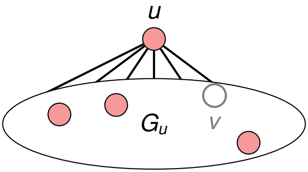

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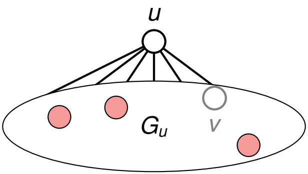
- $\Leftarrow$ Assume G_u has a vertex cover C_u of size $k-1$. Adding u yields a vertex cover of G which is of size k
- $\Rightarrow$ Assume G has a vertex cover C of size k, which contains, say u. Removing u from C yields a vertex cover of G_u which is of size $k-1$. $\square$



A More Efficient Search Algorithm

VERTEX-COVER-SEARCH(G, k)

1: **if** $E = \emptyset$ **return** $\emptyset$
2: **if** $k = 0$ and $E \neq \emptyset$ **return** $\perp$
3: Pick an arbitrary edge $(u, v) \in E$
4: $S_1 =$ VERTEX-COVER-SEARCH$(G_u, k-1)$
5: $S_2 =$ VERTEX-COVER-SEARCH$(G_v, k-1)$
6: **if** $S_1 \neq \perp$ **return** $S_1 \cup \{u\}$
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Correctness follows by the Substructure Lemma and induction.

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exponential in k, but much better than $\Theta(n^k)$ (i.e., still polynomial for $k = O(\log n)$)

Outline

Introduction

Vertex Cover

The Set-Covering Problem

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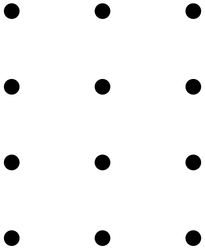

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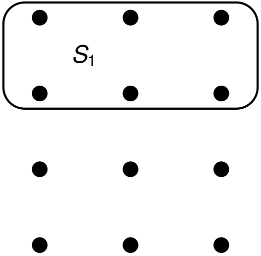

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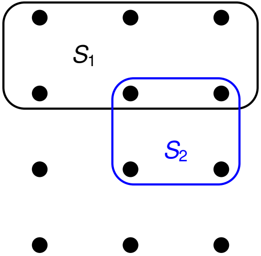

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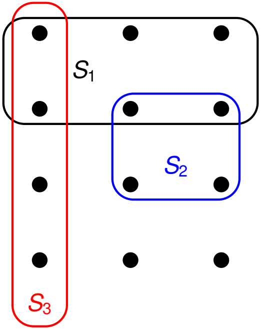

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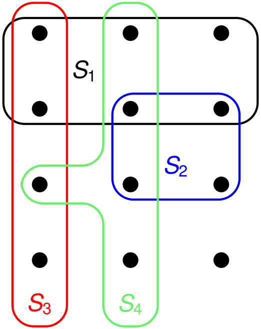

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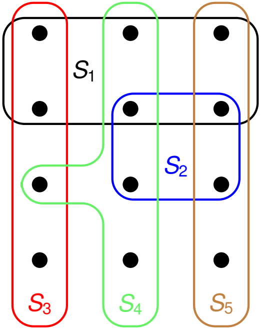

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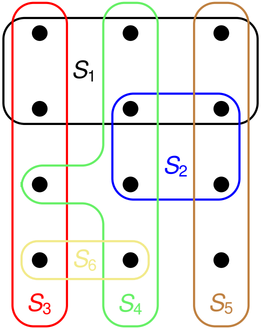

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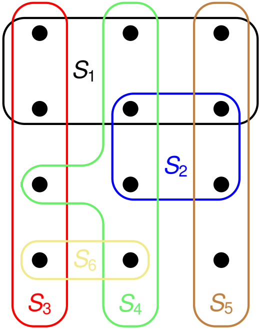

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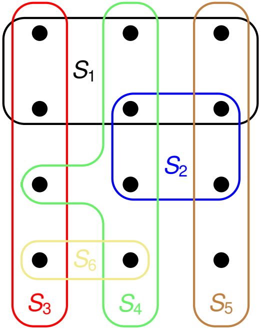

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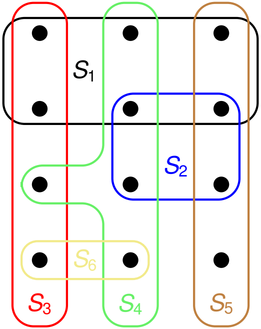

Remarks:

- generalisation of the vertex-cover problem and hence also NP-hard.
- models resource allocation problems, e.g., wireless coverage

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Strategy: Pick the set S that covers the largest number of uncovered elements.

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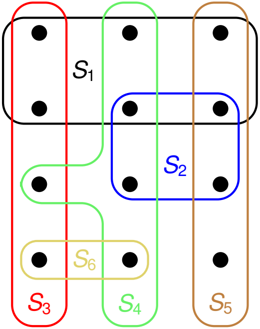

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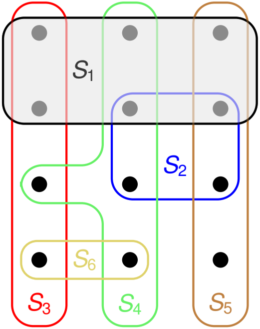

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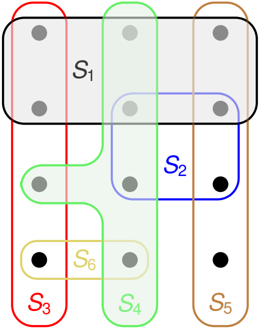

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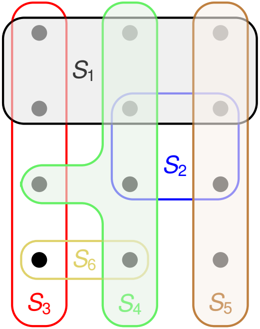

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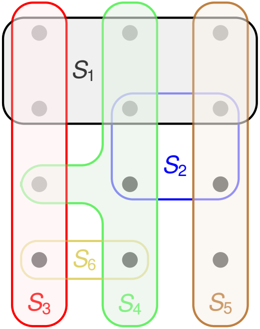

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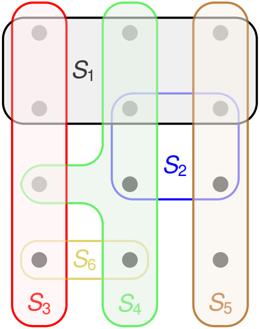

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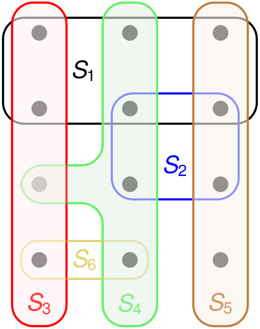

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Optimal cover is $\mathcal{C} = \{S_3, S_4, S_5\}$

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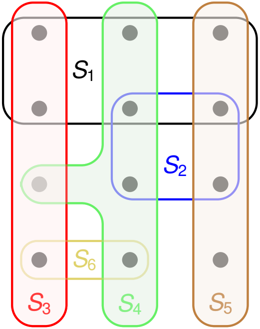

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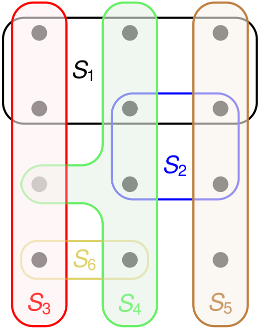

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How good is the approximation ratio?

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Theorem 35.4

GREEDY-SET-COVER is a polynomial-time $\rho(n)$-algorithm, where

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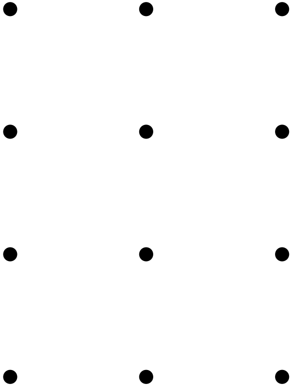
Notice that in the mathematical analysis, S_i is the set chosen in iteration i - not to be confused with the sets $S_1, S_2, \ldots, S_6$ in the example.

Illustration of Costs for Greedy picking S_1, S_4, S_5 and S_3



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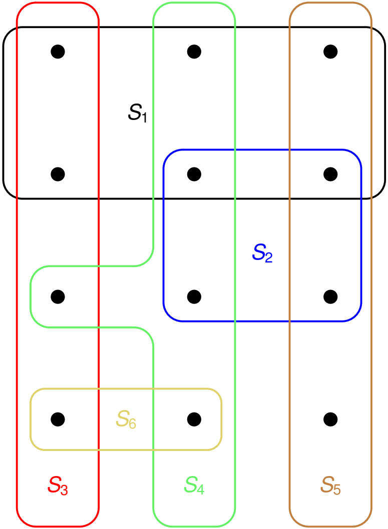

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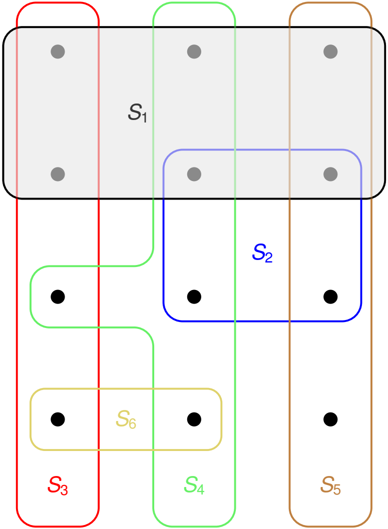

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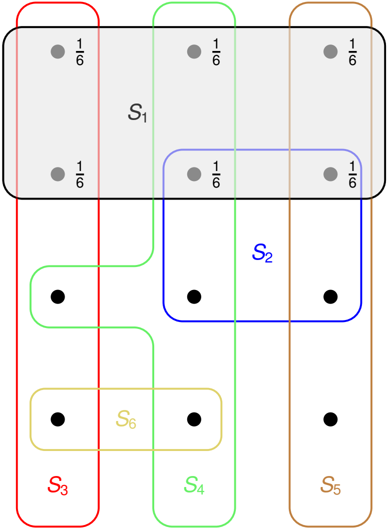

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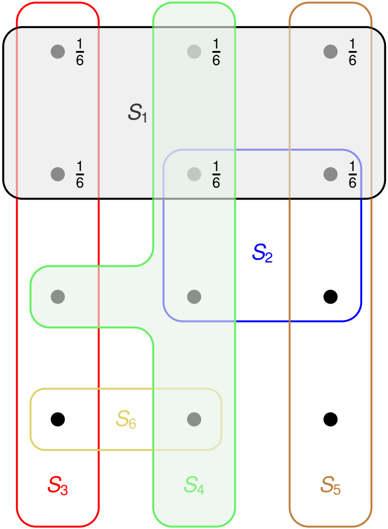

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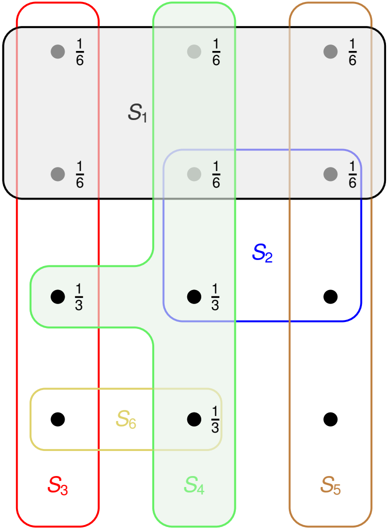

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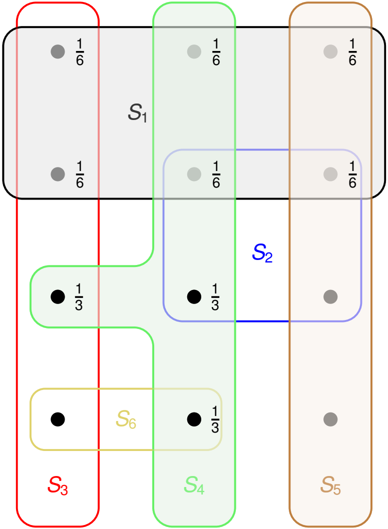

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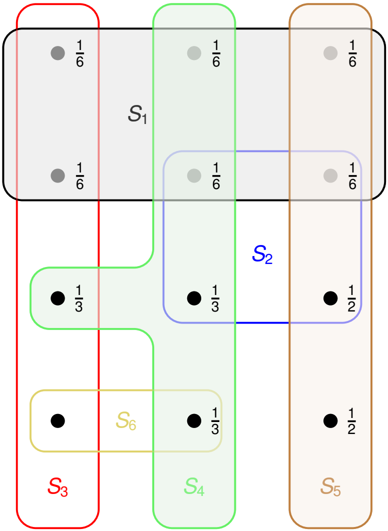

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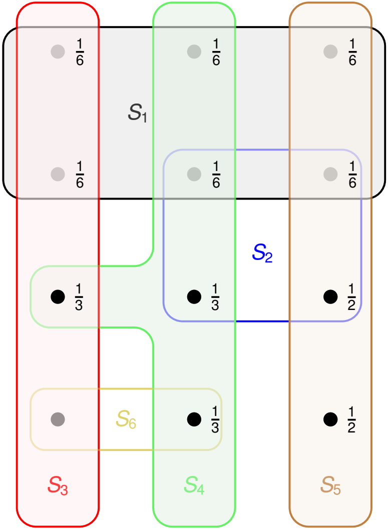

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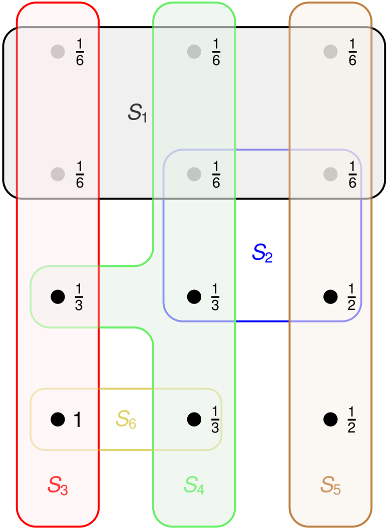

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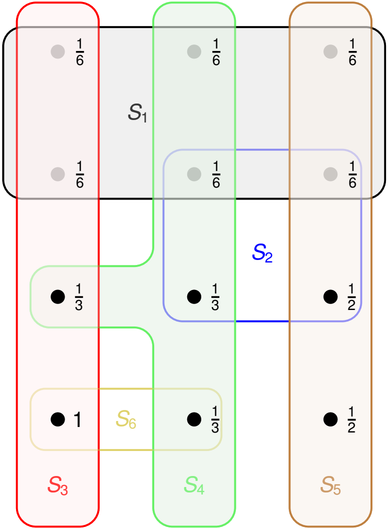

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{2} + \frac{1}{2} + 1 = ??$$

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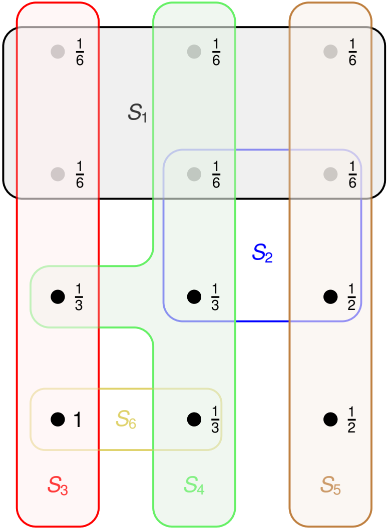

$$\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{2}+\frac{1}{2}+1=4$$

Proof of Theorem 35.4 (1/2)

Definition of cost

If x is covered for the first time by a set S_i, then $c_x := \frac{1}{\left|S_i \setminus (S_1 \cup S_2 \cup \cdots \cup S_{i-1})\right|}$.

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Remaining uncovered elements in S

- For any $S \in \mathcal{F}$ and $i = 1, 2, \ldots, |\mathcal{C}| = k$ let $u_i := |S \setminus (S_1 \cup S_2 \cup \cdots \cup S_i)|$

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Sets chosen by the algorithm

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&= \sum_{i=1}^{k} (H(u_{i-1}) - H(u_i)) = H(u_0) - H(u_k)
\end{aligned}$$

Proof of Theorem 35.4 (2/2)

Proof of the Key Inequality $\boxed{\sum_{x \in S} c_x \leq H(|S|)}$

- For any $S \in \mathcal{F}$ and $i = 1, 2, \ldots, |\mathcal{C}| = k$ let $u_i := |S \setminus (S_1 \cup S_2 \cup \cdots \cup S_i)|$

$\Rightarrow$ $|S| = u_0 \geq u_1 \geq \cdots \geq u_{|\mathcal{C}|} = 0$ and $u_{i-1} - u_i$ counts the items in S covered first time by S_i.

$\Rightarrow$

$$\sum_{x \in S} c_x = \sum_{i=1}^{k} (u_{i-1} - u_i) \cdot \frac{1}{|S_i \setminus (S_1 \cup S_2 \cup \cdots \cup S_{i-1})|}$$

- Further, by definition of the Greedy-Set-Cover:

$$|S_i \setminus (S_1 \cup S_2 \cup \cdots \cup S_{i-1})| \geq |S \setminus (S_1 \cup S_2 \cup \cdots \cup S_{i-1})| = u_{i-1}.$$

- Combining the last inequalities gives:

$$\begin{aligned}
\sum_{x \in S} c_x \leq \sum_{i=1}^{k} (u_{i-1} - u_i) \cdot \frac{1}{u_{i-1}} &= \sum_{i=1}^{k} \sum_{j=u_i+1}^{u_{i-1}} \frac{1}{u_{i-1}} \\
&\leq \sum_{i=1}^{k} \sum_{j=u_i+1}^{u_{i-1}} \frac{1}{j} \\
&= \sum_{i=1}^{k} (H(u_{i-1}) - H(u_i)) = H(u_0) - H(u_k) = H(|S|). \qquad \square
\end{aligned}$$

Set-Covering Problem (Summary)

Theorem 35.4

GREEDY-SET-COVER is a polynomial-time $\rho(n)$-algorithm, where

$$\rho(n) = H(\max\{|S| : S \in \mathcal{F}\}) \leq \ln(n) + 1.$$

Set-Covering Problem (Summary)

The same approach also gives an approximation ratio of $O(\ln(n))$ if there exists a cost function $c : \mathcal{F} \to \mathbb{R}^+$

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- Is the bound on the approximation ratio in Theorem 35.4 tight?
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Lower Bound

Unless P=NP, there is no $c \cdot \ln(n)$ polynomial-time approximation algorithm for some constant $0 < c < 1$.

Example where the solution of Greedy is bad

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- Given any integer $k \geq 3$

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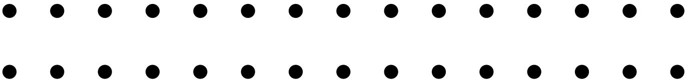

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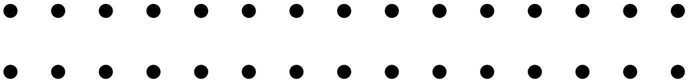

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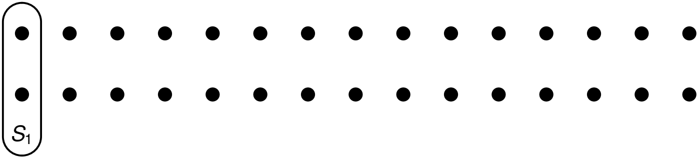

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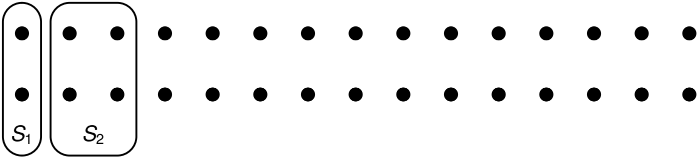

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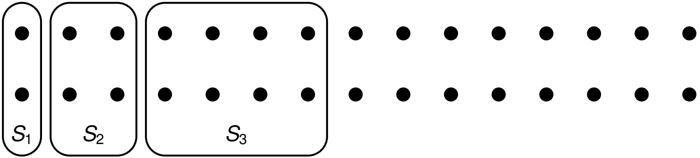

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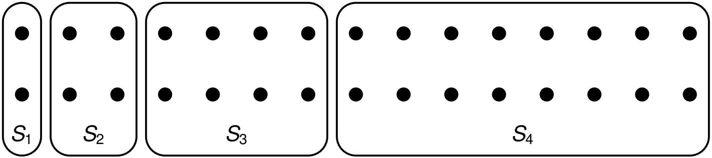

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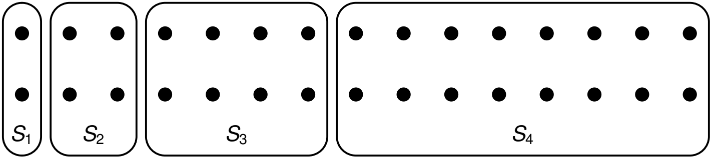

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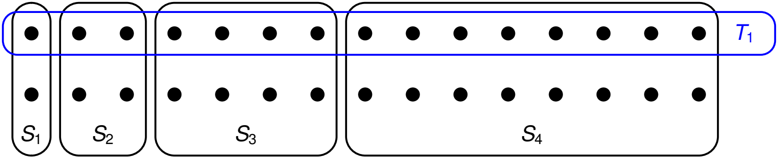

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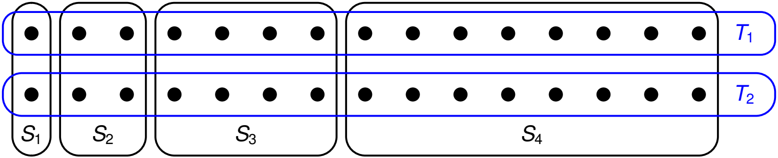

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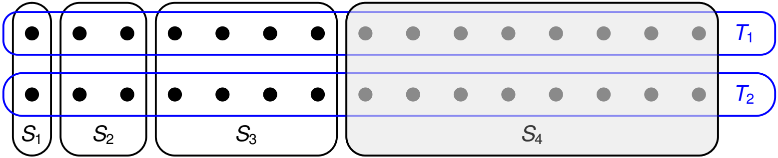

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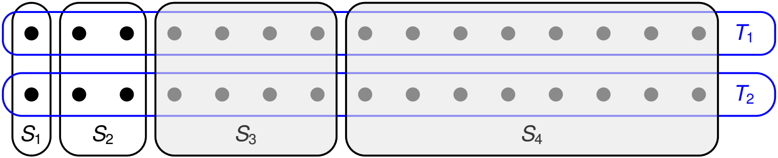

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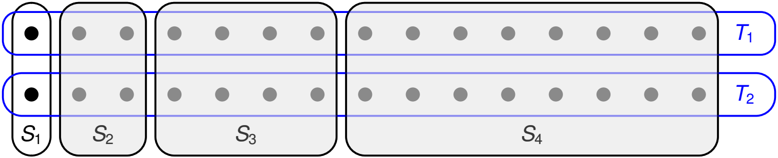

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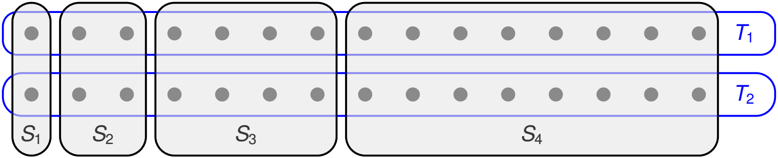

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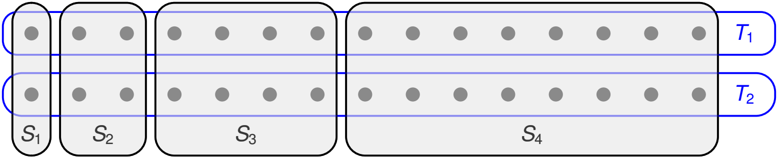

Solution of Greedy consists of k sets.

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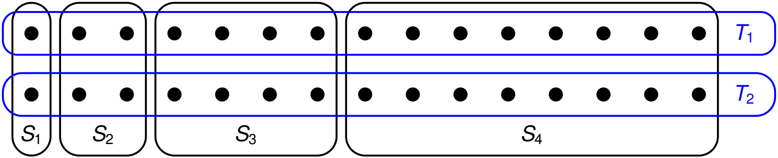

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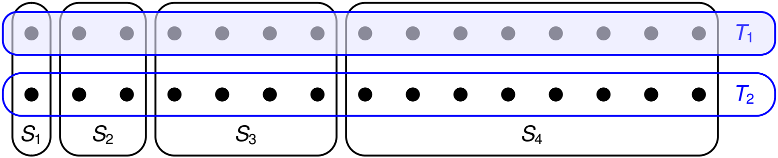

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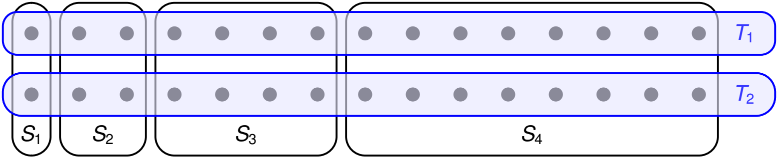

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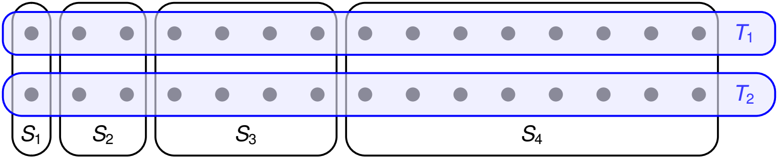

Solution of Greedy consists of k sets.

Optimum consists of 2 sets.

Exercise: Consider the vertex cover problem, restricted to a graph where every vertex has exactly 3 neighbours. Which approximation ratio can we obtain?

1. 1 (i.e., I can solve it exactly!!!)
2. 2
3. $11/6 = 2 - 1/6$
4. $H(n) \leq log(n)$

IV. Approximation Algorithms via Exact Algorithms

Thomas Sauerwald

Easter 2021

UNIVERSITY OF CAMBRIDGE

Outline

The Subset-Sum Problem

Parallel Machine Scheduling

Bonus Material: A PTAS for Parallel Machine Scheduling (non-examinable)

The Subset-Sum Problem

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\ x_i \in S'} x_i \leq t$.

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This problem is NP-hard

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$t = 13$ tons

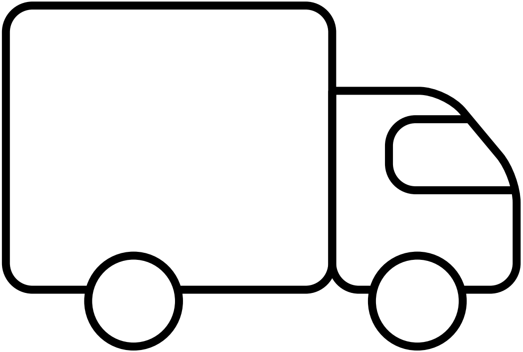
$x_1 = 10$

$x_2 = 4$

$x_3 = 5$

$x_4 = 6$

$x_5 = 1$



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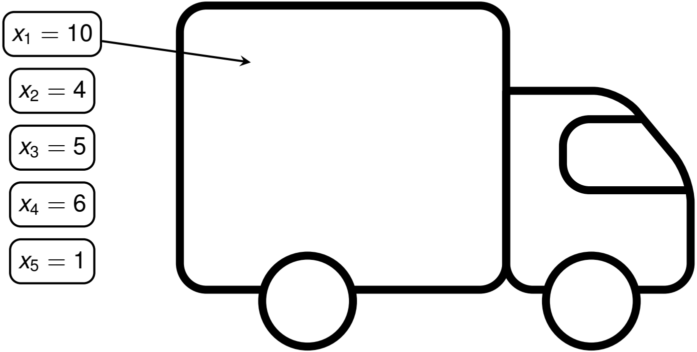

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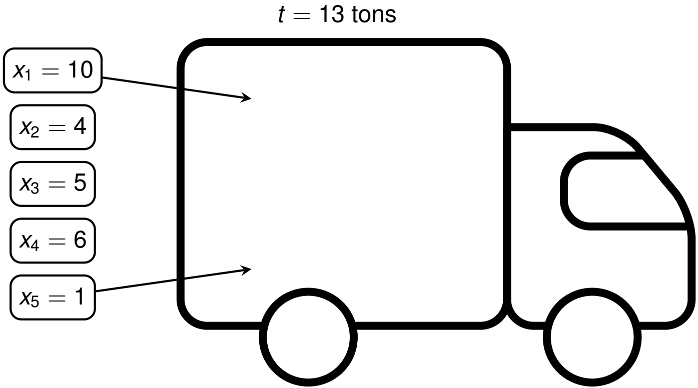

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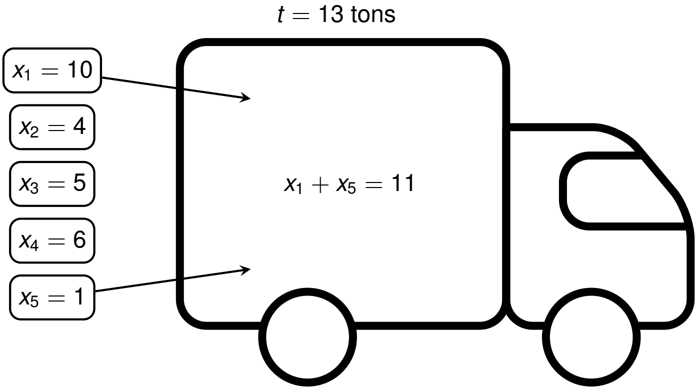

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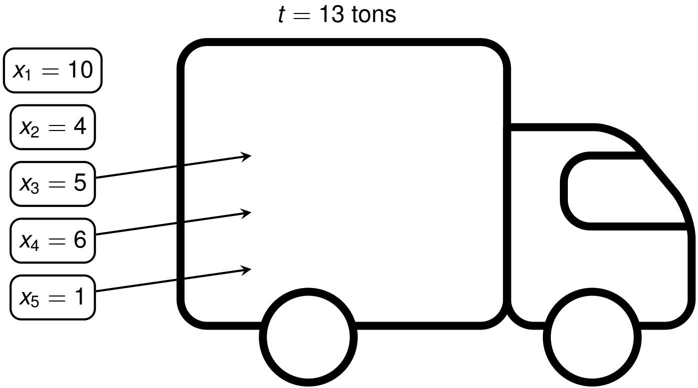

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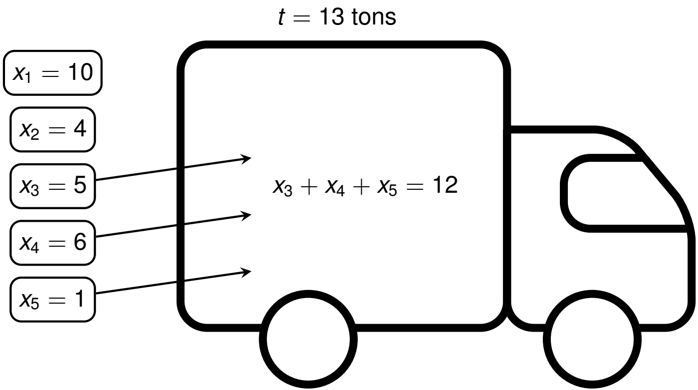

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

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Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

EXACT-SUBSET-SUM(S, t)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
  $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
  remove from L_i every element that is greater than t
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Returns the merged list (in sorted order and without duplicates)

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Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

implementable in time $O(|L_{i-1}|)$ (like Merge-Sort)

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- $S = \{1, 4, 5\}$, $t = 10$

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- $L_0 = \langle 0 \rangle$
- $L_1 = \langle 0, 1 \rangle$

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

EXACT-SUBSET-SUM(S, t)

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n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    remove from L_i every element that is greater than t
return the largest element in L_n
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can be shown by induction on n

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There are 2^i subsets of $\{x_1, x_2, \ldots, x_i\}$.

Better runtime if t and/or $|L_i|$ are small.

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Towards a FPTAS

Idea: Don't need to maintain two values in L which are close to each other.

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```
TRIM(L, δ)
let m be the length of L
L' = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L'
        last = y_i
return L'
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TRIM works in time $\Theta(m)$, if L is given in sorted order.

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
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$$\delta = 0.1$$

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

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$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

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Illustration of the Trim Operation

TRIM(L, δ)

```
1  let m be the length of L
2  L' = ⟨y_1⟩
3  last = y_1
4  for i = 2 to m
5      if y_i > last · (1 + δ)      // y_i ≥ last because L is sorted
6          append y_i onto the end of L'
7          last = y_i
8  return L'
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The FPTAS

APPROX-SUBSET-SUM(S, t, ϵ)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    $L_i = \text{TRIM}(L_i, \epsilon/2n)$
    remove from L_i every element that is greater than t
let z^* be the largest value in L_n
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Repeated application of TRIM to make sure L_i's remain short.

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    remove from L_i every element that is greater than t
return the largest element in L_n

- We must bound the inaccuracy introduced by repeated trimming

The FPTAS

APPROX-SUBSET-SUM(S, t, ϵ)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    $L_i = \text{TRIM}(L_i, \epsilon/2n)$
    remove from L_i every element that is greater than t
let z^* be the largest value in L_n
return z^*

Repeated application of TRIM to make sure L_i's remain short.

EXACT-SUBSET-SUM(S, t)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    remove from L_i every element that is greater than t
return the largest element in L_n

- We must bound the inaccuracy introduced by repeated trimming
- We must show that the algorithm is polynomial time

The FPTAS

APPROX-SUBSET-SUM(S, t, ϵ)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
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EXACT-SUBSET-SUM(S, t)

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return the largest element in L_n

- We must bound the inaccuracy introduced by repeated trimming
- We must show that the algorithm is polynomial time

Solution is a careful choice of δ!

Running through an Example (CLRS3)

APPROX-SUBSET-SUM(S, t, ϵ)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
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Running through an Example (CLRS3)

APPROX-SUBSET-SUM(S, t, ϵ)

$n = |S|$
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    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
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    remove from L_i every element that is greater than t
let z^* be the largest value in L_n
return z^*

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

Running through an Example (CLRS3)

APPROX-SUBSET-SUM(S, t, ϵ)

```
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
      L_i = TRIM(L_i, ϵ/2n)
      remove from L_i every element that is greater than t
let z* be the largest value in L_n
return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

Running through an Example (CLRS3)

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$

Running through an Example (CLRS3)

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$

Running through an Example (CLRS3)

```
APPROX-SUBSET-SUM(S, t, ϵ)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ϵ/2n)
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```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$

Running through an Example (CLRS3)

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i−1}, L_{i−1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

Running through an Example (CLRS3)

APPROX-SUBSET-SUM(S, t, ϵ)

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
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7  let z* be the largest value in L_n
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- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$

Running through an Example (CLRS3)

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101\rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0\rangle$
- line 4: $L_1 = \langle 0, 104\rangle$
- line 5: $L_1 = \langle 0, 104\rangle$
- line 6: $L_1 = \langle 0, 104\rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206\rangle$
- line 5: $L_2 = \langle 0, 102, 206\rangle$

Running through an Example (CLRS3)

APPROX-SUBSET-SUM(S, t, ϵ)

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
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```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$

Running through an Example (CLRS3)

APPROX-SUBSET-SUM(S, t, ϵ)

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4       L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5       L_i = TRIM(L_i, ϵ/2n)
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- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

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- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$

Running through an Example (CLRS3)

APPROX-SUBSET-SUM(S, t, ϵ)

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$

Running through an Example (CLRS3)

```
APPROX-SUBSET-SUM(S, t, ϵ)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
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```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$

Running through an Example (CLRS3)

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$
- line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404 \rangle$

Running through an Example (CLRS3)

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$
- line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404 \rangle$
- line 5: $L_4 = \langle 0, 101, 201, 302, 404 \rangle$

Running through an Example (CLRS3)

APPROX-SUBSET-SUM(S, t, ϵ)

1 $n = |S|$
2 $L_0 = \langle 0 \rangle$
3 **for** $i = 1$ **to** n
4 $\quad L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
5 $\quad L_i = \text{TRIM}(L_i, \epsilon/2n)$
6 $\quad$ remove from L_i every element that is greater than t
7 let z^* be the largest value in L_n
8 **return** z^*

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$
- line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404 \rangle$
- line 5: $L_4 = \langle 0, 101, 201, 302, 404 \rangle$
- line 6: $L_4 = \langle 0, 101, 201, 302 \rangle$

Running through an Example (CLRS3)

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$
- line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404 \rangle$
- line 5: $L_4 = \langle 0, 101, 201, 302, 404 \rangle$
- line 6: $L_4 = \langle 0, 101, 201, 302 \rangle$

Returned solution $z^* = 302$, which is 2% within the optimum $307 = 104 + 102 + 101$

Reminder: Performance Ratios for Approximation Algorithms

Approximation Ratio

An algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size n, the cost C of the returned solution and optimal cost C^* satisfy:

$$\max\left(\frac{C}{C^*}, \frac{C^*}{C}\right) \leq \rho(n).$$

For many problems: tradeoff between runtime and approximation ratio.

Approximation Schemes

An approximation scheme is an approximation algorithm, which given any input and $\epsilon > 0$, is a $(1+\epsilon)$-approximation algorithm.

- It is a polynomial-time approximation scheme (PTAS) if for any fixed $\epsilon > 0$, the runtime is polynomial in n. For example, $O(n^{2/\epsilon})$.
- It is a fully polynomial-time approximation scheme (FPTAS) if the runtime is polynomial in both $1/\epsilon$ and n. For example, $O((1/\epsilon)^2 \cdot n^3)$.

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

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- Returned solution z^* is a valid solution ✓
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APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i'$ s.t.:

Analysis of APPROX-SUBSET-SUM

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Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i'$ s.t.:

$$\frac{y}{(1+\epsilon/(2n))^i} \leq z \leq y$$

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- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i'$ s.t.:

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Can be shown by induction on i

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- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L'_i$ s.t.:

$$\frac{y}{(1+\epsilon/(2n))^i} \leq z \leq y \quad \overset{y=y^*, i=n}{\Rightarrow}$$

Can be shown by induction on i

Analysis of APPROX-SUBSET-SUM

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- Returned solution z^* is a valid solution ✓
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- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i'$ s.t.:

$$\frac{y}{(1+\epsilon/(2n))^i} \leq z \leq y \quad \stackrel{y=y^*, i=n}{\Rightarrow} \quad \frac{y^*}{(1+\epsilon/(2n))^n} \leq z \leq y^*$$

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- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i'$ s.t.:

$$\frac{y}{(1+\epsilon/(2n))^i} \leq z \leq y \quad \overset{y=y^*, i=n}{\Rightarrow} \quad \frac{y^*}{(1+\epsilon/(2n))^n} \leq z \leq y^*$$

Can be shown by induction on i

$$\frac{y^*}{z} \leq \left(1+\frac{\epsilon}{2n}\right)^n,$$

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

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Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i'$ s.t.:

$$\frac{y}{(1+\epsilon/(2n))^i} \leq z \leq y \quad \overset{y=y^*, i=n}{\Rightarrow} \quad \frac{y^*}{(1+\epsilon/(2n))^n} \leq z \leq y^*$$

Can be shown by induction on i

$$\frac{y^*}{z} \leq \left(1+\frac{\epsilon}{2n}\right)^n,$$

and now using the fact that $\left(1+\frac{\epsilon/2}{n}\right)^n \overset{n\to\infty}{\longrightarrow} e^{\epsilon/2}$ yields

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i'$ s.t.:

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Analysis of Approx-Subset-Sum

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Approx-Subset-Sum is a FPTAS for the subset-sum problem.

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- Strategy: Derive a bound on $|L_i|$ (running time is linear in $|L_i|$)
- After trimming, two successive elements z and z' satisfy $z'/z \geq 1 + \epsilon/(2n)$

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Need $\log(t)$ bits to represent t and n bits to represent S

Concluding Remarks

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\ x_i \in S'} x_i \leq t$.

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- Given: Items $i = 1, 2, \ldots, n$ with weights w_i and values v_i, and integer t

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A more general problem than Subset-Sum

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Algorithm very similar to APPROX-SUBSET-SUM

Theorem

There is a FPTAS for the Knapsack problem.

Outline

The Subset-Sum Problem

Parallel Machine Scheduling

Bonus Material: A PTAS for Parallel Machine Scheduling (non-examinable)

Parallel Machine Scheduling

Machine Scheduling Problem

- Given: n jobs $J_1, J_2, \ldots, J_n$ with processing times $p_1, p_2, \ldots, p_n$, and m identical machines $M_1, M_2, \ldots, M_m$

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- J_1: $p_1 = 2$
- J_2: $p_2 = 12$
- J_3: $p_3 = 6$
- J_4: $p_4 = 4$

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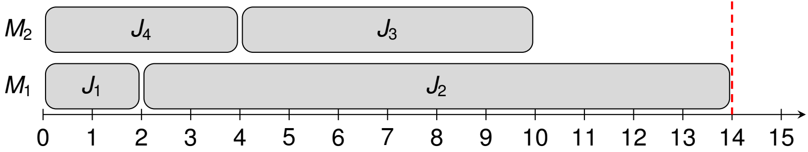

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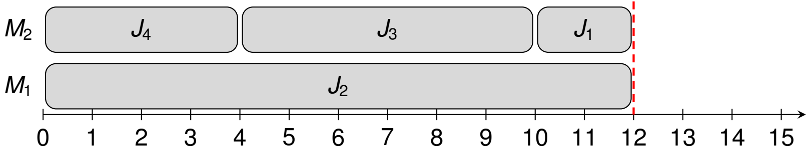

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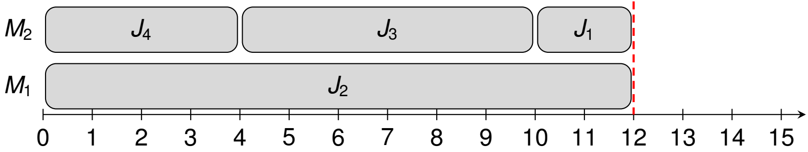
For the analysis, it will be convenient to denote by C_i the completion time of a machine i.



NP-Completeness of Parallel Machine Scheduling

Lemma

Parallel Machine Scheduling is NP-complete even if there are only two machines.

Proof Idea: Polynomial time reduction from NUMBER-PARTITIONING.

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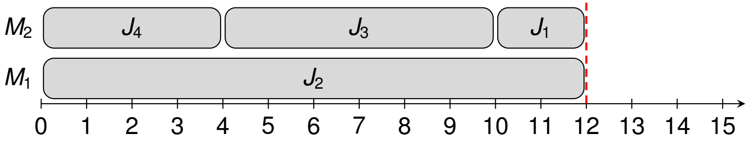

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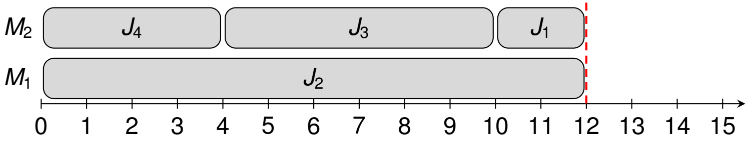

LIST SCHEDULING$(J_1, J_2, \ldots, J_n, m)$
1: **while** there exists an unassigned job
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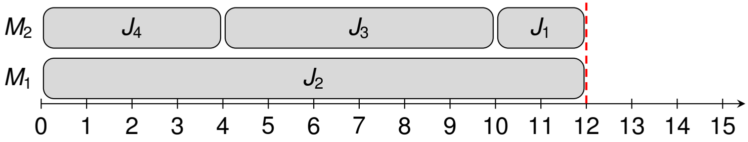

Equivalent to the following Online Algorithm [CLRS3]:
Whenever a machine is idle, schedule the next job on that machine.

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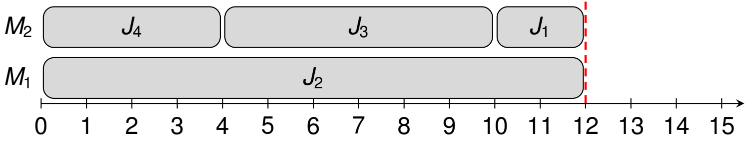

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How good is this most basic Greedy Approach?

List Scheduling Analysis (Observations)

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Ex 35-5 a.&b.

a. The optimal makespan is at least as large as the greatest processing time, that is,

$$C^*_{\max} \geq \max_{1 \leq k \leq n} p_k .$$

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$\Rightarrow$ One machine must have a load of at least $\frac{1}{m} \cdot \sum_{k=1}^{n} p_k$ $\square$

List Scheduling Analysis (Final Step)

Ex 35-5 d. (Graham 1966)

For the schedule returned by the greedy algorithm it holds that

$$C_{\max} \leq \frac{1}{m} \sum_{k=1}^{n} p_k + \max_{1 \leq k \leq n} p_k.$$

Hence list scheduling is a poly-time 2-approximation algorithm.

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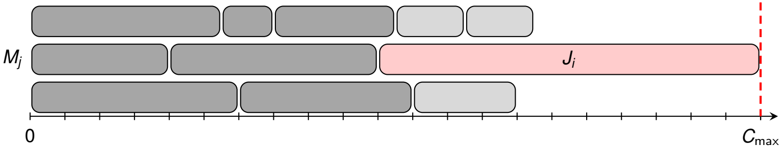

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Ex 35-5 d. (Graham 1966)

For the schedule returned by the greedy algorithm it holds that

$$C_{\max} \leq \frac{1}{m} \sum_{k=1}^{n} p_k + \max_{1 \leq k \leq n} p_k.$$

Hence list scheduling is a poly-time 2-approximation algorithm.

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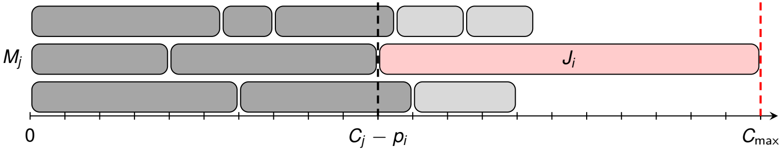

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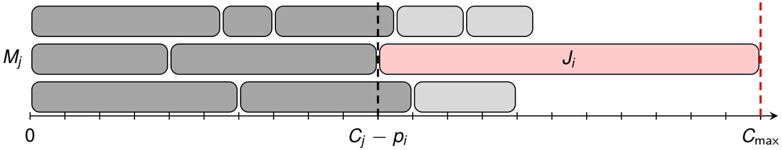

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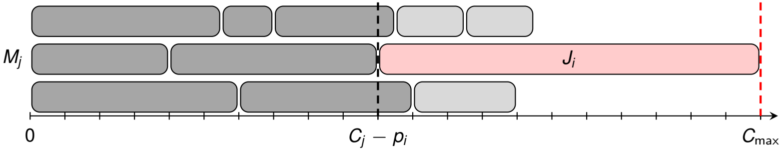

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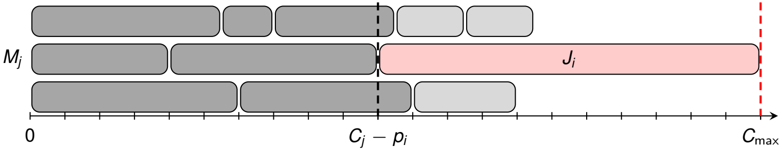

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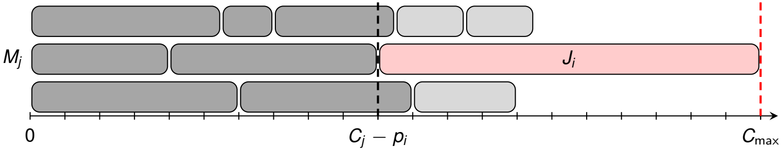

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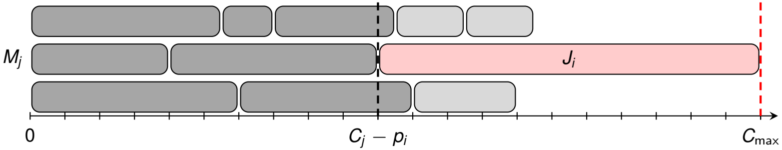

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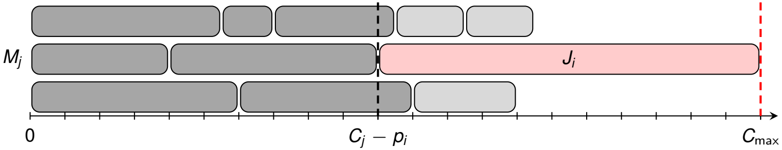

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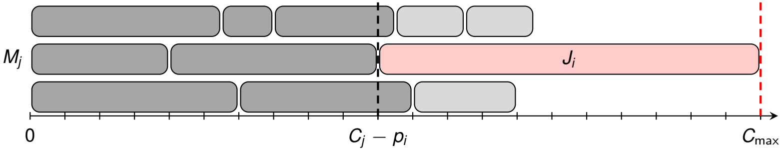

Improving Greedy

Analysis can be shown to be almost tight. Is there a better algorithm?

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The problem of the List-Scheduling Approach were the large jobs

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LEAST PROCESSING TIME$(J_1, J_2, \ldots, J_n, m)$

```
Sort jobs decreasingly in their processing times
for i = 1 to m
    C_i = 0
    S_i = ∅
end for
for j = 1 to n
    i = argmin_{1≤k≤m} C_k
    S_i = S_i ∪ {j}, C_i = C_i + p_j
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return S_1, ..., S_m
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Runtime:

- $O(n \log n)$ for sorting
- $O(n \log m)$ for extracting (and re-inserting) the minimum (use priority queue).

Analysis of Improved Greedy

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

This can be shown to be tight (see next slide).

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Proof (of approximation ratio $3/2$).

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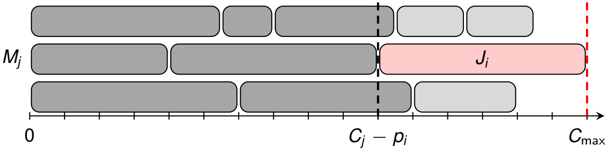

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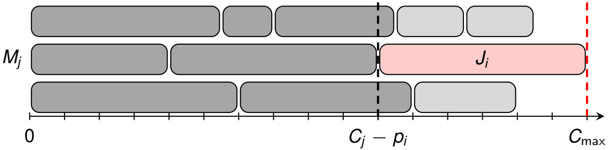

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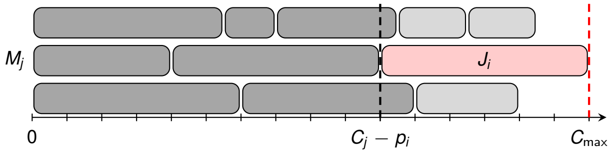
This is for the case $i \geq m+1$ (otherwise, an even stronger inequality holds)



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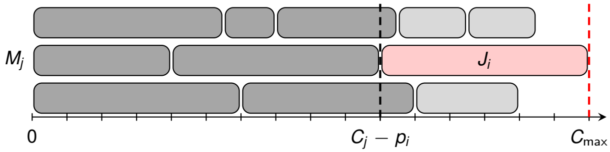

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Proof of an instance which shows tightness:

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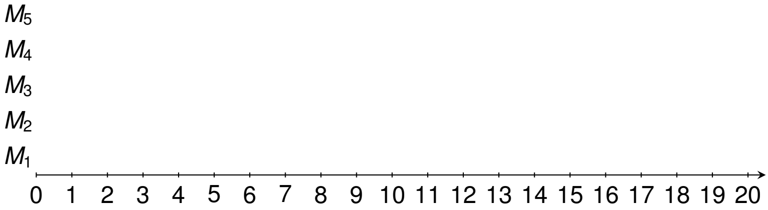

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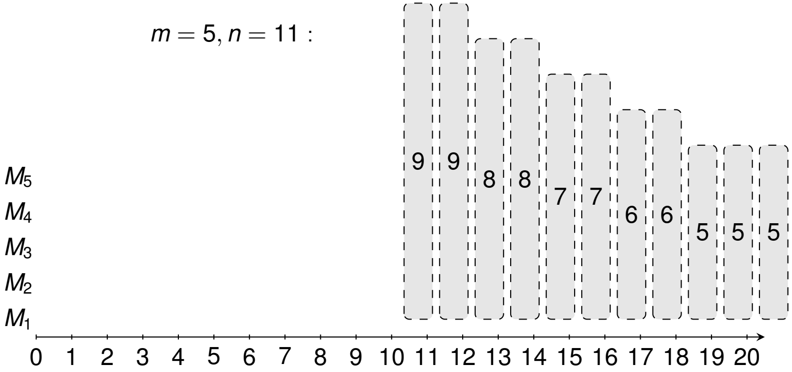

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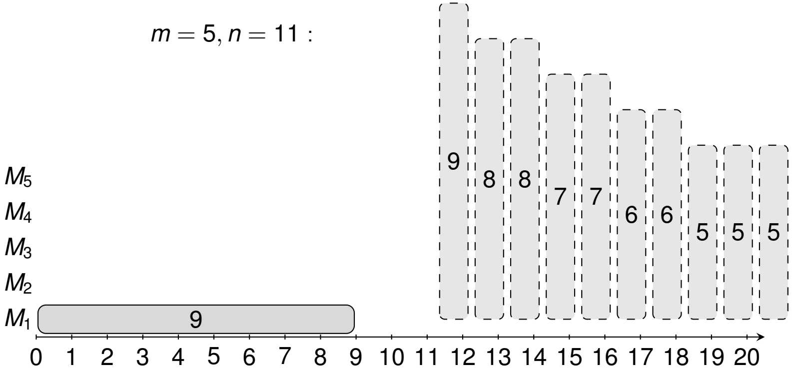

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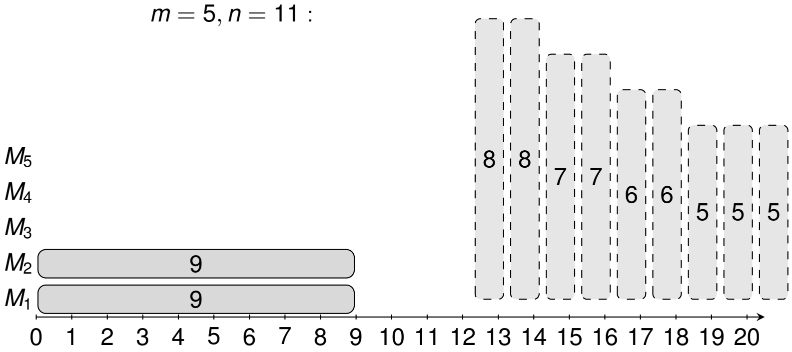

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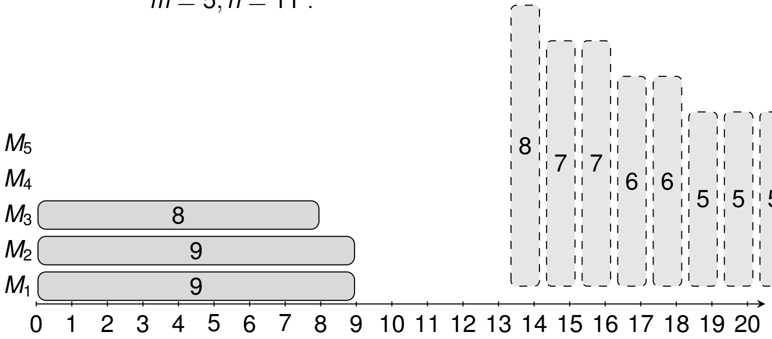

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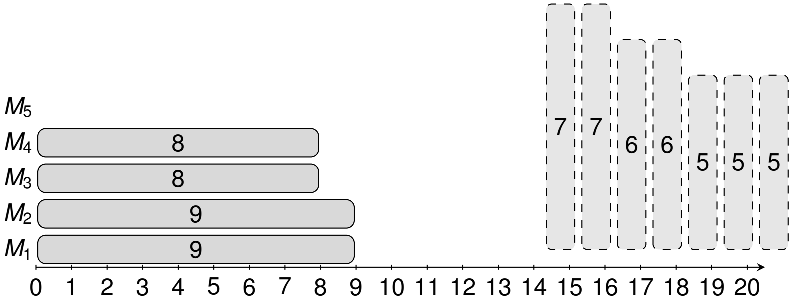

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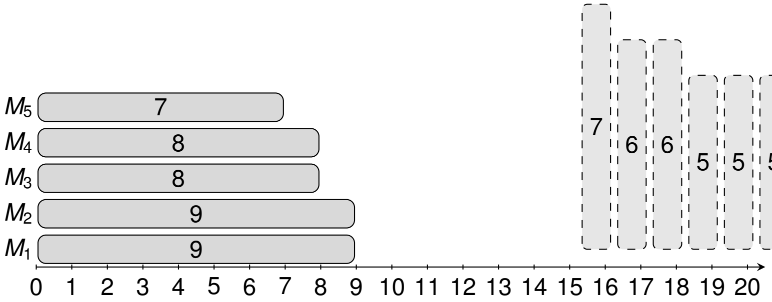

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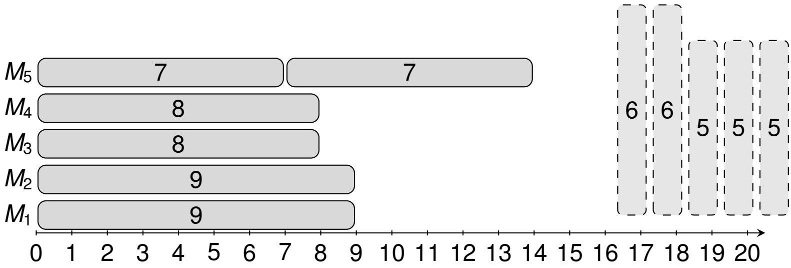

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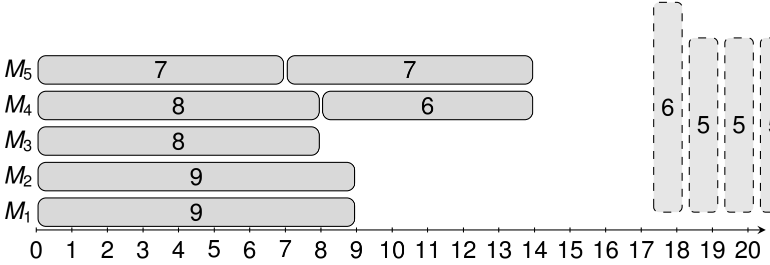

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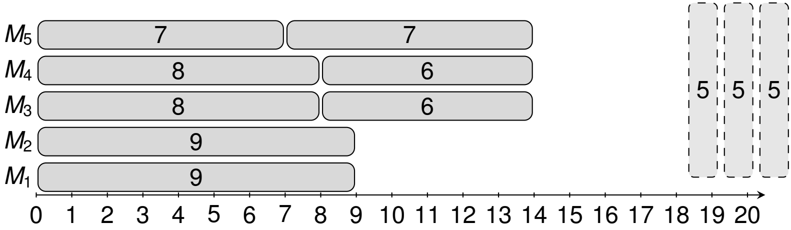

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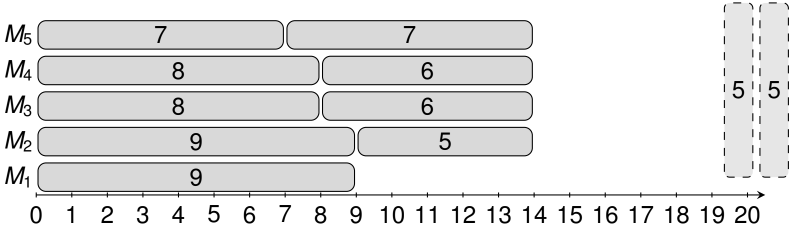

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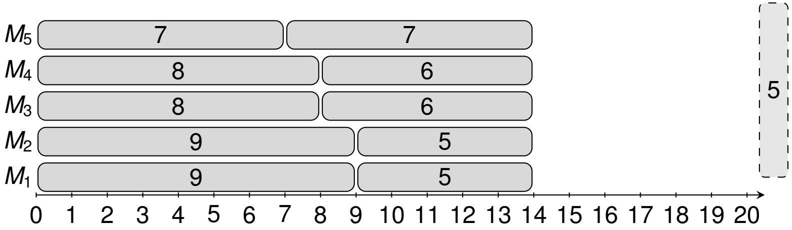

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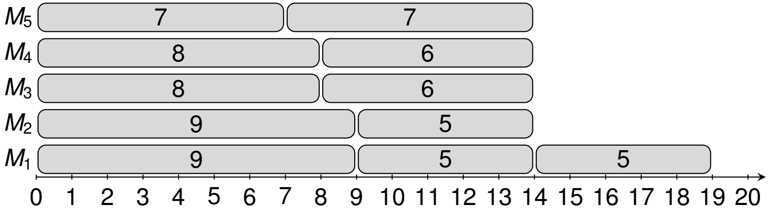

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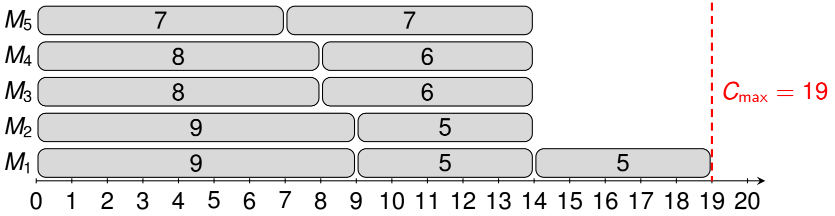

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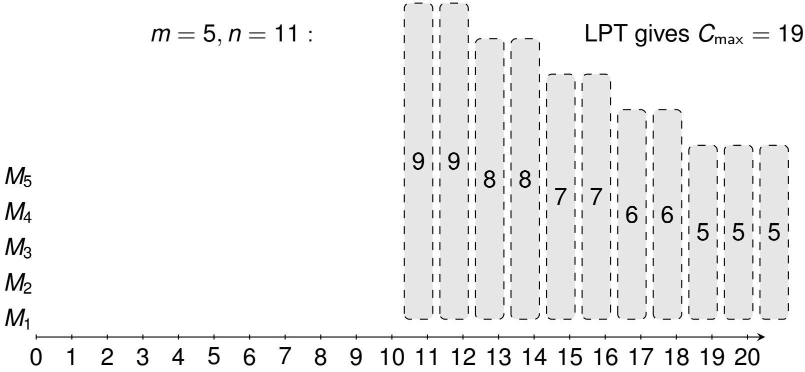

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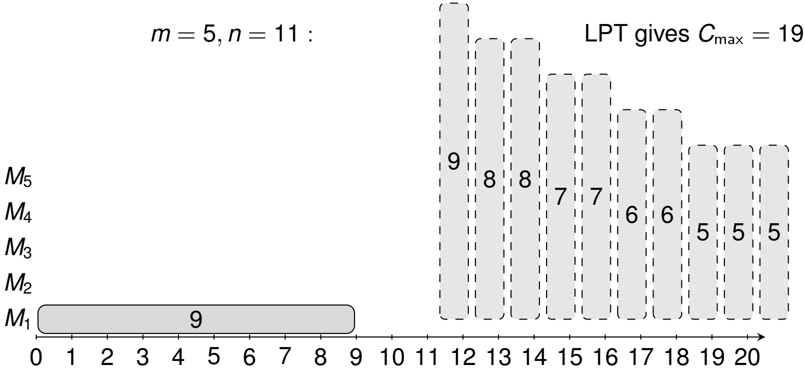

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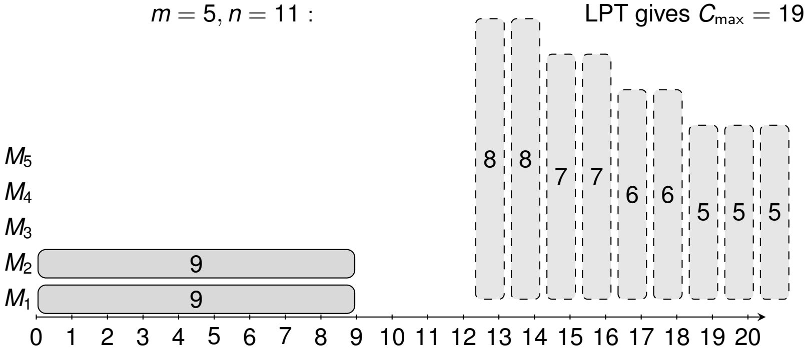

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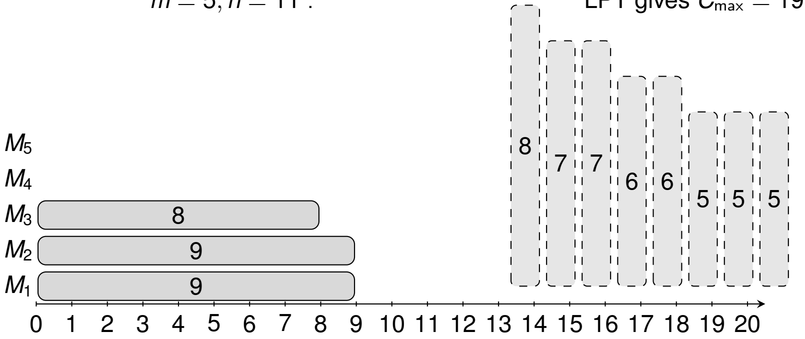

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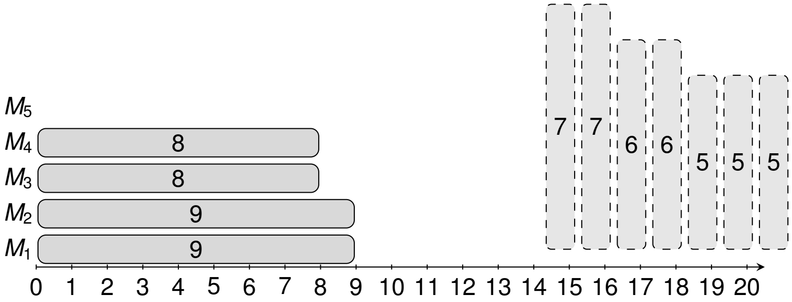

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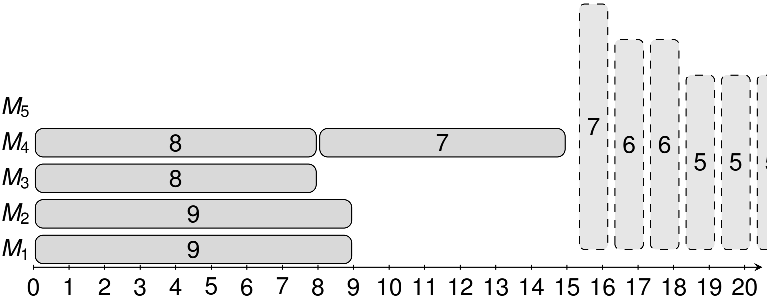

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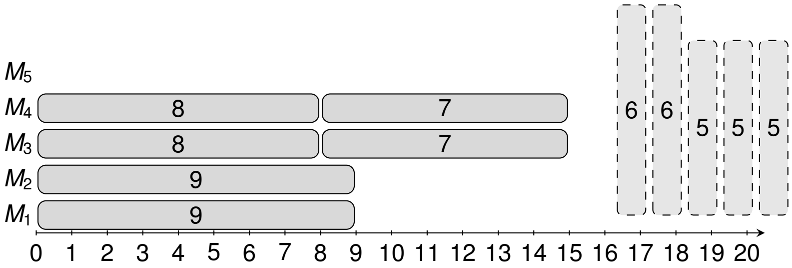

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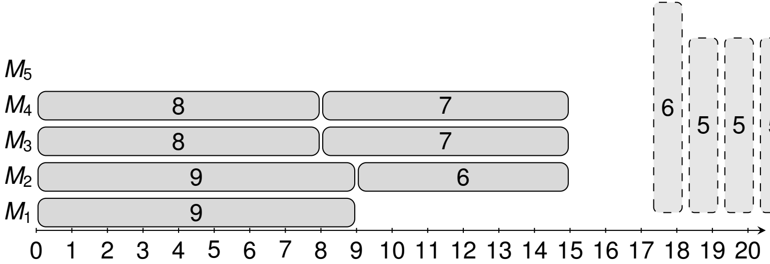

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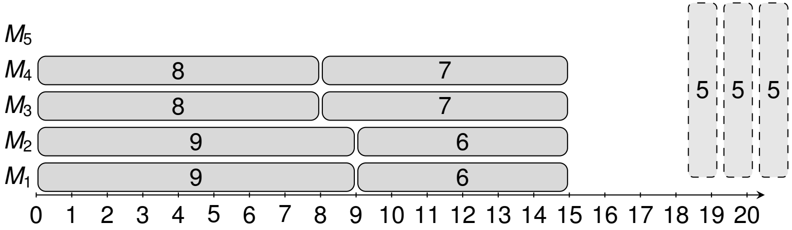

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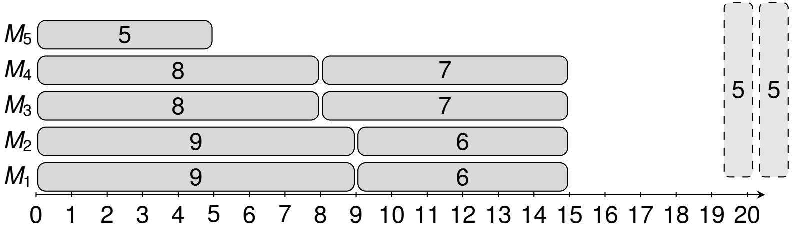

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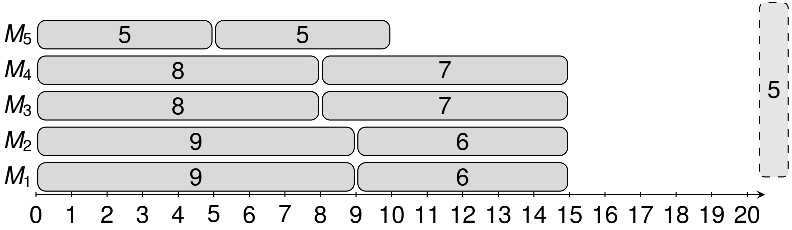

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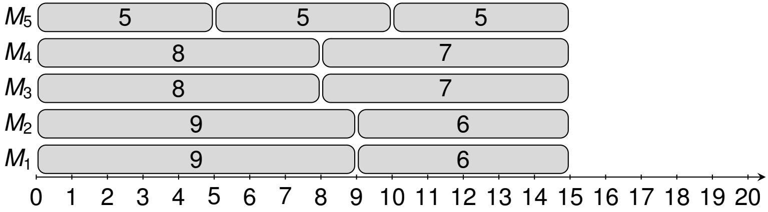

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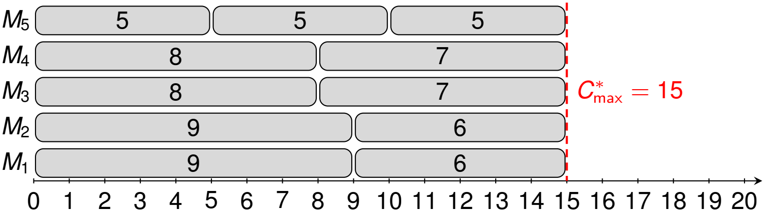

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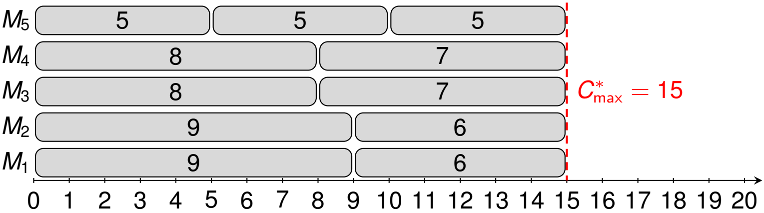

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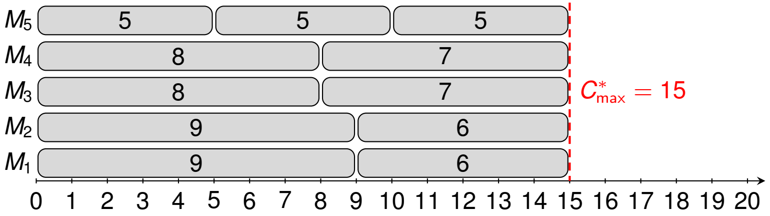

Conclusion

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List scheduling has an approximation ratio of 2.

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No!

Because for sufficiently small approximation ratio $1 + \epsilon$, the computed solution has to be optimal, and Parallel Machine Scheduling is strongly NP-hard.

Exercise (easy): Run the LPT algorithm on three machines and jobs having processing times $\{3, 4, 4, 3, 5, 3, 5\}$. Which allocation do you get?

1. $[3, 3, 5], [4, 5], [4, 3]$
2. $[5, 3], [5, 4], [4, 3, 3]$
3. $[3, 3, 3], [5, 4], [5, 4]$

Outline

The Subset-Sum Problem

Parallel Machine Scheduling

Bonus Material: A PTAS for Parallel Machine Scheduling (non-examinable)

A PTAS for Parallel Machine Scheduling

Basic Idea: For $(1+\epsilon)$-approximation, don't have to work with exact p_k's.

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SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$

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the "well-known" formula

Implementation of Subroutine

SUBROUTINE($J_1, J_2, \ldots, J_n, m, T$)

1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$

2: Or: **Return** there is no solution with makespan $< T$

Observation

Divide jobs into two groups: $J_{\text{small}} = \{i \colon p_i \leq \epsilon \cdot T\}$ and $J_{\text{large}} = [n] \setminus J_{\text{small}}$. Given a solution for J_{large} only with makespan $(1+\epsilon) \cdot T$, then greedily placing J_{small} yields a solution with makespan $(1+\epsilon) \cdot \max\{T, C^*_{\max}\}$.

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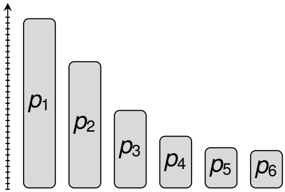

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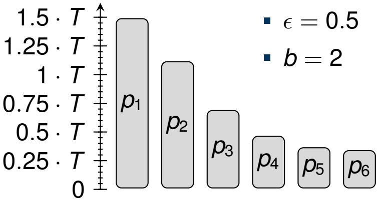

- $\epsilon = 0.5$
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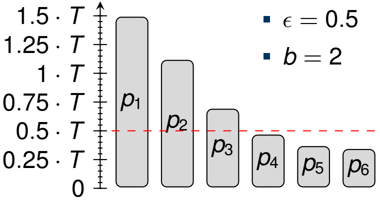

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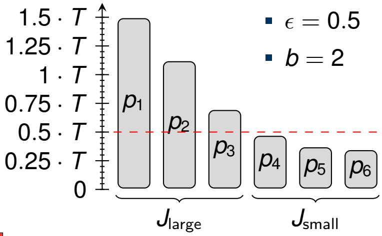

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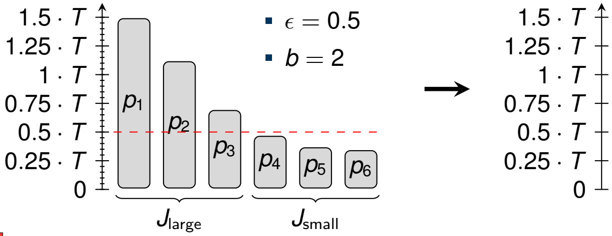

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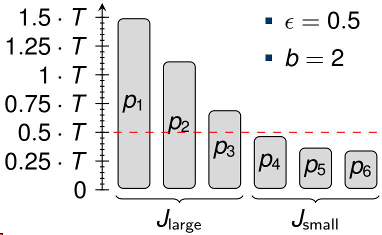

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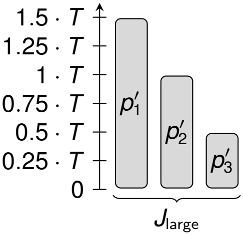

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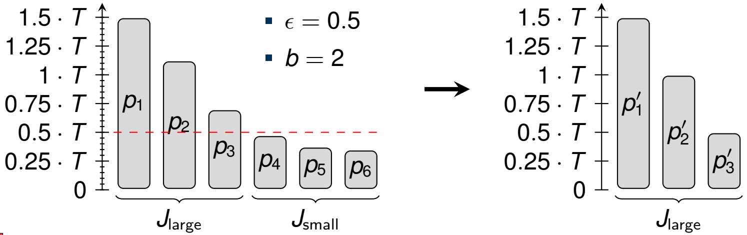
Can assume there are no jobs with $p_j \geq T$!



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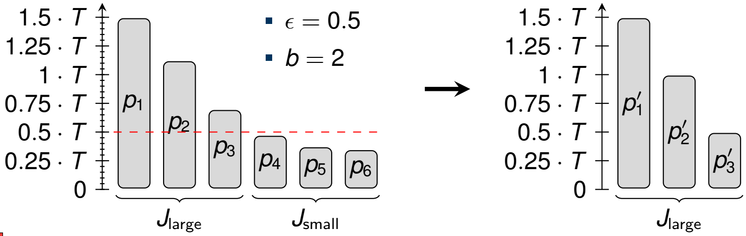

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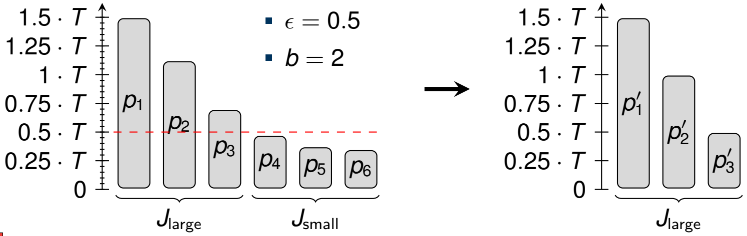
Assignments to one machine with makespan $\leq T$.



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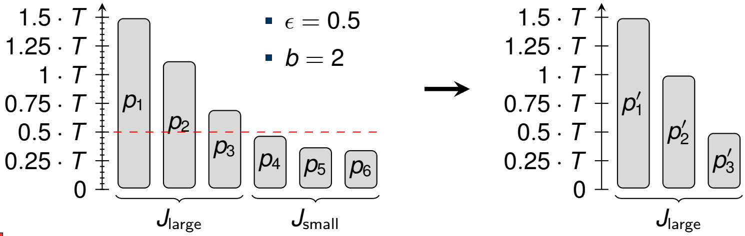

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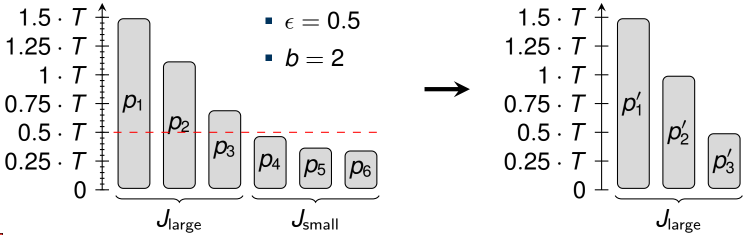

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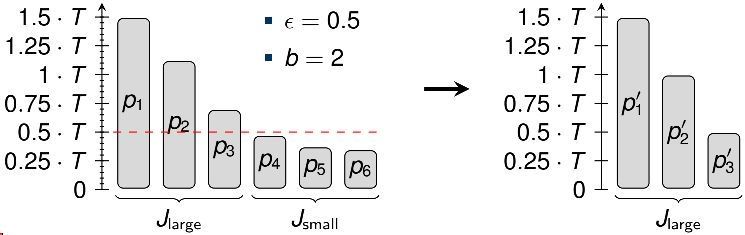

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Assign some jobs to one machine, and then use as few machines as possible for the rest.

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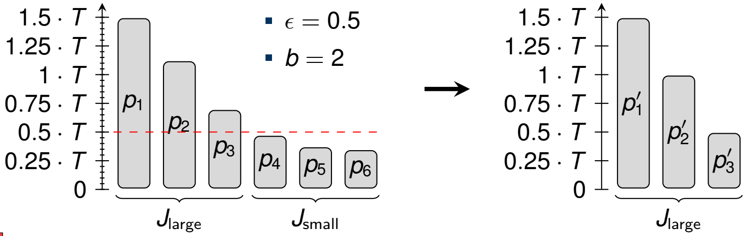

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$$\begin{aligned} C_{\max} &\leq T + b \cdot \max_{i \in J_{\text{large}}} (p_i - p_i') \\ &\leq T + b \cdot \frac{T}{b^2} \leq (1+\epsilon) \cdot T. \qquad \square \end{aligned}$$

V. Approx. Algorithms: Travelling Salesman Problem

Thomas Sauerwald

Easter 2021

UNIVERSITY OF CAMBRIDGE

Outline

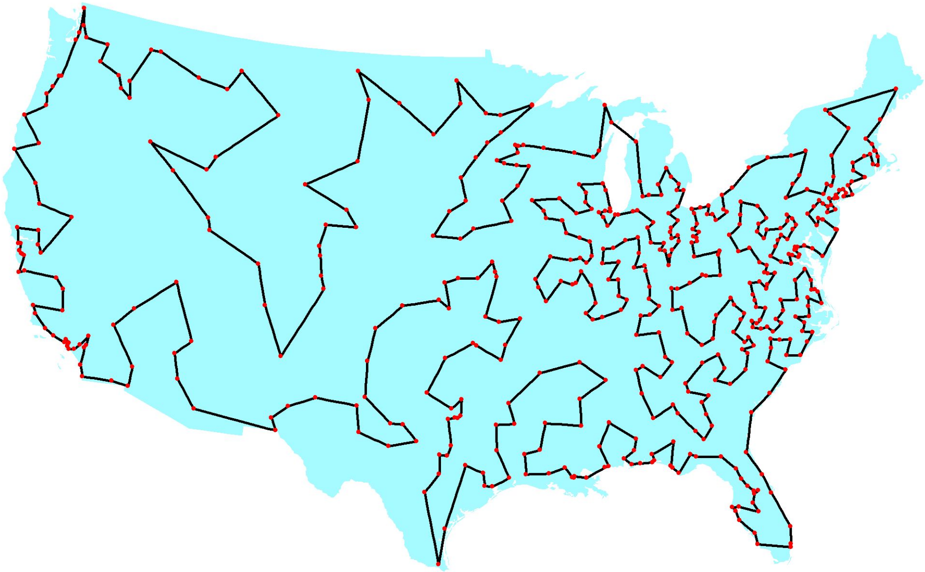
Introduction

General TSP

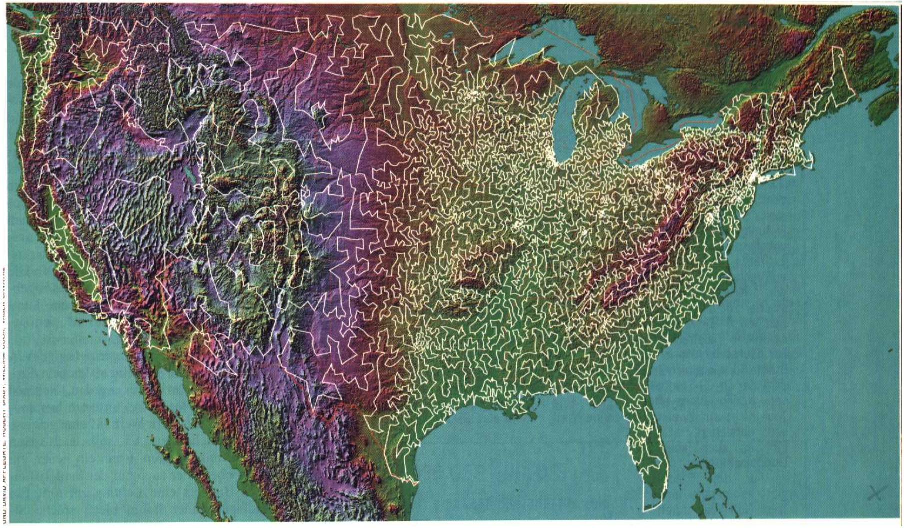
Metric TSP

33 city contest (1964)

532 cities (1987 [Padberg, Rinaldi])



13,509 cities (1999 [Applegate, Bixby, Chavatal, Cook])



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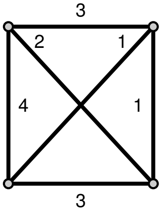

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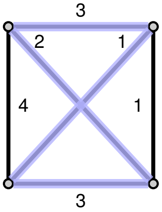

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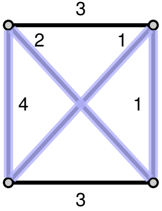

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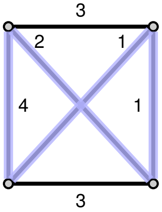

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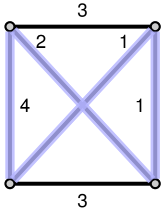

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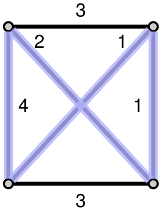

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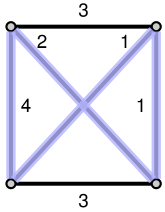

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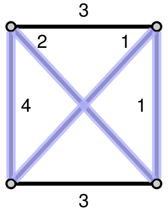

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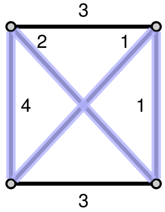

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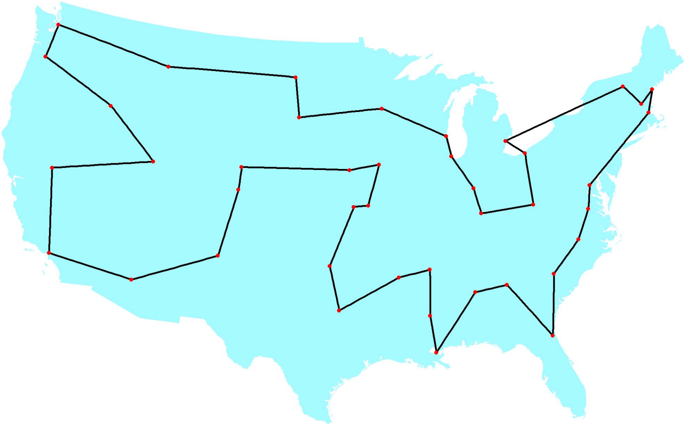
- Euclidean TSP: cities are points in the Euclidean space, costs are equal to their (rounded) Euclidean distance

History of the TSP problem (1954)

Dantzig, Fulkerson and Johnson found an optimal tour through 42 cities.



`http://www.math.uwaterloo.ca/tsp/history/img/dantzig_big.html`

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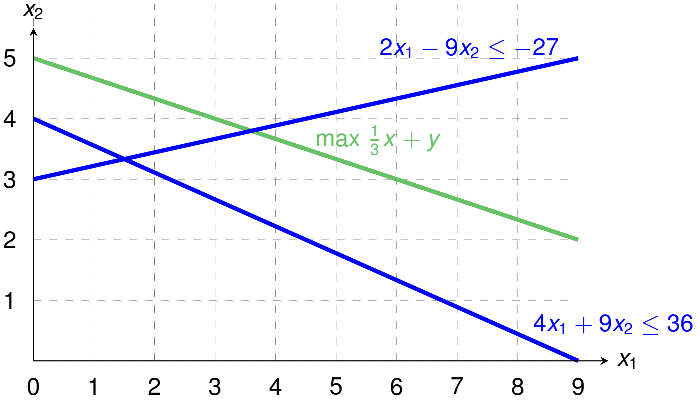

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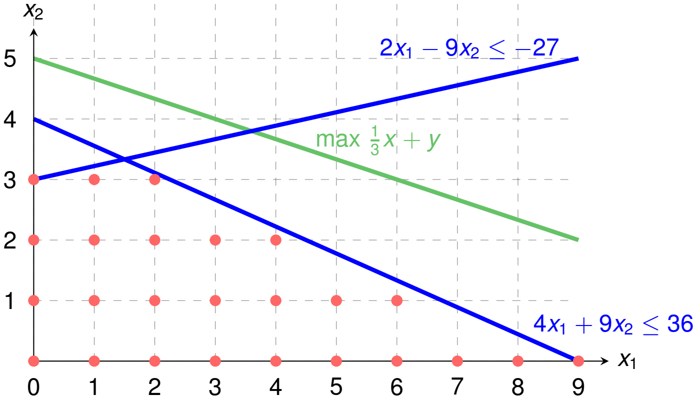

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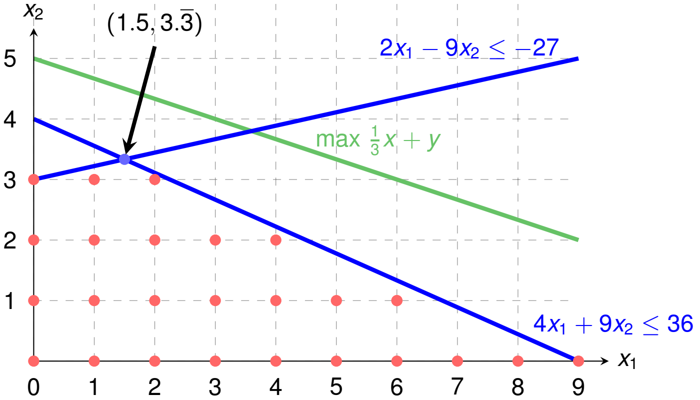

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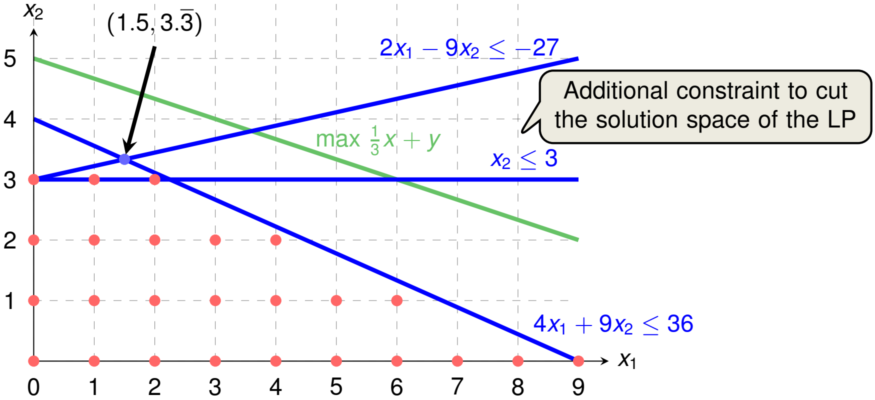

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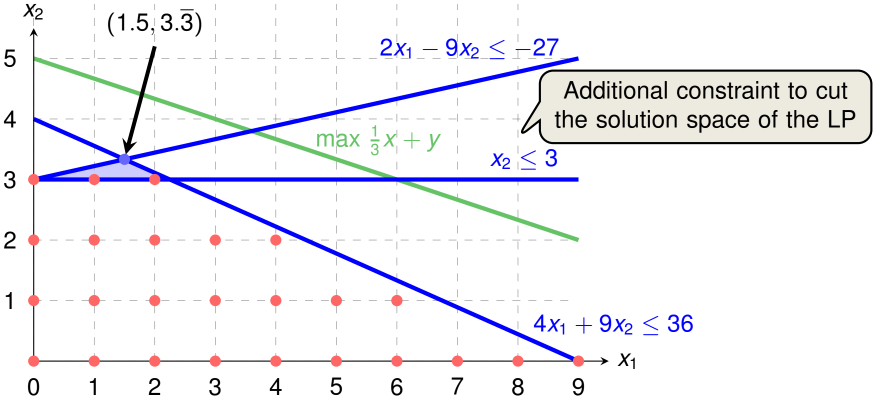

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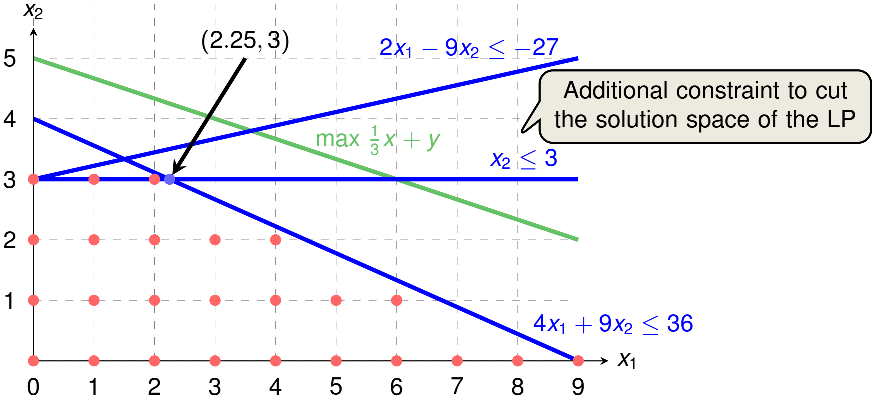

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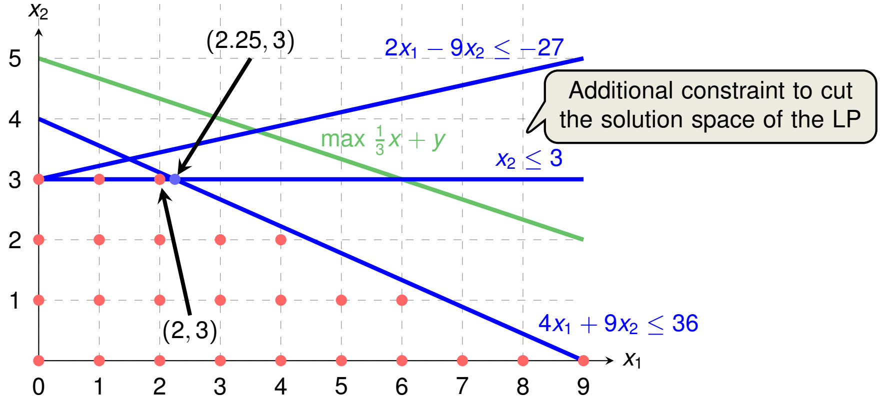

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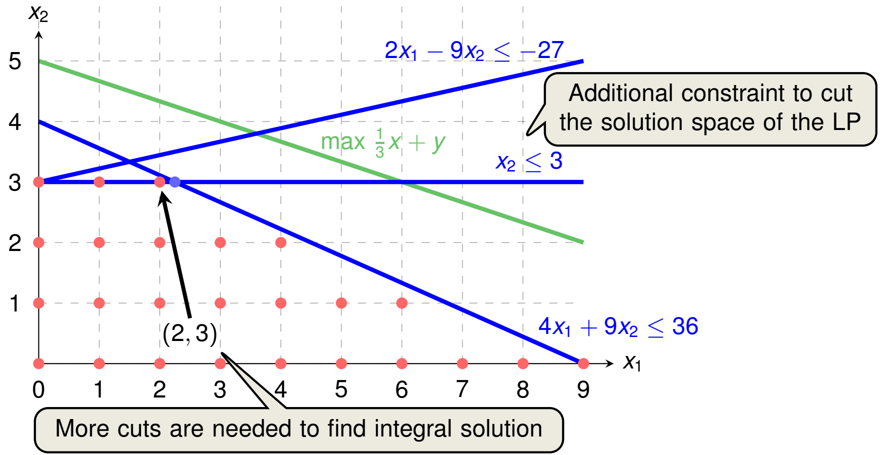

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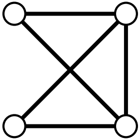
$G = (V, E)$ 

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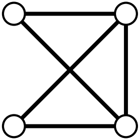
$G = (V, E)$ 

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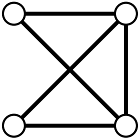

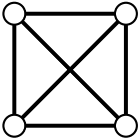

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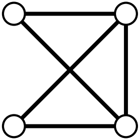
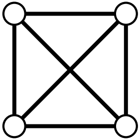
$G = (V, E)$ 

 $G' = (V, E')$

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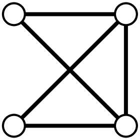

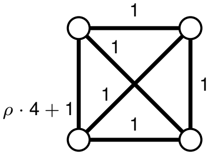

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Large weight will render this edge useless!

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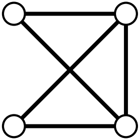

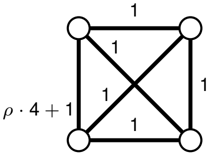

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Can create representations of G' and c in time polynomial in $|V|$ and $|E|$!

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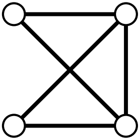

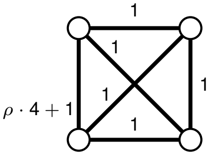

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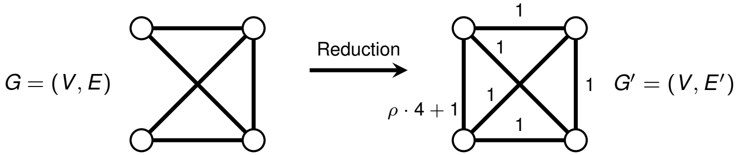

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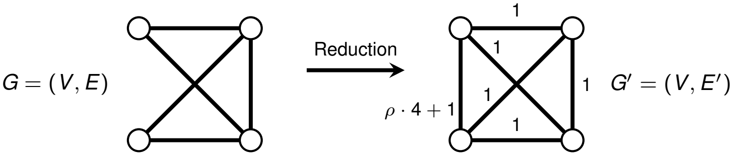
- If G has a hamiltonian cycle H, then (G', c) contains a tour of cost $|V|$



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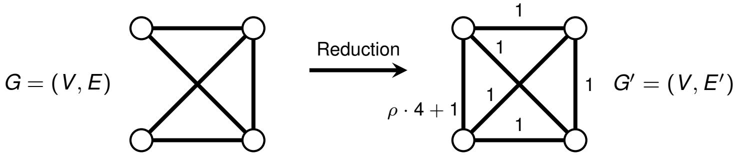

Hardness of Approximation

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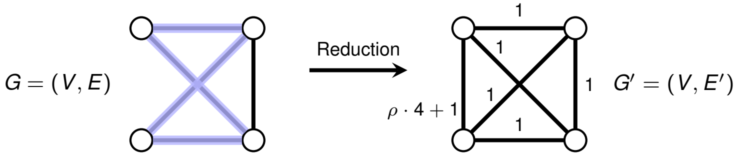

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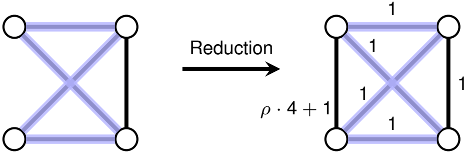

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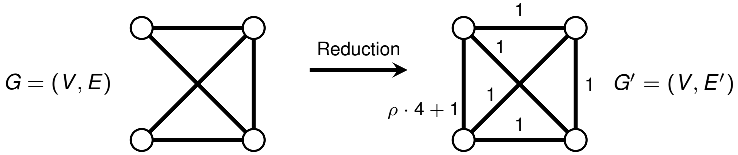

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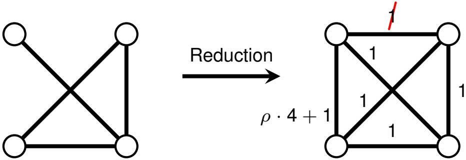

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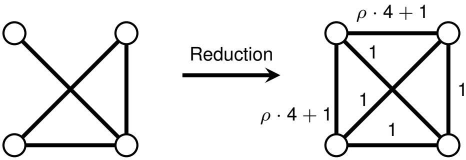

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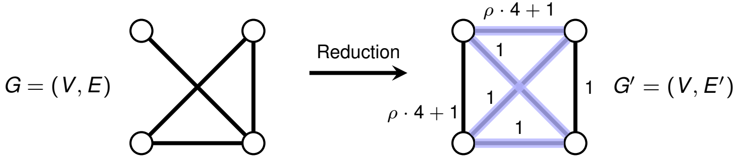

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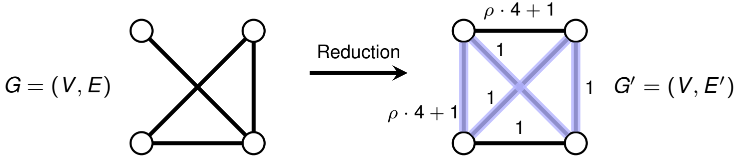

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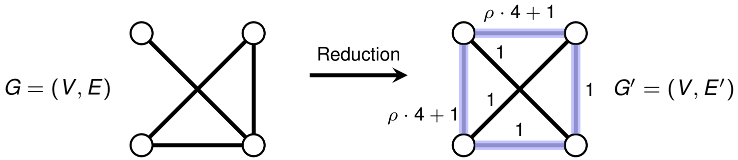

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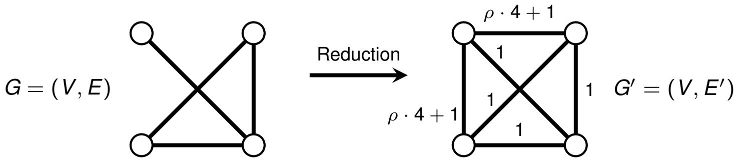

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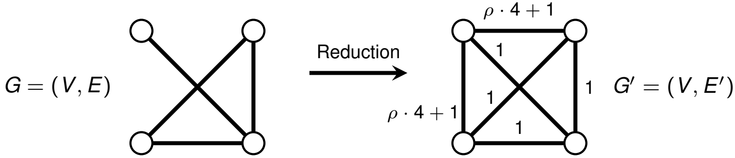

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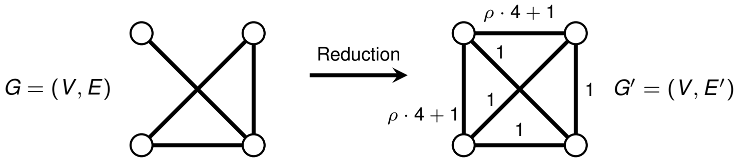

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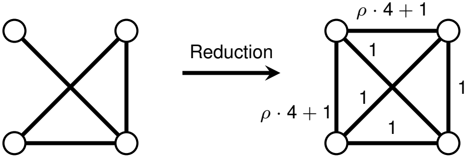

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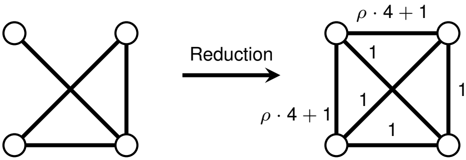

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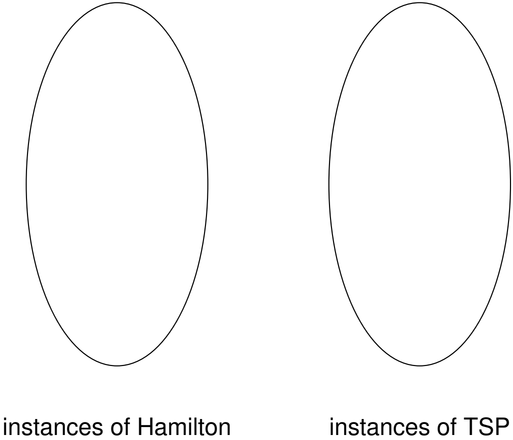
Proof of Theorem 35.3 from a higher perspective



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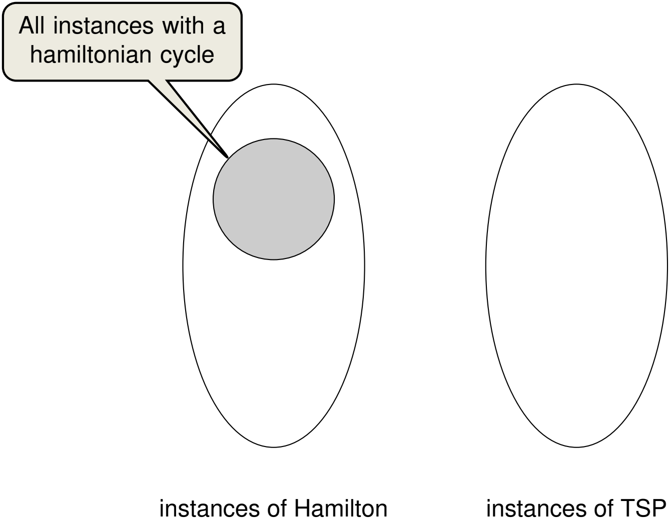

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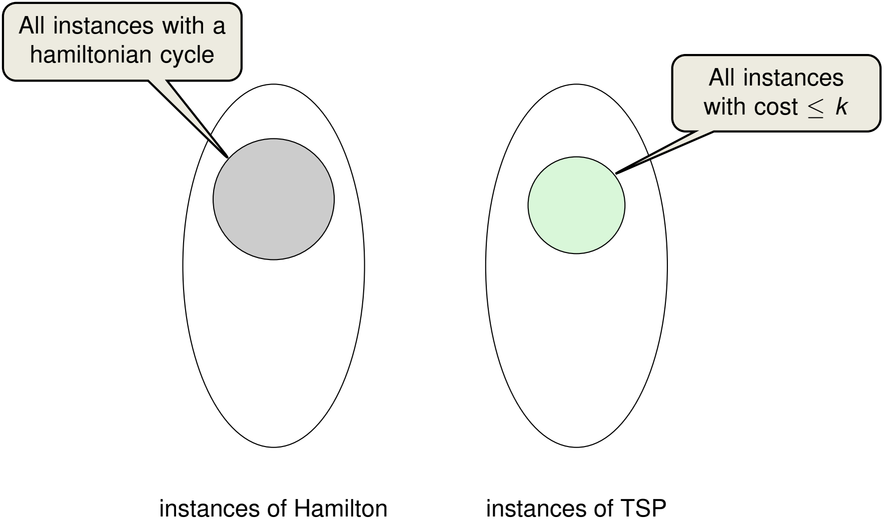

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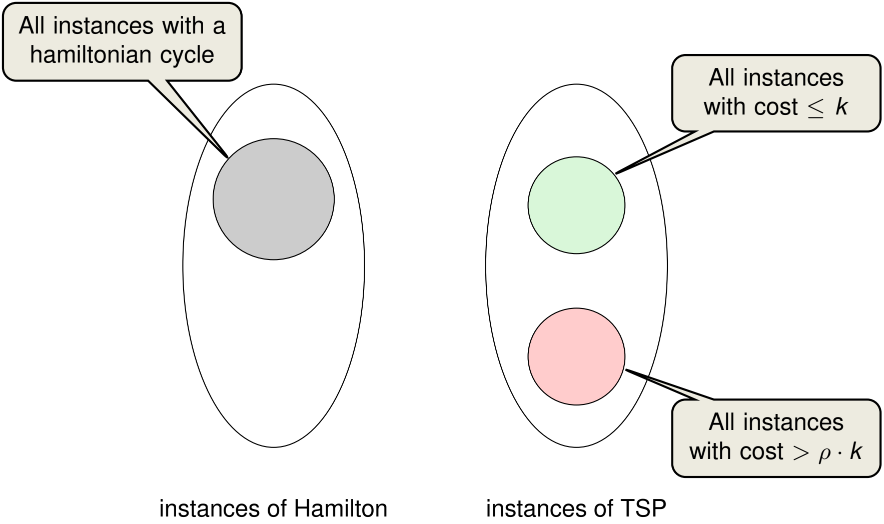

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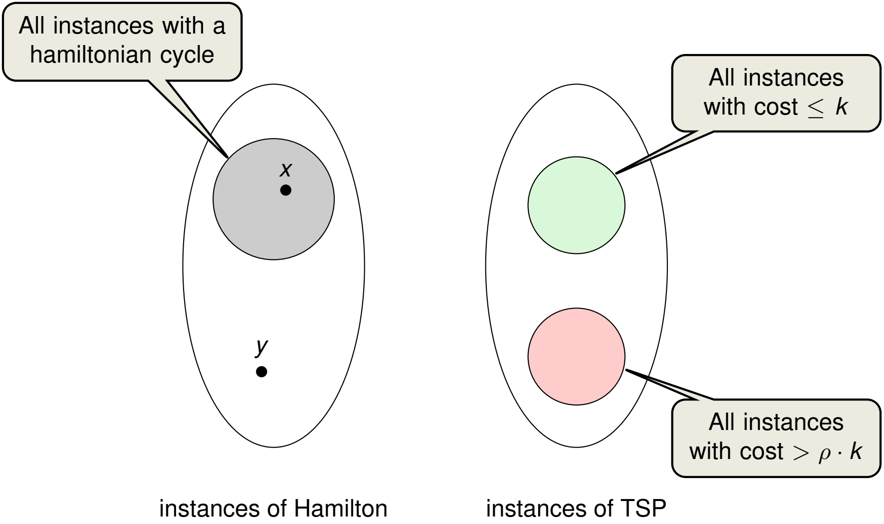

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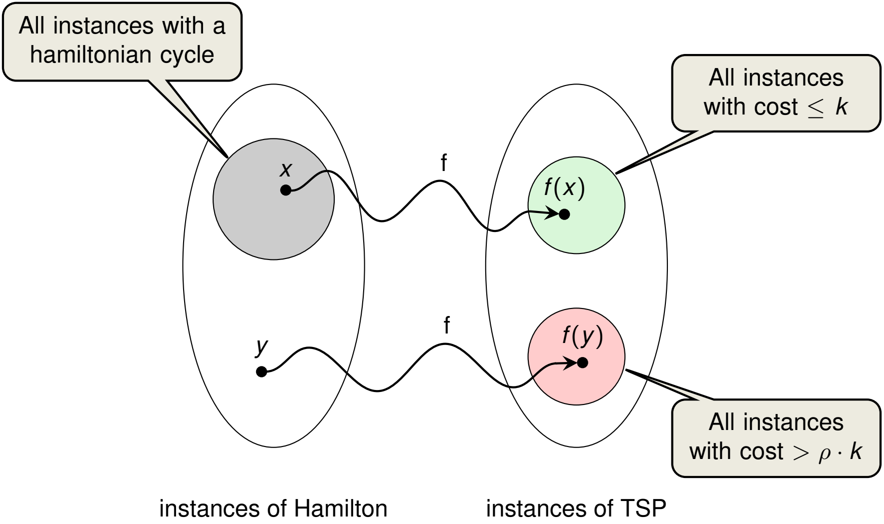

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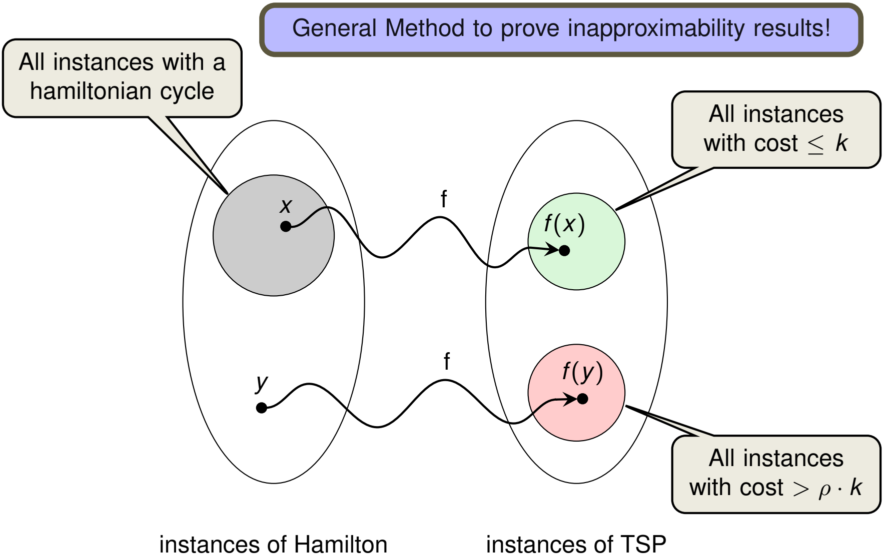

Outline

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General TSP

Metric TSP

Metric TSP (TSP Problem with the Triangle Inequality)

Idea: First compute an MST, and then create a tour based on the tree.

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APPROX-TSP-TOUR(G, c)

select a vertex $r \in G.V$ to be a "root" vertex
compute a minimum spanning tree T_{min} for G from root r
    using MST-PRIM(G, c, r)
let H be a list of vertices, ordered according to when they are first visited
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Runtime is dominated by MST-PRIM, which is $\Theta(V^2)$.

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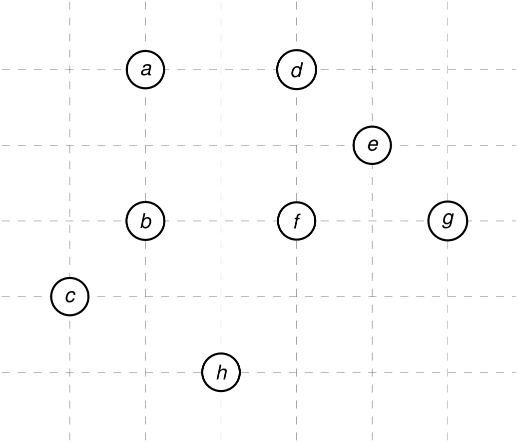
Remember: In the Metric-TSP problem, G is a complete graph.

Run of APPROX-TSP-TOUR



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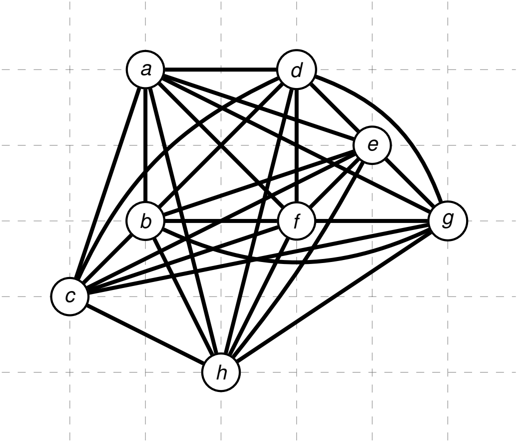

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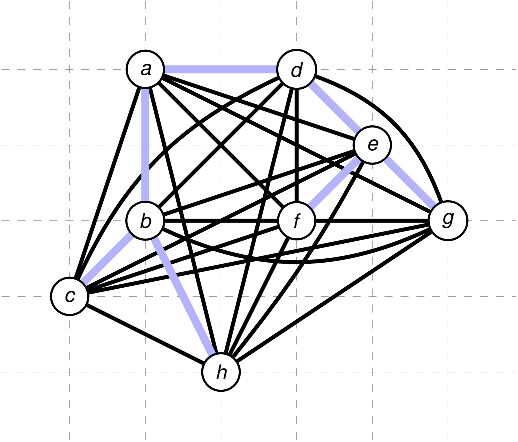

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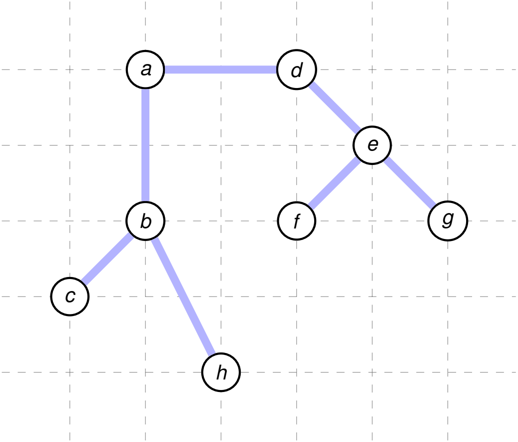

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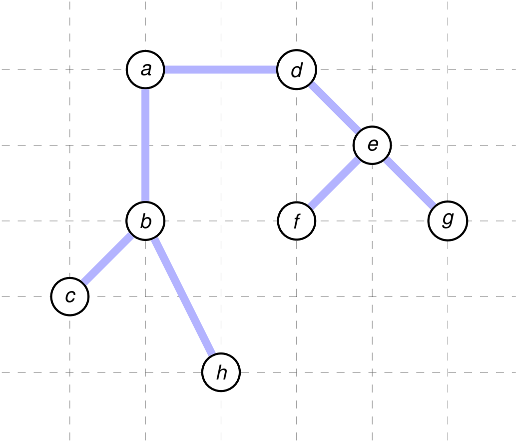

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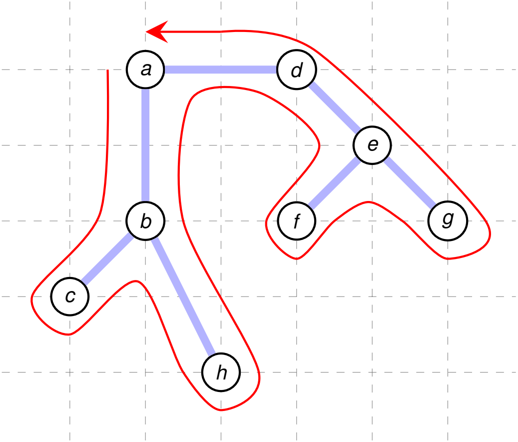

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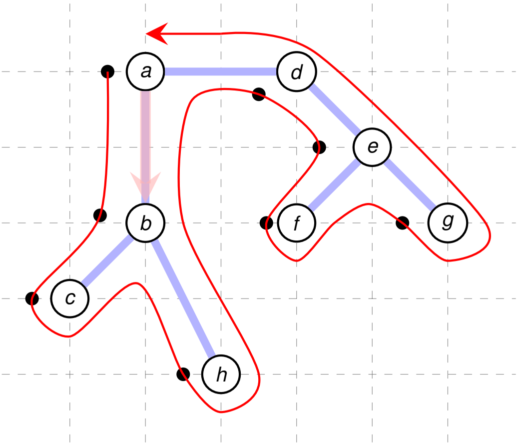

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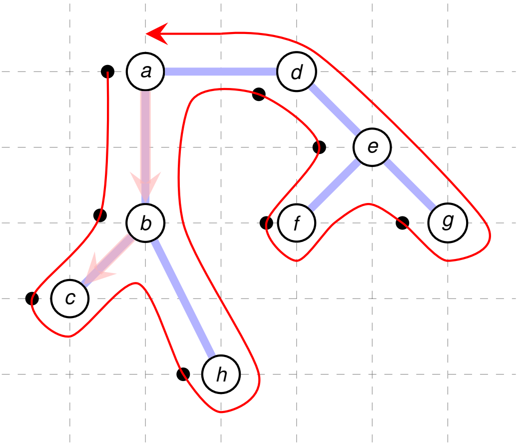

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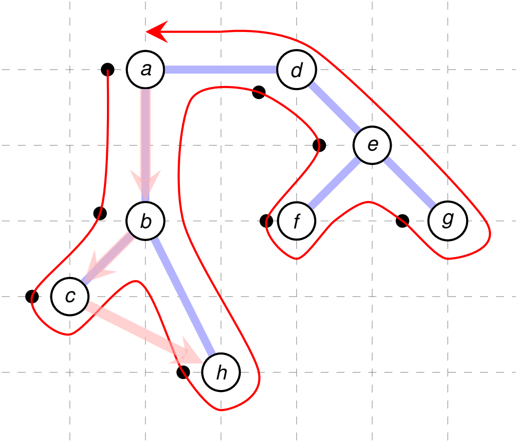

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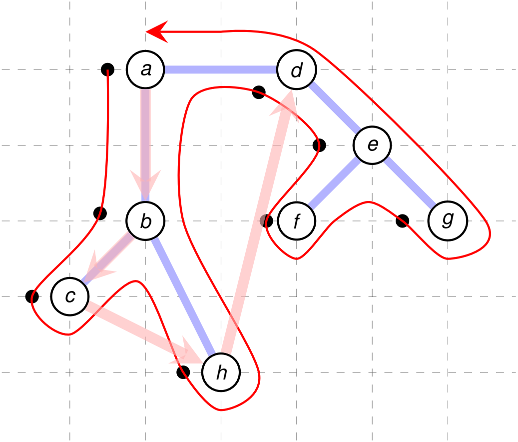

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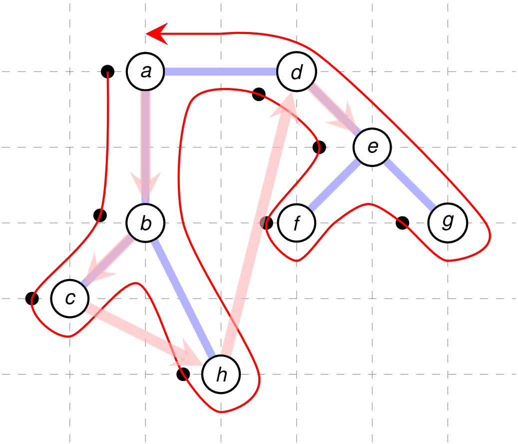

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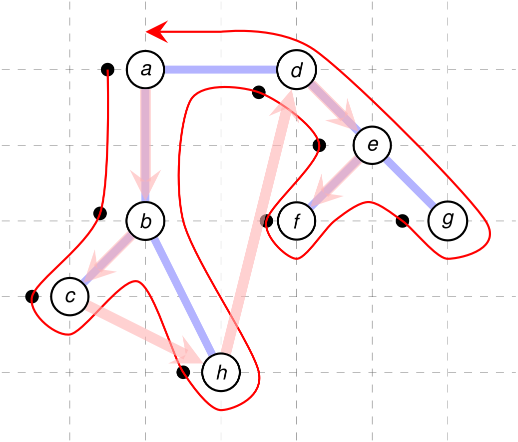

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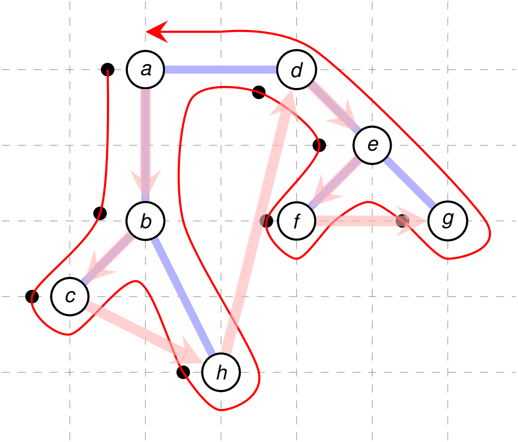

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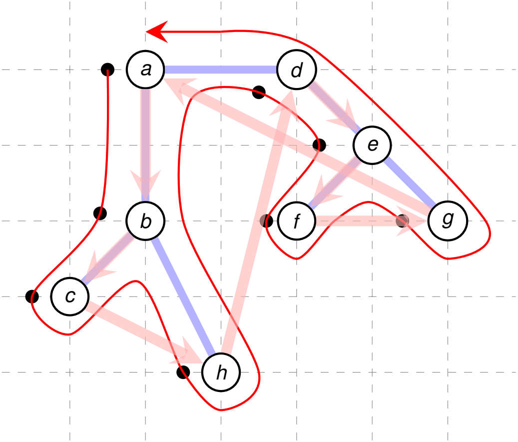
Run of APPROX-TSP-TOUR



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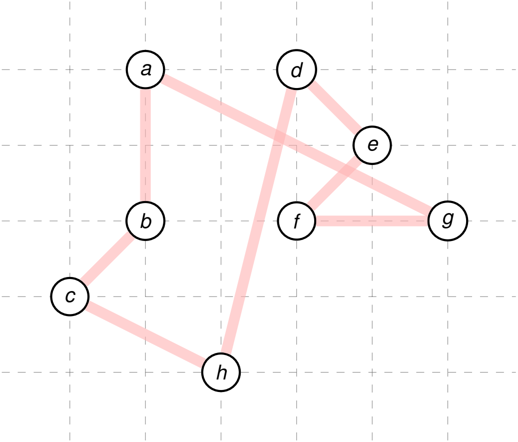

1. Compute MST T_{min} ✓
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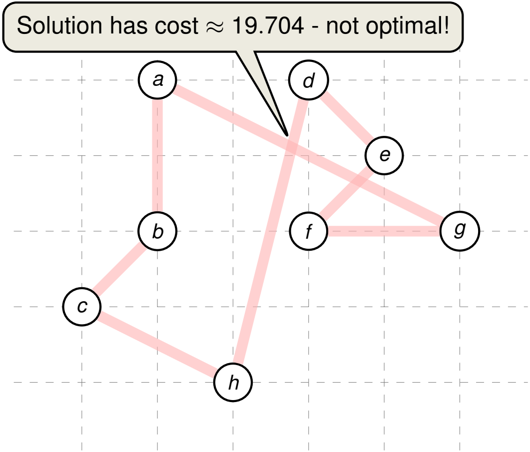

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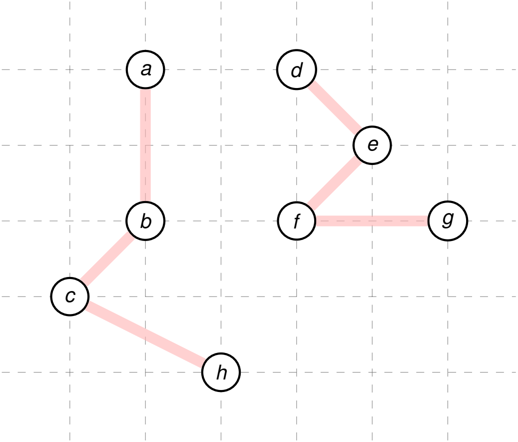

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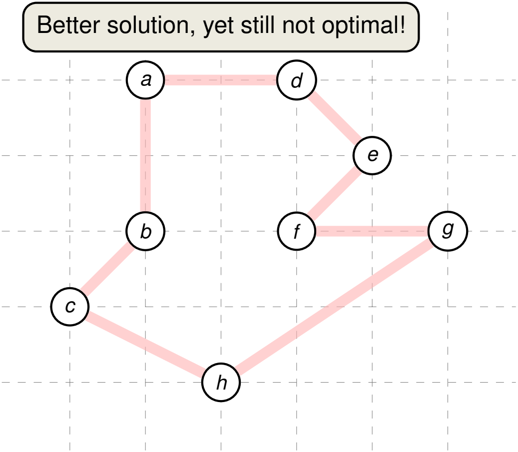

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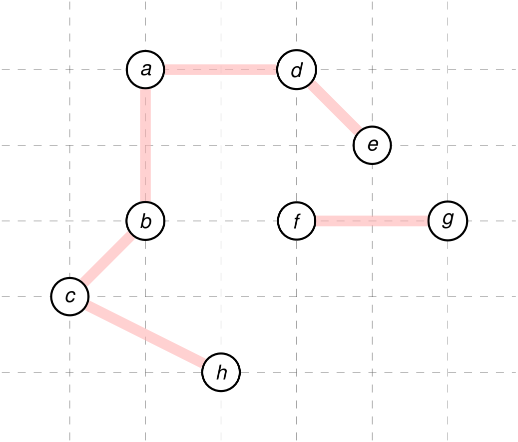

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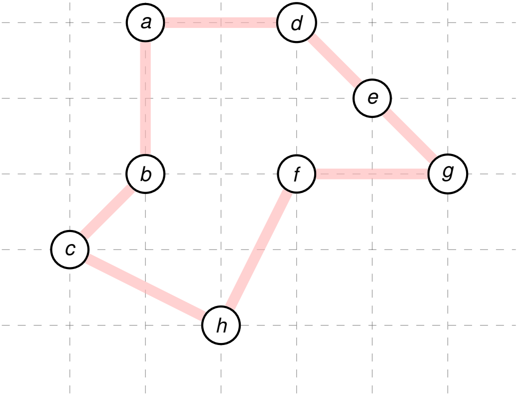
Run of APPROX-TSP-TOUR

This is the optimal solution (cost $\approx$ 14.715).



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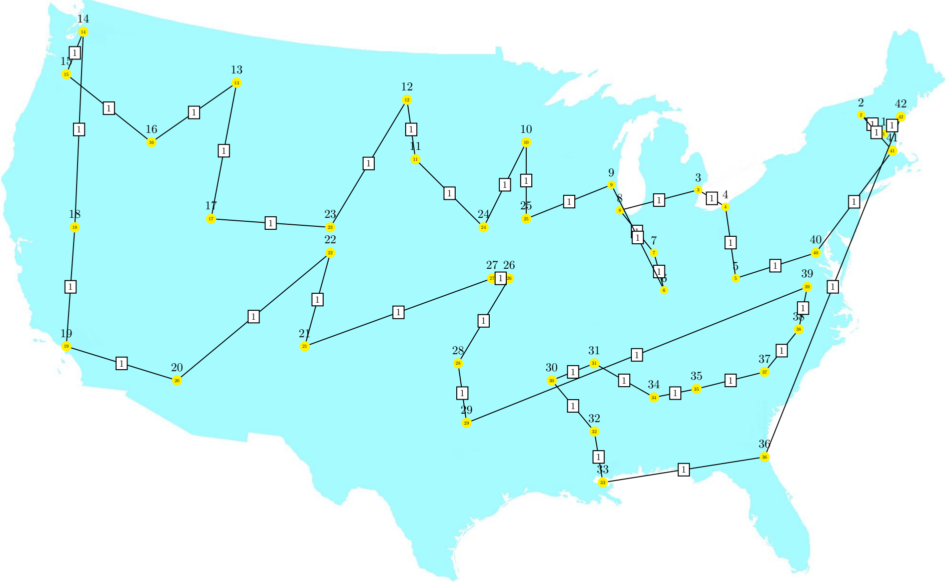
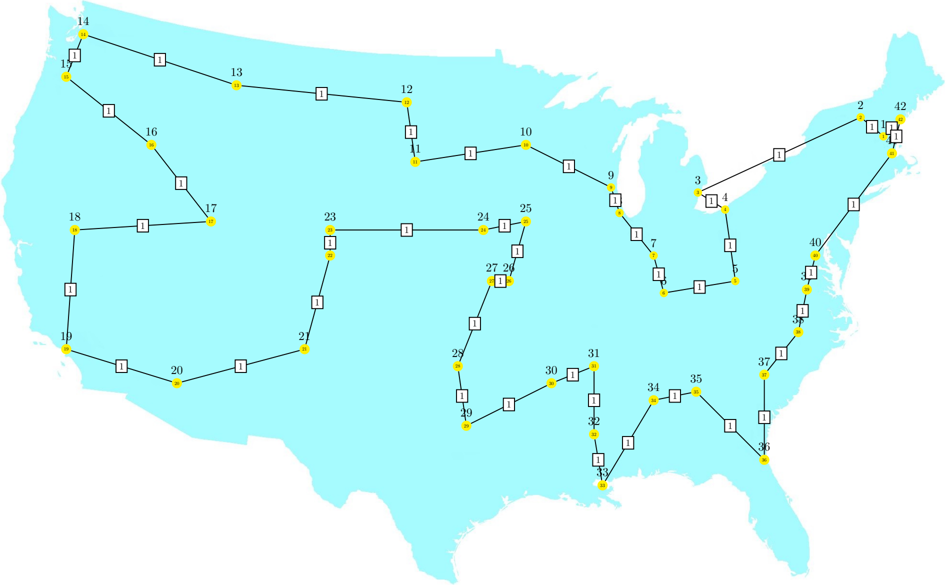
Approximate Solution: Objective 921



Optimal Solution: Objective 699



Proof of the Approximation Ratio

Theorem 35.2

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

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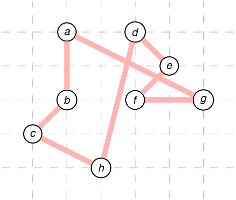

solution H of APPROX-TSP

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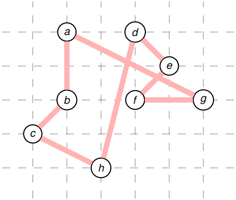

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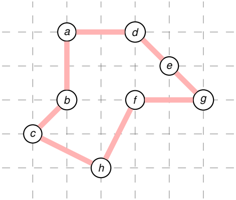

optimal solution H^*

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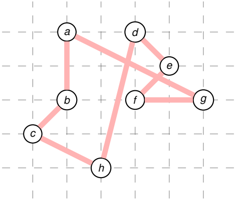

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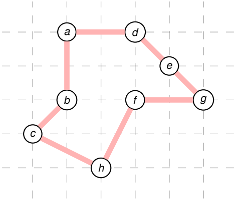

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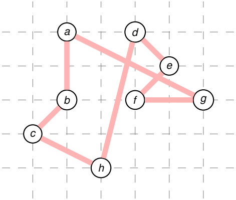

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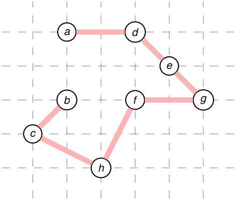

spanning tree T as a subset of H^*

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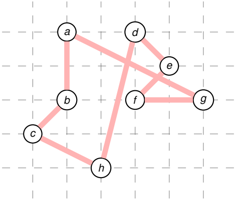

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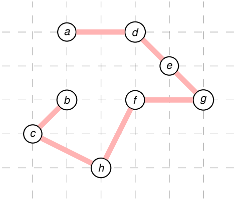

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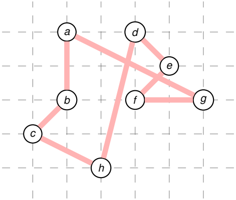

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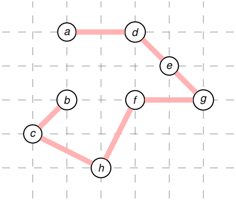

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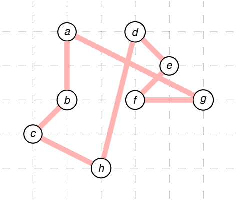
exploiting that all edge costs are non-negative!



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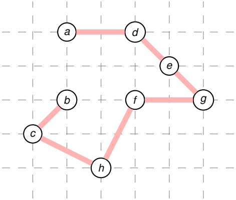

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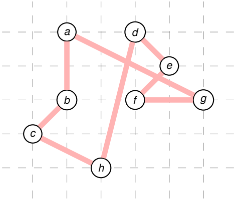

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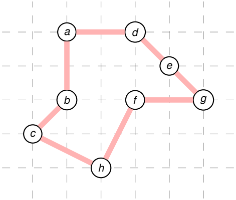

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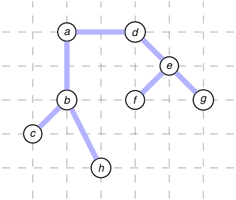
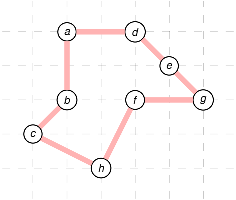


minimum spanning tree $T_{\min}$



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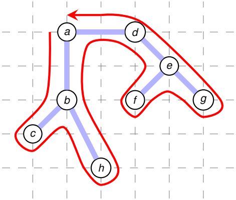
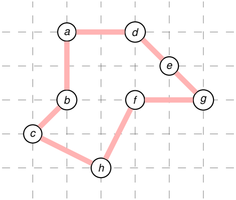


Walk $W = (a, b, c, b, h, b, a, d, e, f, e, g, e, d, a)$



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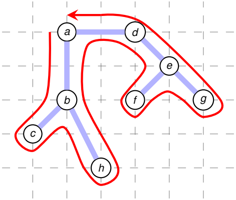

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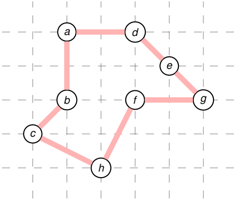

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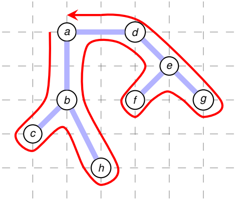

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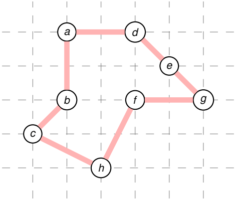

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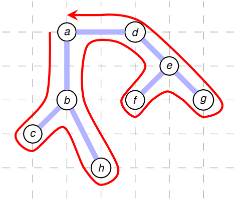

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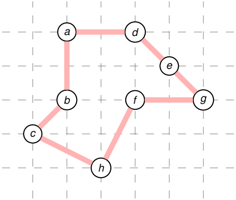

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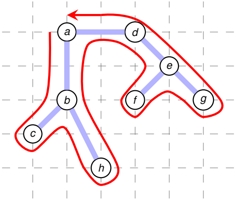

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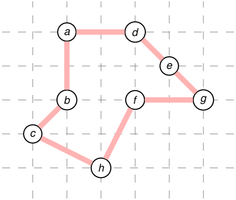

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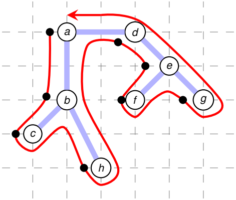

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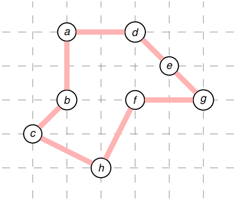

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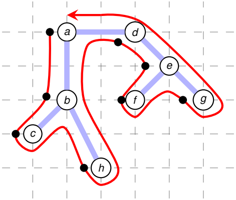

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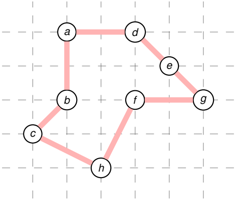

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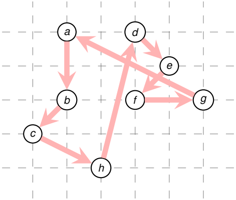
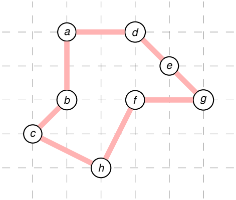


Tour $H = (a, b, c, h, d, e, f, g, a)$



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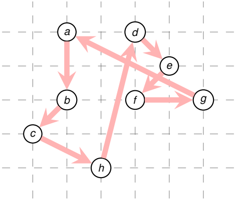
exploiting triangle inequality!

- Deleting duplicate vertices from W yields a tour H with smaller cost:



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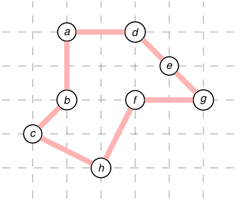

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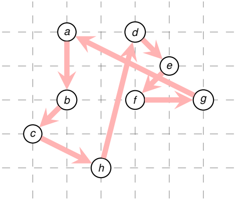

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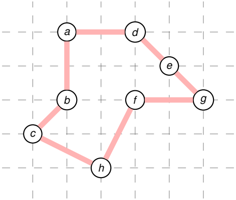

optimal solution H^*

Proof of the Approximation Ratio

Theorem 35.2

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

Proof:

- Consider the optimal tour H^* and remove an arbitrary edge

$\Rightarrow$ yields a spanning tree T and $c(T_{\min}) \leq c(T) \leq c(H^*)$

- Let W be the full walk of the minimum spanning tree $T_{\min}$ (including repeated visits)

$\Rightarrow$ Full walk traverses every edge exactly twice, so

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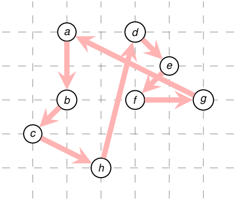

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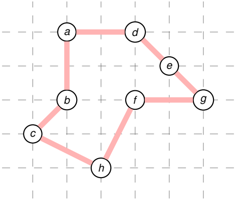

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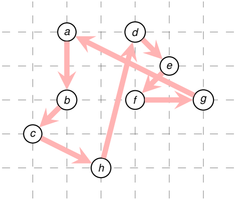

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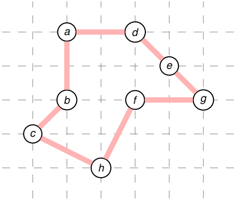

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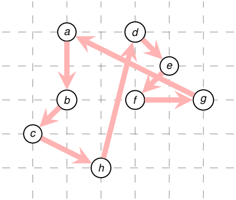
$$c(H) \leq c(W) \leq 2c(H^*) \qquad \square$$



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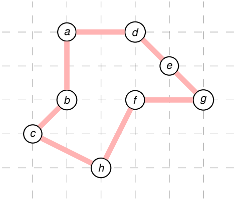

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Christofides Algorithm

Theorem 35.2

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Christofides Algorithm

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APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

Can we get a better approximation ratio?

Christofides Algorithm

Theorem 35.2

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

Can we get a better approximation ratio?

CHRISTOFIDES(G, c)
select a vertex $r \in G.V$ to be a "root" vertex
compute a minimum spanning tree T_{min} for G from root r
    using MST-PRIM(G, c, r)
compute a perfect matching M_{min} with minimum weight in the complete graph
    over the odd-degree vertices in T_{min}
let H be a list of vertices, ordered according to when they are first visited
    in a Eulearian circuit of $T_{\text{min}} \cup M_{\text{min}}$
return the hamiltonian cycle H

Christofides Algorithm

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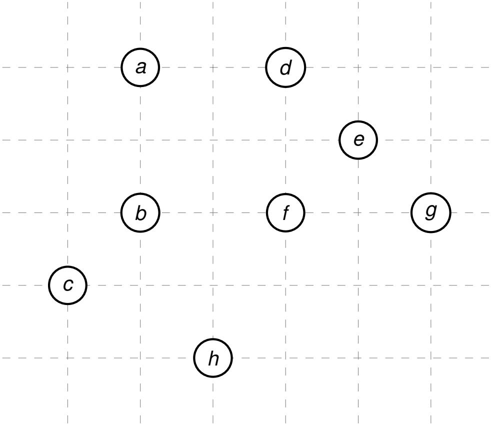
Theorem (Christofides'76)

There is a polynomial-time $\frac{3}{2}$-approximation algorithm for the travelling salesman problem with the triangle inequality.

Run of CHRISTOFIDES



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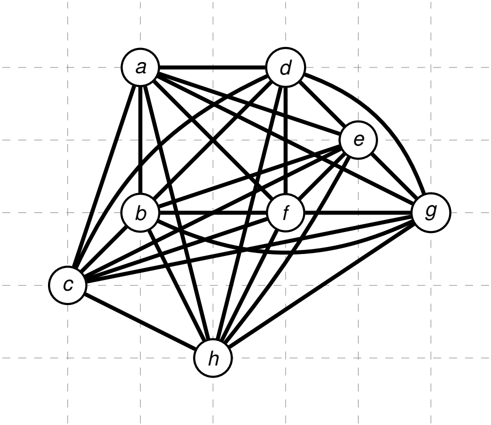
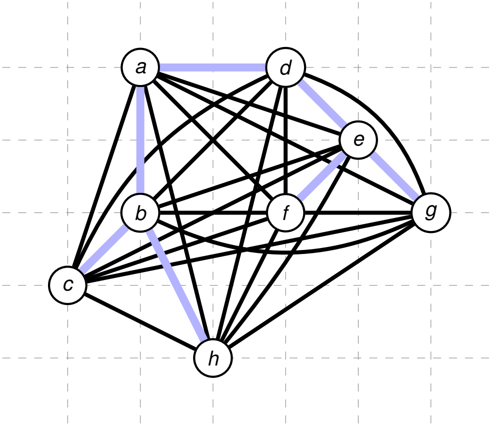


1. Compute MST $T_{\min}$

Run of CHRISTOFIDES



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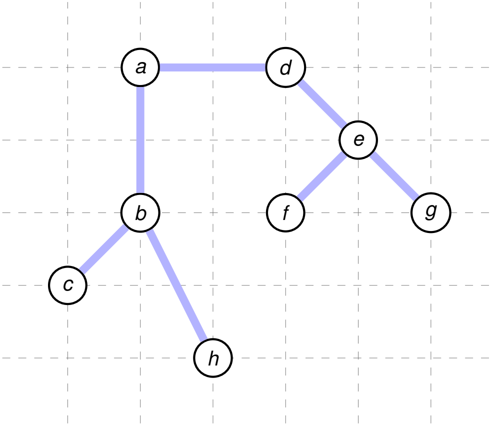

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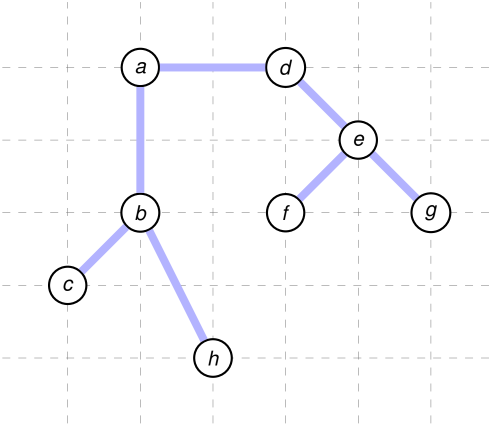
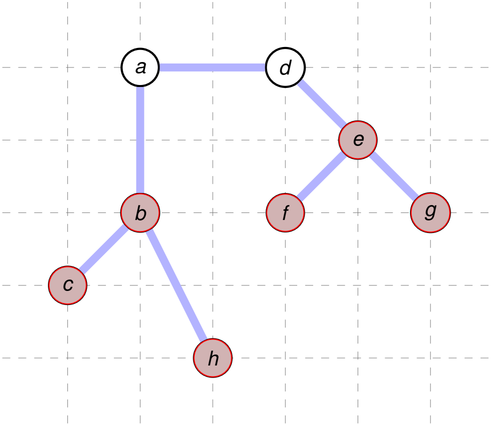


1. Compute MST $T_{\min}$ ✓
2. Add a minimum-weight perfect matching $M_{\min}$ of the odd vertices in $T_{\min}$

Run of CHRISTOFIDES



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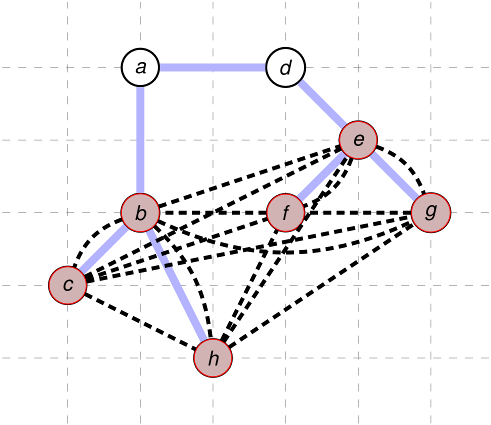

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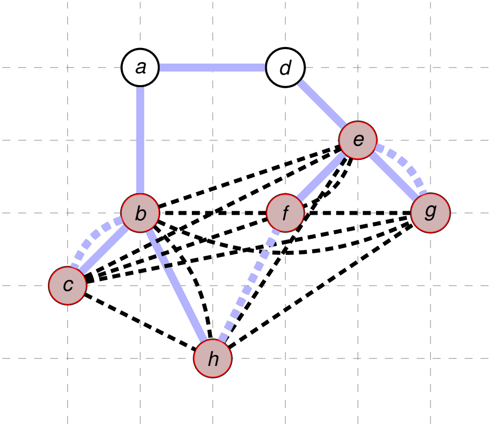

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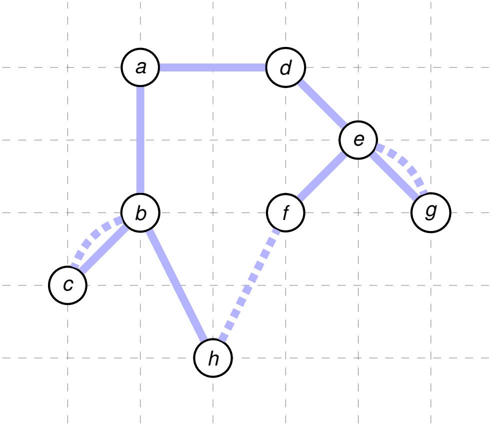

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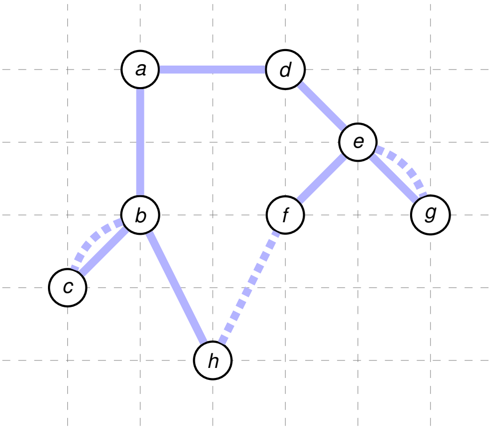

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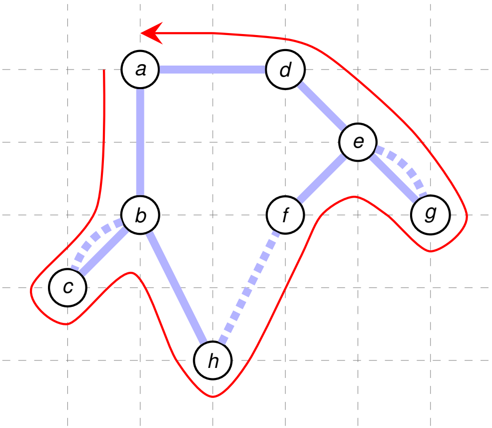
3. Find an Eulerian Circuit in $T_{\min} \cup M_{\min}$

All vertices in $T_{\min} \cup M_{\min}$ have even degree!

Run of CHRISTOFIDES



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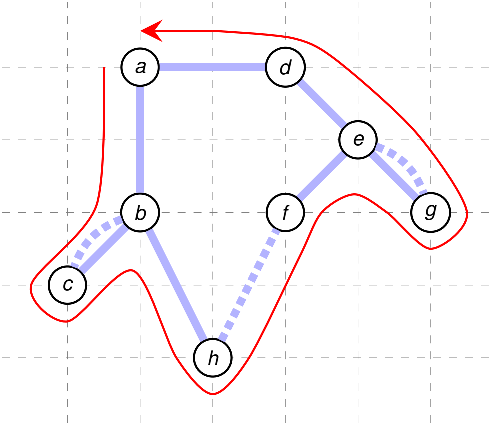

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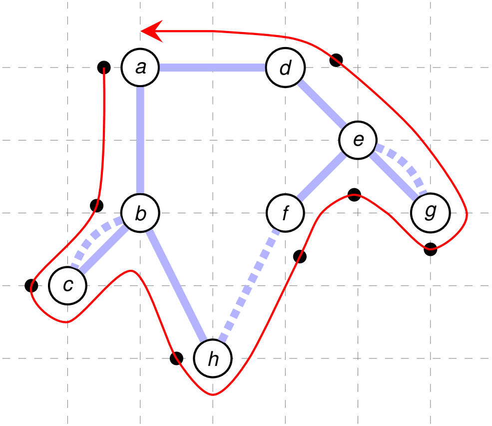
4. Transform the Circuit into a Hamiltonian Cycle

Run of CHRISTOFIDES



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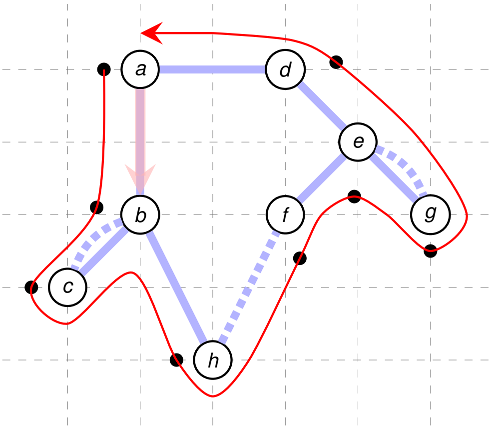

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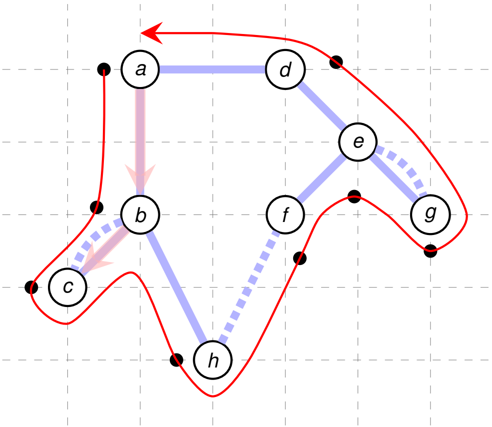

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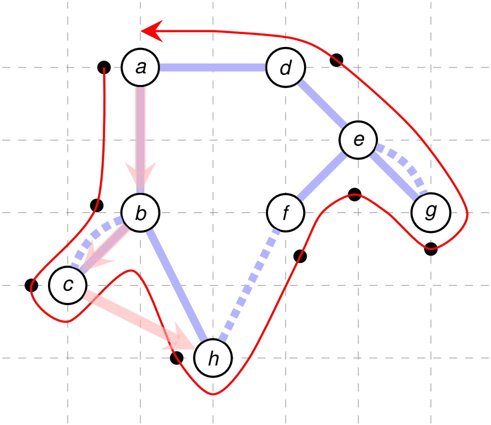

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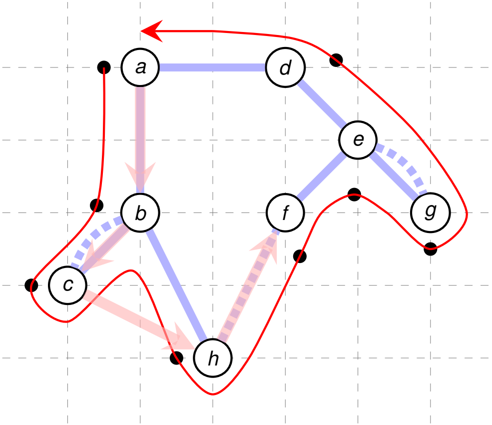

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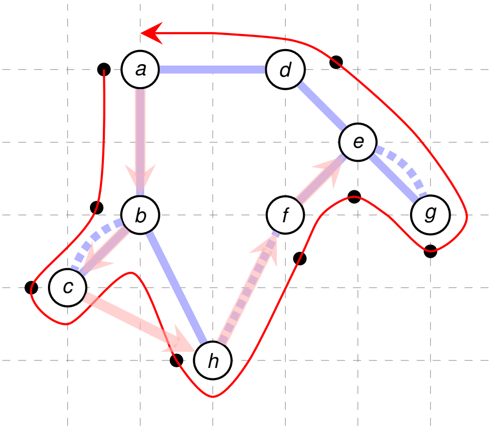

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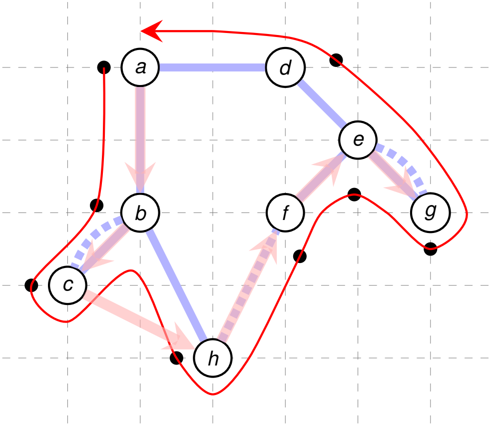

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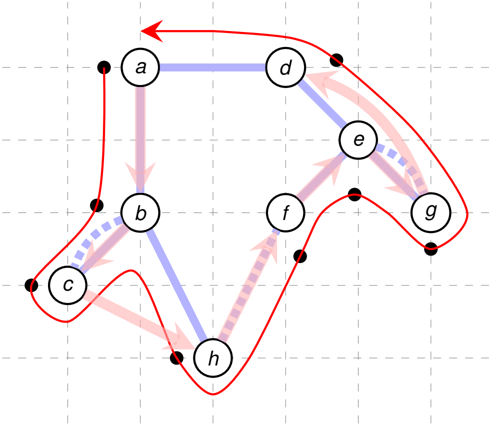

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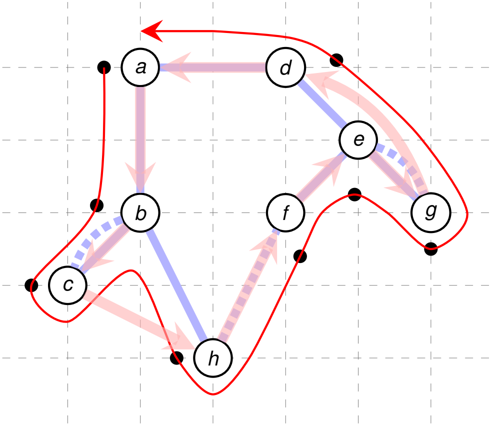

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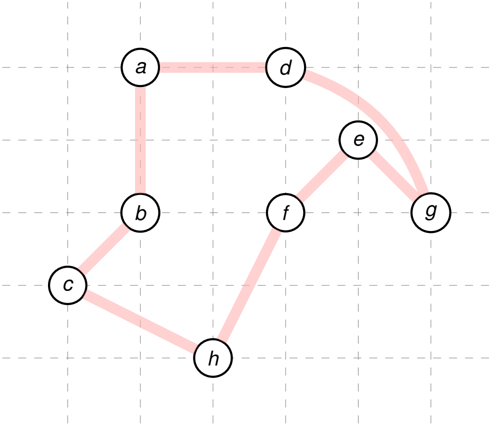

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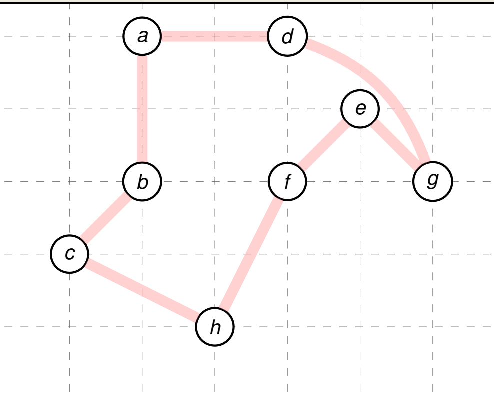
Solution has cost ≈ 15.54 - within 10% of the optimum!



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Proof of the Approximation Ratio

Theorem (Christofides'76)

There is a polynomial-time $\frac{3}{2}$-approximation algorithm for the travelling salesman problem with the triangle inequality.

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(1)

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- By shortcutting and the triangle inequality,

(2)

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- Combining 1 with 2 yields

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- Let H^*_{odd} be an optimal tour on the odd-degree vertices in $T_{\min}$
- Taking edges alternately, we obtain two matchings M_1 and M_2 such that $c(M_1) + c(M_2) = c(H^*_{odd})$

Number of odd-degree vertices is even!

- By shortcutting and the triangle inequality,

$$c(M_{\min}) \leq \frac{1}{2} c({H_{odd}}^*) \leq \frac{1}{2} c(H^*). \tag{2}$$

- Combining 1 with 2 yields

$$c(W) \leq c(H^*) + c(M_{\min}) \leq c(H^*) + \frac{1}{2} c(H^*) = \frac{3}{2} c(H^*).$$

Concluding Remarks

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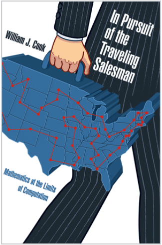

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Exercise: Prove that the approximation ratio of APPROX-TSP-TOUR satisfies $\rho(n) < 2$.

Hint: Consider the effect of the shortcutting, but note that edge costs might be zero!

VI. Approx. Algorithms: Randomisation and Rounding

Thomas Sauerwald

Easter 2021

UNIVERSITY OF CAMBRIDGE

Outline

Performance Ratios for Randomised Approximation Algorithms

Approximation Ratio

A randomised algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size n, the expected cost C of the returned solution and optimal cost C^* satisfy:

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Approximation Schemes

An approximation scheme is an approximation algorithm, which given any input and $\epsilon > 0$, is a $(1+\epsilon)$-approximation algorithm.

- It is a polynomial-time approximation scheme (PTAS) if for any fixed $\epsilon > 0$, the runtime is polynomial in n. For example, $O(n^{2/\epsilon})$.
- It is a fully polynomial-time approximation scheme (FPTAS) if the runtime is polynomial in both $1/\epsilon$ and n. For example, $O((1/\epsilon)^2 \cdot n^3)$.

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extends in the natural way to randomised algorithms

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Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Weighted Set Cover

MAX-CNF

Conclusion

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Idea: What about assigning each variable uniformly and independently at random?

Analysis

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

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Probabilistic Method: powerful tool to show existence of a non-obvious property.

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Probabilistic Method: powerful tool to show existence of a non-obvious property.

Corollary

Any instance of MAX-3-CNF with at most 7 clauses is satisfiable.

Interesting Implications

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised $8/7$-approximation algorithm.

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Follows from the previous Corollary.

Expected Approximation Ratio

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Y is defined as in the previous proof.

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Algorithm: Assign x_1 so that the conditional expectation is maximized and recurse.

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GREEDY-3-CNF(ϕ, n, m)

for $j = 1, 2, \ldots, n$
    Compute $\mathbf{E}[\,Y \mid x_1 = v_1 \ldots, x_{j-1} = v_{j-1}, x_j = 1\,]$
    Compute $\mathbf{E}[\,Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 0\,]$
    Let $x_j = v_j$ so that the conditional expectation is maximized
return the assignment $v_1, v_2, \ldots, v_n$

Analysis of GREEDY-3-CNF(ϕ, n, m)

Theorem

GREEDY-3-CNF(ϕ, n, m) is a polynomial-time $8/7$-approximation.

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Run of GREEDY-3-CNF(φ, n, m)

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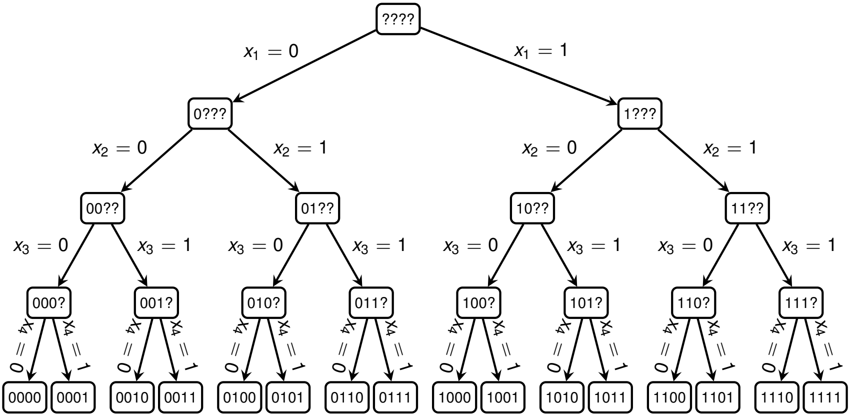

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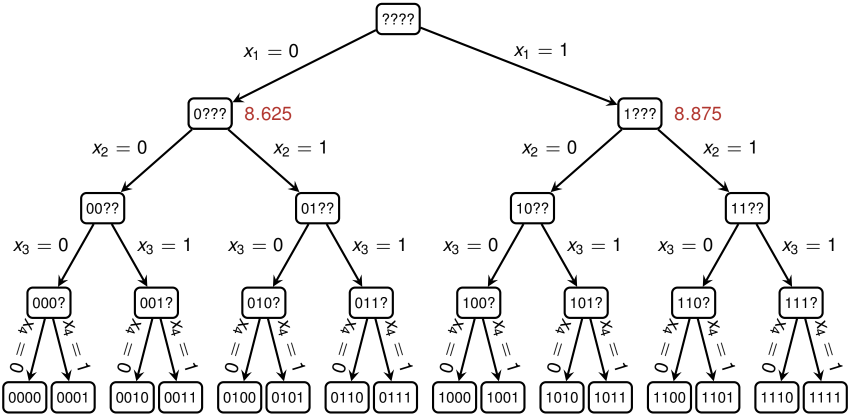
$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



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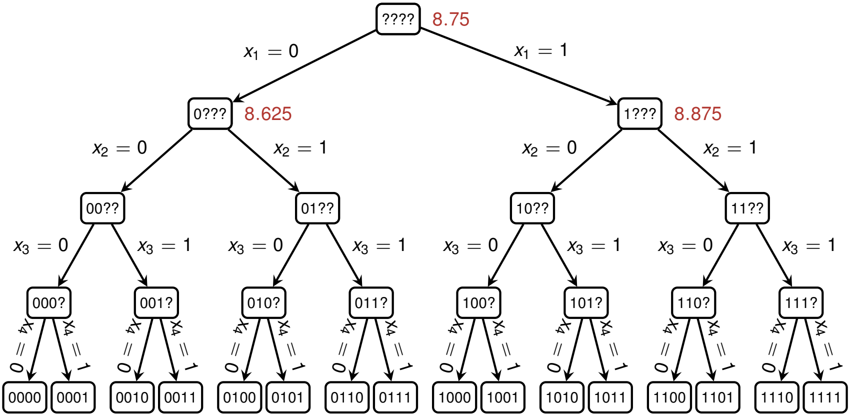

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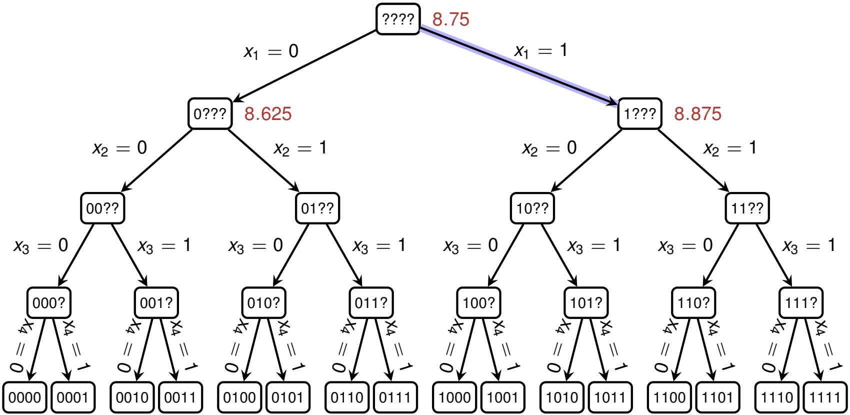
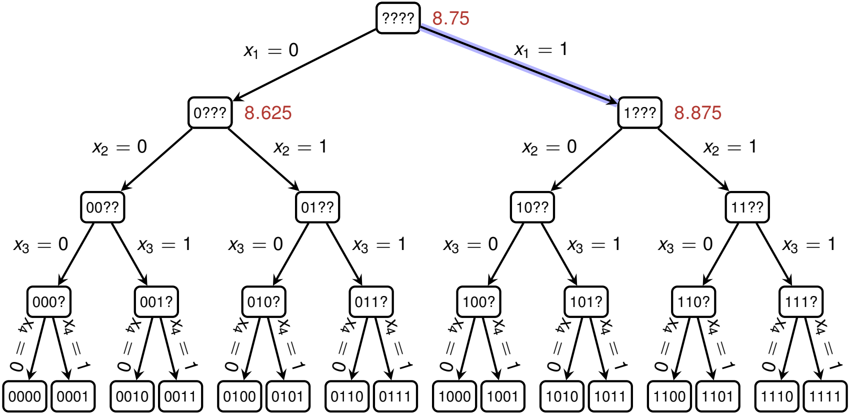


Run of GREEDY-3-CNF(φ, n, m)

$\cancel{(x_1 \vee x_2 \vee x_3)} \wedge \cancel{(x_1 \vee \overline{x_2} \vee \overline{x_4})} \wedge \cancel{(x_1 \vee x_2 \vee \overline{x_4})} \wedge (\cancel{\overline{x_1}} \vee \overline{x_3} \vee x_4) \wedge \cancel{(x_1 \vee x_2 \vee \overline{x_3})} \wedge (\cancel{\overline{x_1}} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\cancel{\overline{x_1}} \vee x_2 \vee x_3) \wedge (\cancel{\overline{x_1}} \vee \overline{x_2} \vee x_3) \wedge \cancel{(x_1 \vee x_3 \vee \overline{x_4})} \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



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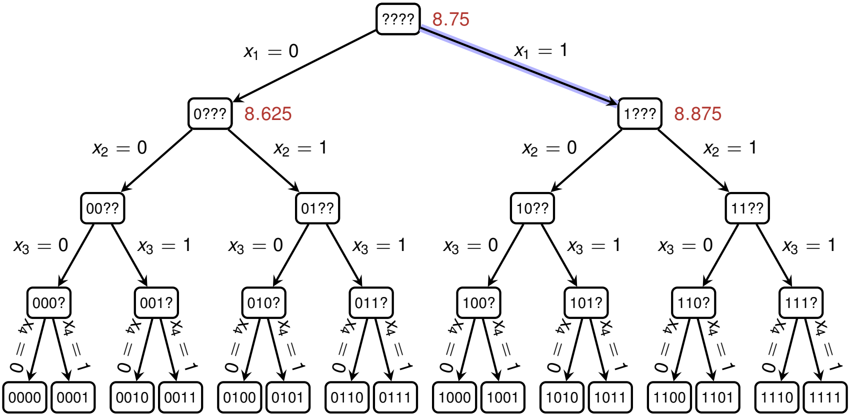

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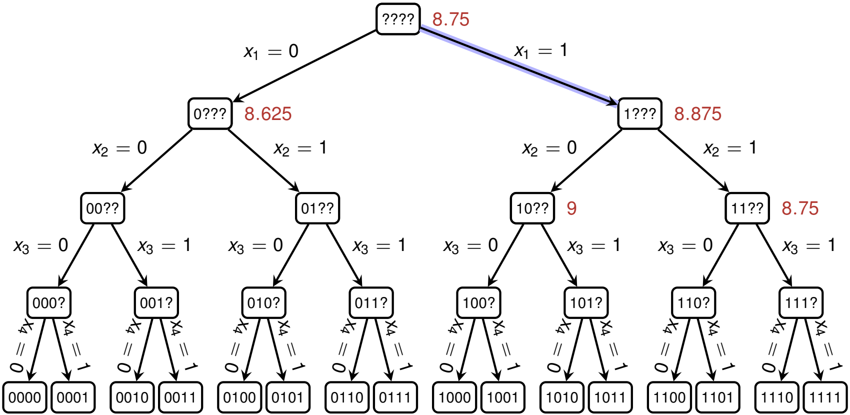

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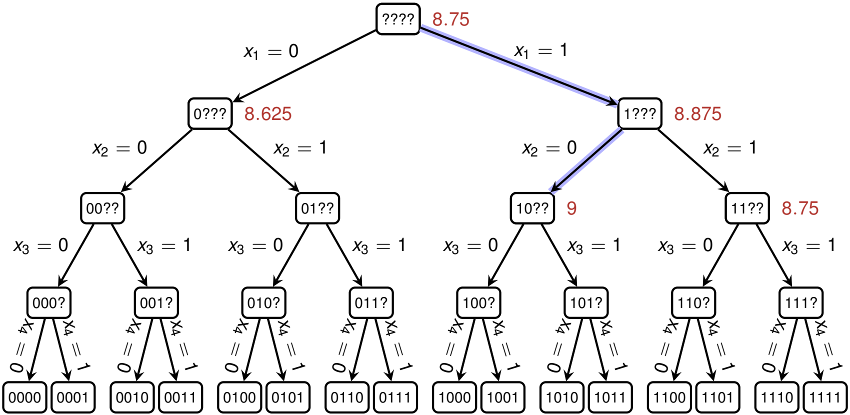

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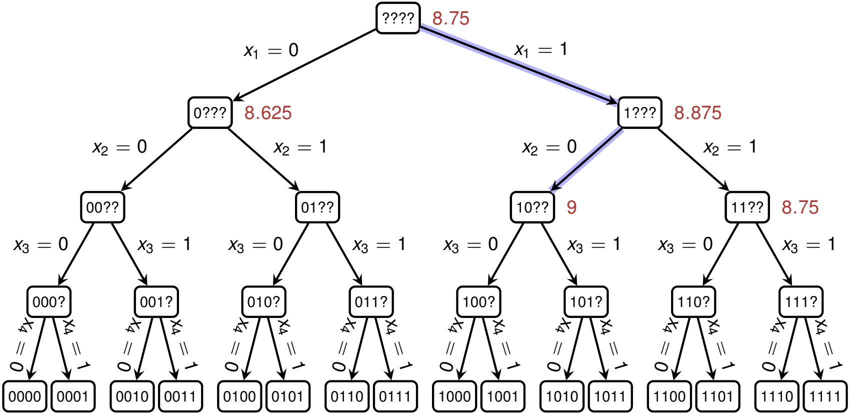
$$1 \wedge 1 \wedge 1 \wedge (\overline{x_3} \vee x_4) \wedge 1 \wedge \cancel{(\overline{x_2} \vee \overline{x_3})} \wedge (\cancel{x_2} \vee x_3) \wedge \cancel{(\overline{x_2} \vee \overline{x_3})} \wedge 1 \wedge (\cancel{x_2} \vee \overline{x_3} \vee \overline{x_4})$$



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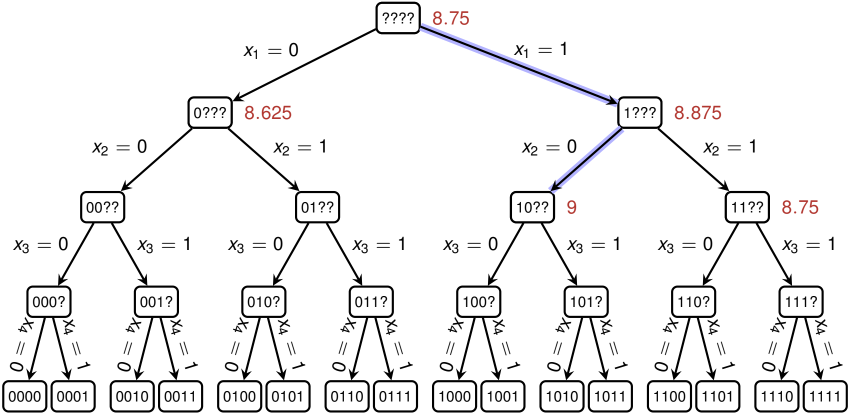

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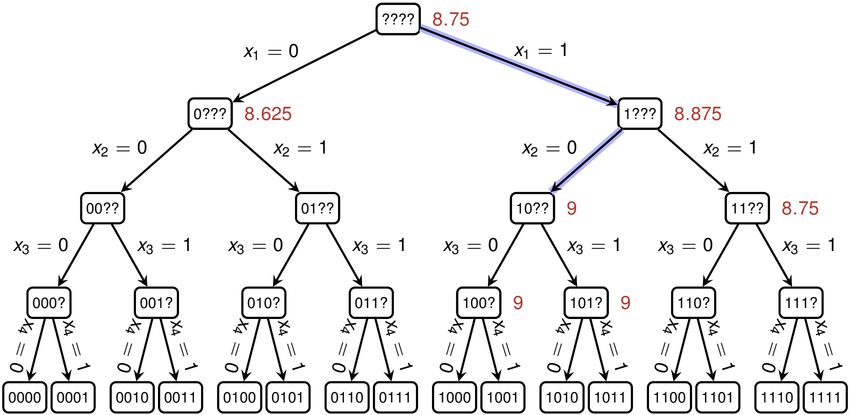

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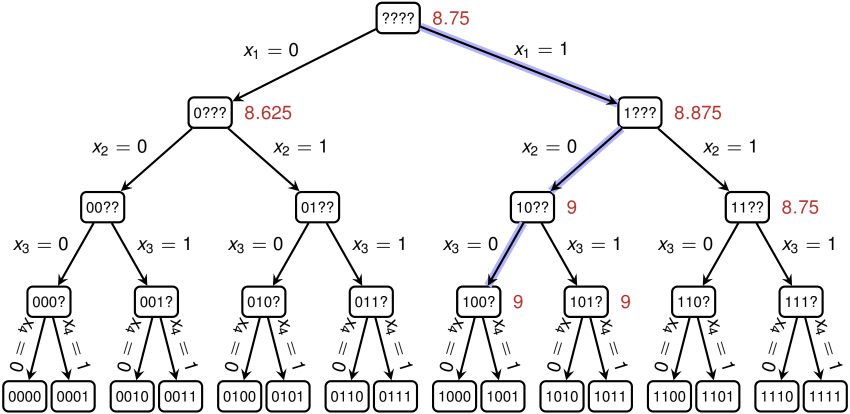

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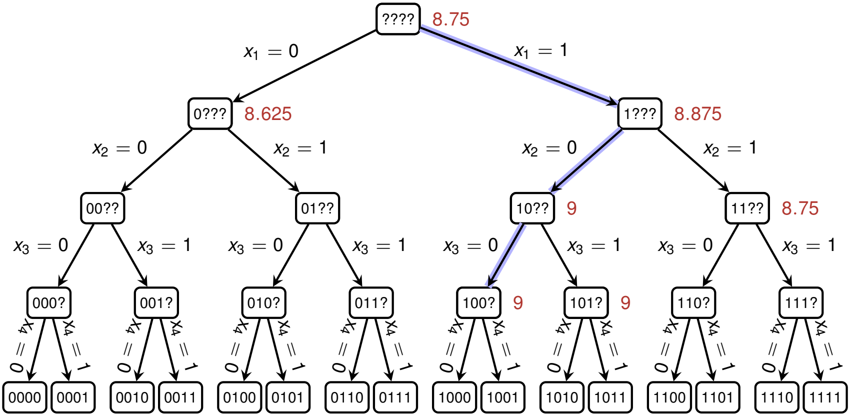

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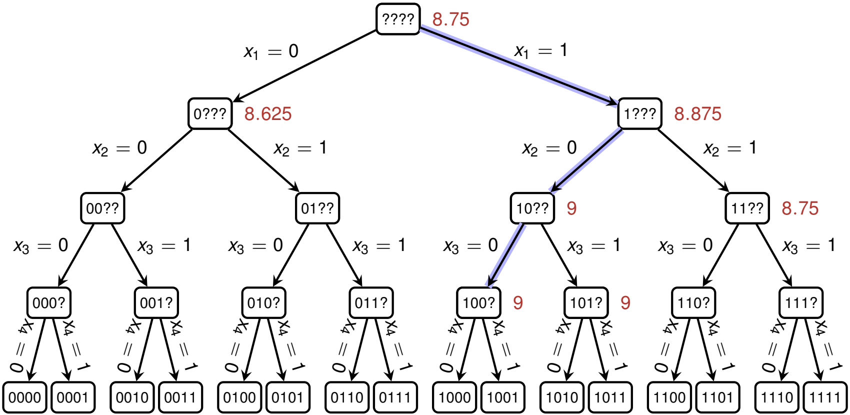

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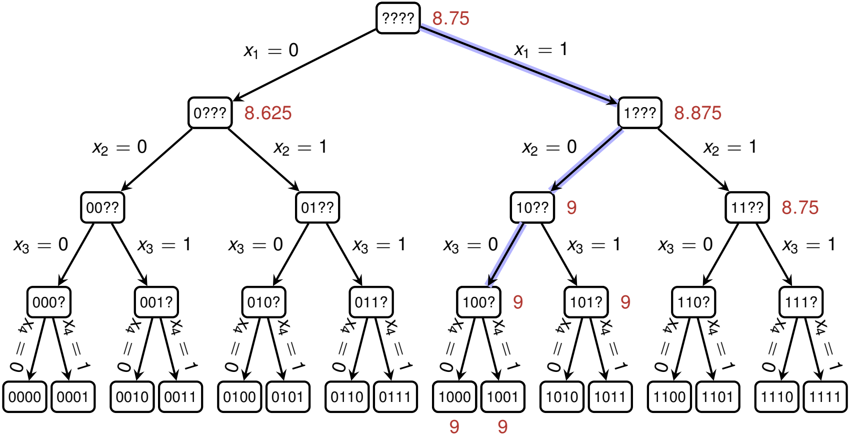

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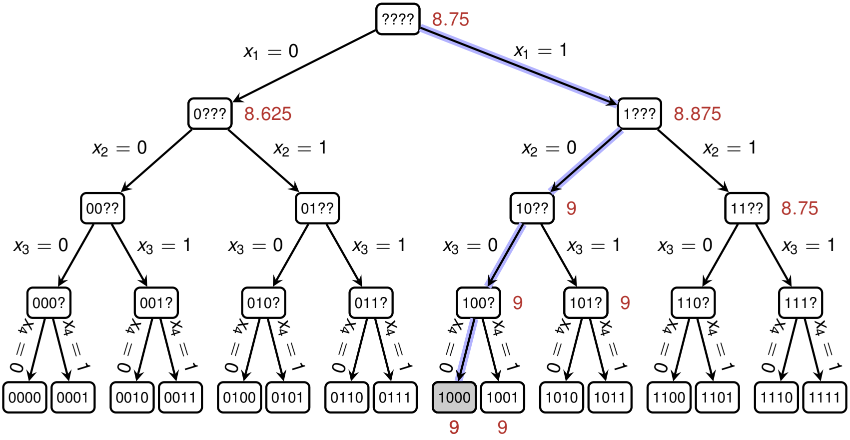

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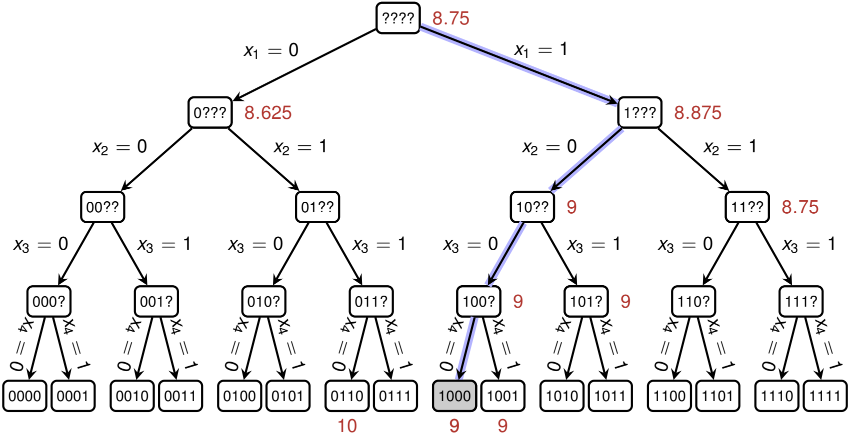

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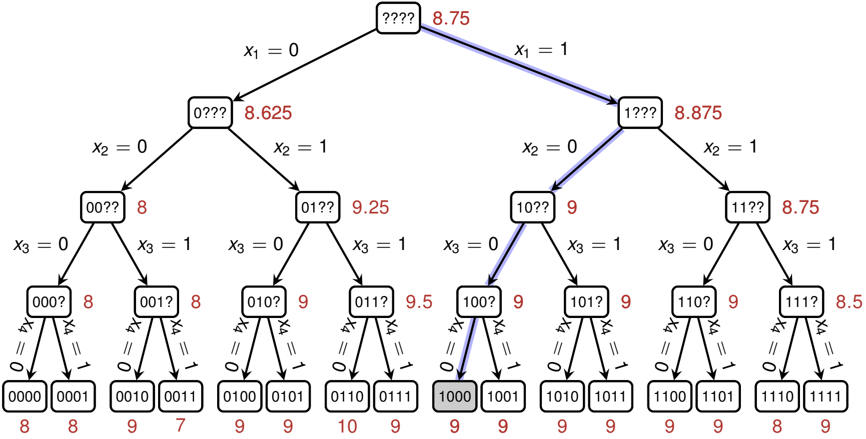

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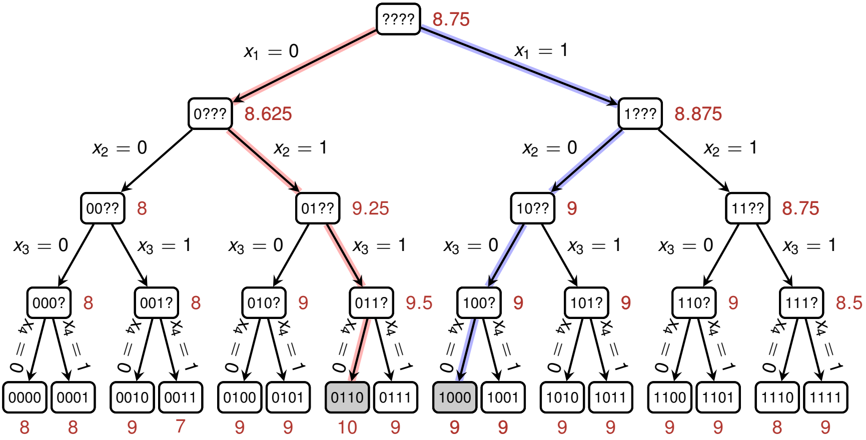

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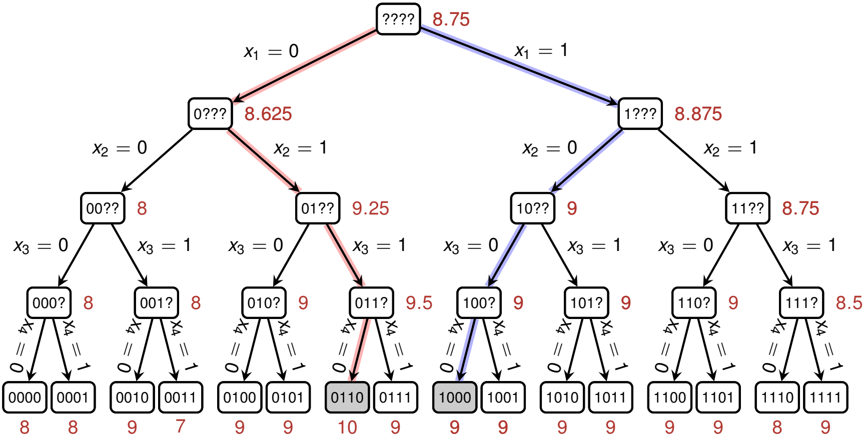

Returned solution satisfies 9 out of 10 clauses, but the formula is satisfiable.

MAX-3-CNF: Concluding Remarks

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

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GREEDY-3-CNF(ϕ, n, m) is a polynomial-time $8/7$-approximation.

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Essentially there is nothing smarter than just guessing!

Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Weighted Set Cover

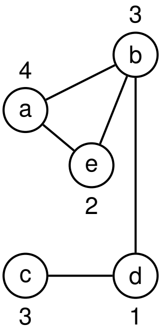
MAX-CNF

Conclusion

The Weighted Vertex-Cover Problem

Vertex Cover Problem

- Given: Undirected, vertex-weighted graph $G = (V, E)$
- Goal: Find a minimum-weight subset $V' \subseteq V$ such that if $(u, v) \in E(G)$, then $u \in V'$ or $v \in V'$.



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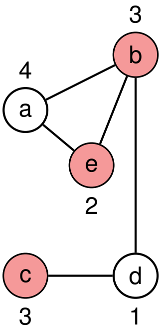

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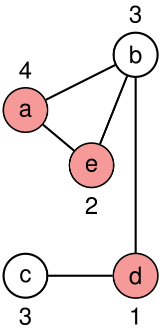

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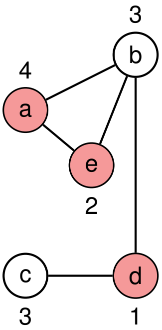

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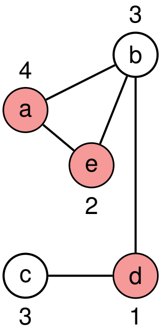

Applications:

The Weighted Vertex-Cover Problem

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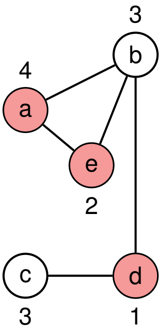

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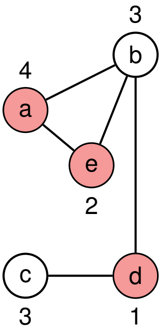

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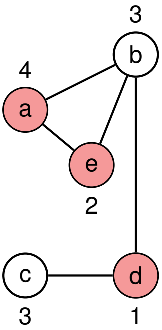

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- Weight of a vertex could be salary of a person
- Perform all tasks with the minimal amount of resources

The Greedy Approach from (Unweighted) Vertex Cover

APPROX-VERTEX-COVER(G)

```
C = ∅
E′ = G.E
while E′ ≠ ∅
    let (u, v) be an arbitrary edge of E′
    C = C ∪ {u, v}
    remove from E′ every edge incident on either u or v
return C
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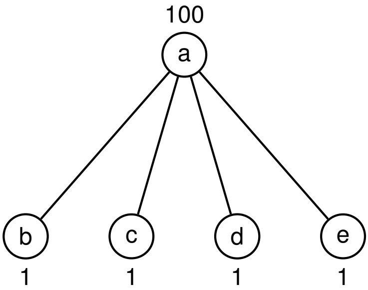

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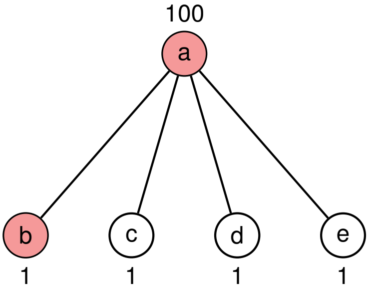

Computed solution has weight 101

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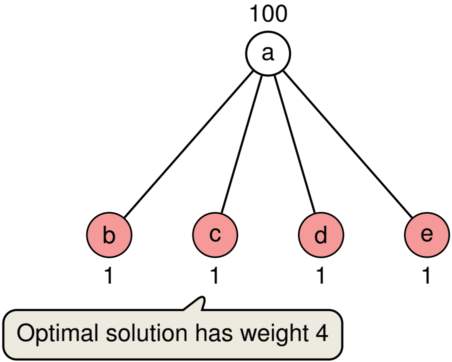

Invoking an (Integer) Linear Program

Idea: Round the solution of an associated linear program.

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0-1 Integer Program

$$\begin{array}{llcll} \text{minimize} & \displaystyle\sum_{v \in V} w(v)x(v) & & & \\ \text{subject to} & x(u) + x(v) & \geq & 1 & \text{for each } (u, v) \in E \\ & x(v) & \in & \{0, 1\} & \text{for each } v \in V \end{array}$$

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optimum is a lower bound on the optimal weight of a minimum weight-cover.

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Rounding Rule: if $x(v) \geq 1/2$ then round up, otherwise round down.

The Algorithm

APPROX-MIN-WEIGHT-VC(G, w)

$C = \emptyset$
compute $\bar{x}$, an optimal solution to the linear program
for each $v \in V$
    if $\bar{x}(v) \geq 1/2$
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Theorem 35.7

APPROX-MIN-WEIGHT-VC is a polynomial-time 2-approximation algorithm for the minimum-weight vertex-cover problem.

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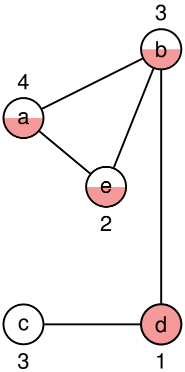
is polynomial-time because we can solve the linear program in polynomial time

Example of APPROX-MIN-WEIGHT-VC

$\overline{x}(a) = \overline{x}(b) = \overline{x}(e) = \frac{1}{2}, \overline{x}(d) = 1, \overline{x}(c) = 0$



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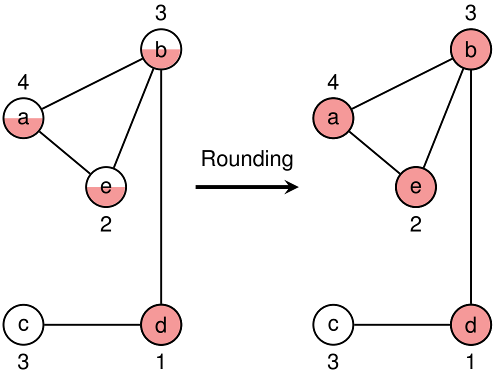

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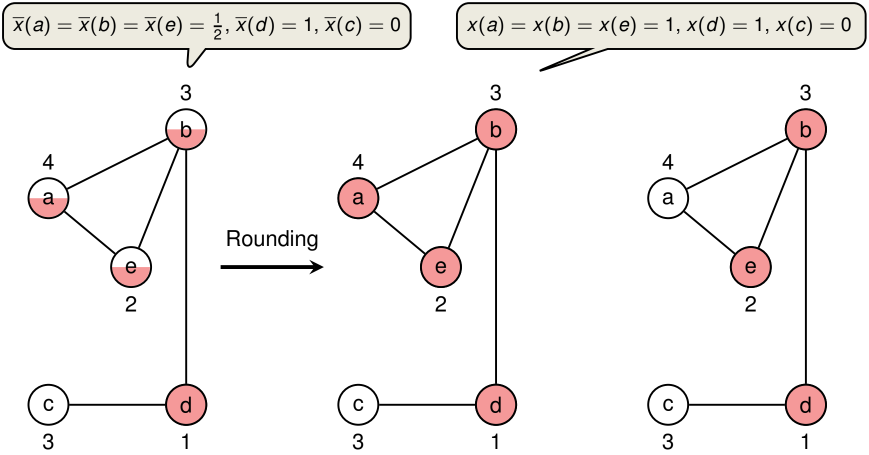

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optimal solution with weight $= 6$

Approximation Ratio

Proof (Approximation Ratio is 2 and Correctness):

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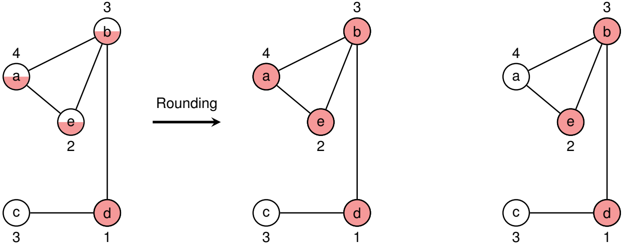

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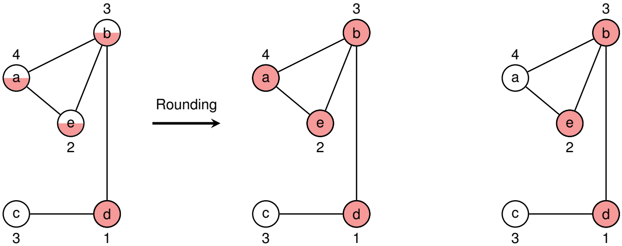

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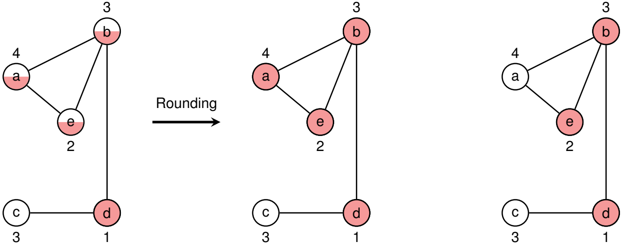

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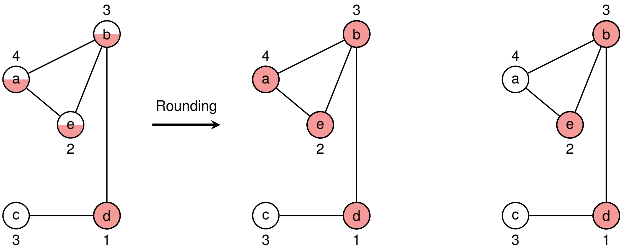

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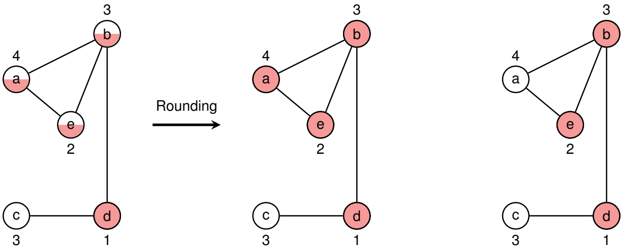

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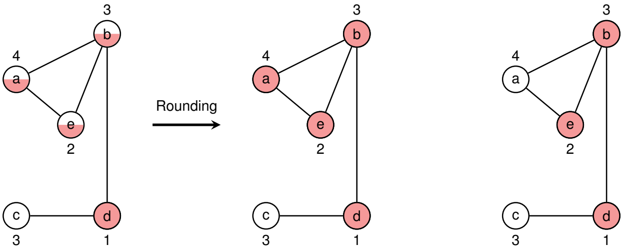

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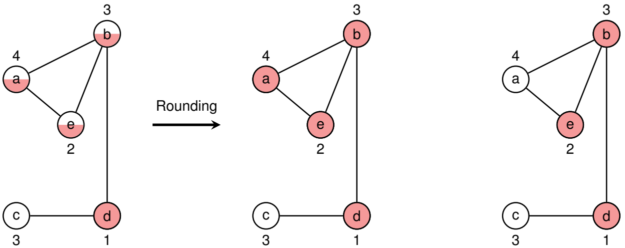

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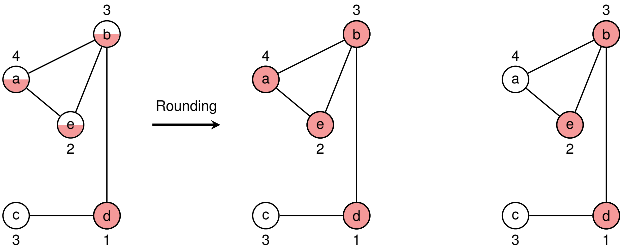

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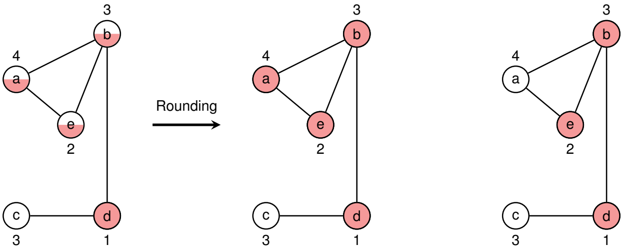

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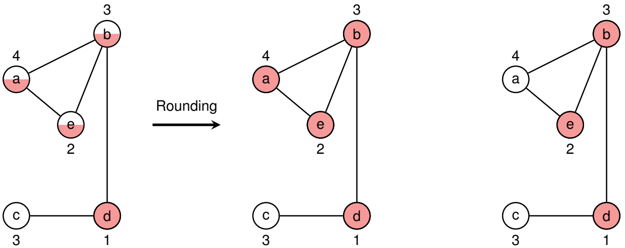

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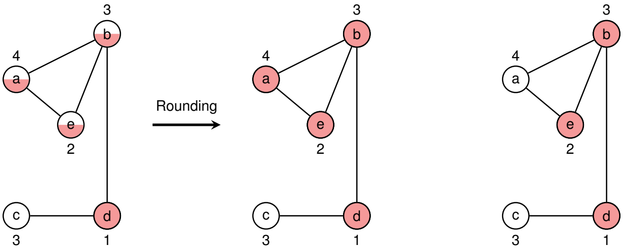

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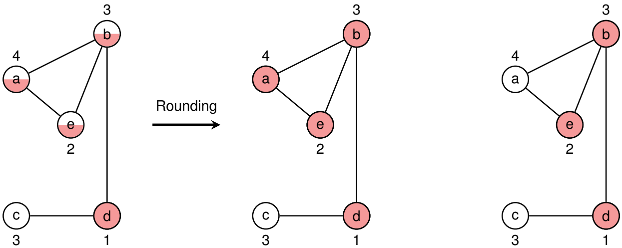

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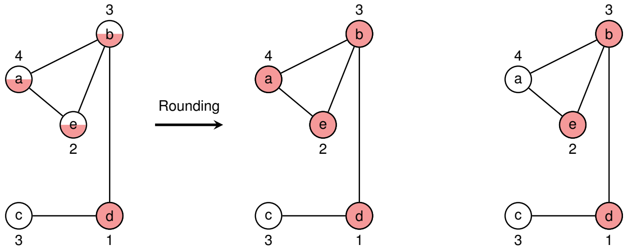

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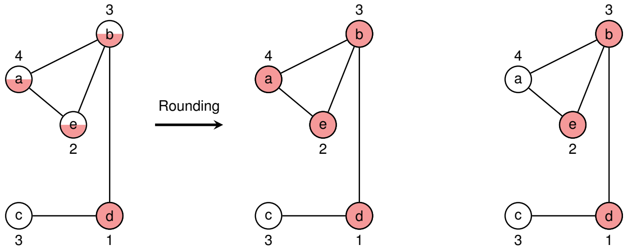

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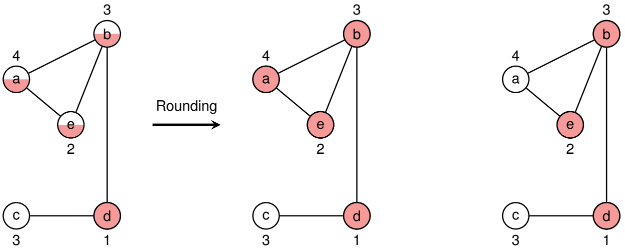

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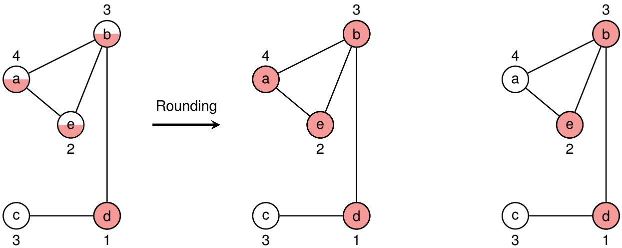

Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Weighted Set Cover

MAX-CNF

Conclusion

The Weighted Set-Covering Problem

Set Cover Problem

- Given: set X and a family of subsets $\mathcal{F}$, and a cost function $c : \mathcal{F} \to \mathbb{R}^+$
- Goal: Find a minimum-cost subset $\mathcal{C} \subseteq \mathcal{F}$

$$\text{s.t.} \qquad X = \bigcup_{S \in \mathcal{C}} S.$$

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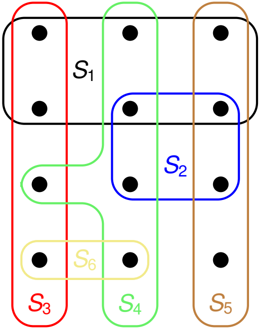

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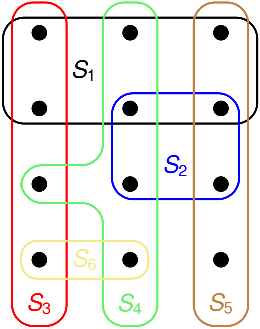

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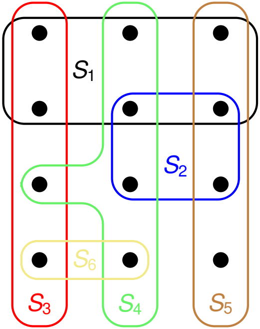

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Remarks:

- generalisation of the weighted vertex-cover problem
- models resource allocation problems

Setting up an Integer Program

Exercise: Try to formulate the integer program and linear program of the weighted SET-COVER problem (solution on next slide!)

Setting up an Integer Program

0-1 Integer Program

$$
\begin{array}{llcll}
\text{minimize} & \displaystyle\sum_{S \in \mathcal{F}} c(S) y(S) & & & \\
\text{subject to} & \displaystyle\sum_{S \in \mathcal{F}\colon\, x \in S} y(S) & \geq & 1 & \text{for each } x \in X \\
& y(S) & \in & \{0, 1\} & \text{for each } S \in \mathcal{F}
\end{array}
$$

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Linear Program

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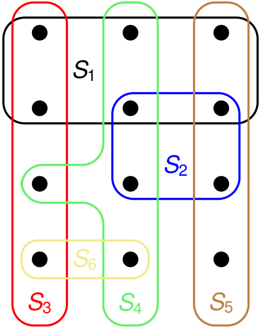
Back to the Example



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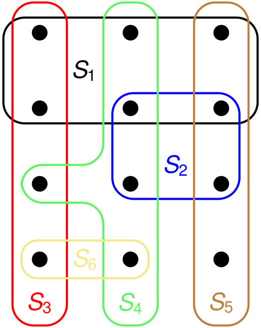

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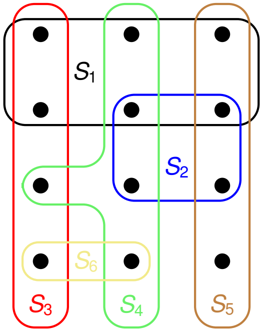

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Cost equals 8.5

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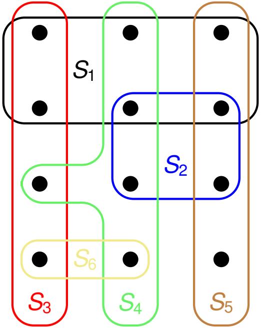

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The strategy employed for Vertex-Cover would take all 6 sets!

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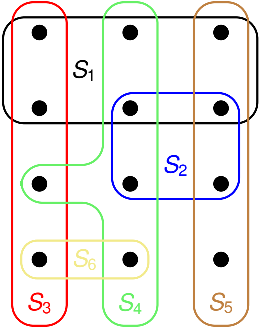

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Even worse: If all y's were below $1/2$, we would not even return a valid cover!

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Idea: Interpret the y-values as probabilities for picking the respective set.

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- Let $\mathcal{C} \subseteq \mathcal{F}$ be a random set with each set S being included independently with probability $y(S)$.
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- Therefore, $\mathbf{E}[\,\bar{y}(S)\,] = y(S)$.

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- The probability that an element $x \in X$ is covered satisfies

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Proof of Lemma

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- **Step 2**: The probability for an element to be (not) covered

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WEIGHTED SET COVER-LP$(X, \mathcal{F}, c)$

```
compute y, an optimal solution to the linear program
C = ∅
repeat 2 ln n times
    for each S ∈ F
        let C = C ∪ {S} with probability y(S)
return C
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return $\mathcal{C}$

clearly runs in polynomial-time!

Analysis of WEIGHTED SET COVER-LP

Theorem

- With probability at least $1 - \frac{1}{n}$, the returned set $\mathcal{C}$ is a valid cover of X.
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Typical Approach for Designing Approximation Algorithms based on LPs

Outline

MAX-CNF

Recall:

MAX-3-CNF Satisfiability

- Given: 3-CNF formula, e.g.: $(x_1 \vee x_3 \vee \overline{x_4}) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge \cdots$
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- Allowing arbitrary clause lengths makes the problem more interesting (we will see that simply guessing is not the best!)
- a nice concluding example where we can practice previously learned approaches

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Assign each variable true or false uniformly and independently at random.

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First solve a linear program and use fractional values for a **biased** coin flip.

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- In the corresponding LP each $\in \{0, 1\}$ is replaced by $\in [0, 1]$
- Let (y^*, z^*) be the optimal solution of the LP
- Obtain an integer solution y through randomised rounding of y^*

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For any clause i of length ℓ,

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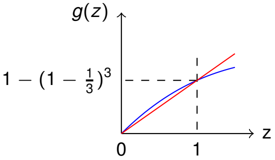

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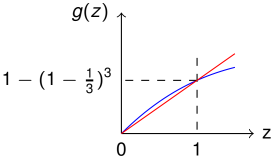

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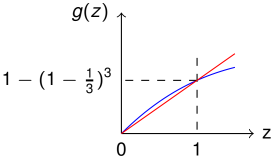

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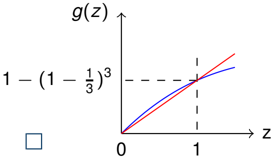

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By Lemma

Since $(1 - 1/x)^x \leq 1/e$

LP solution at least as good as optimum

Approach 3: Hybrid Algorithm

Summary

- Approach 1 (Guessing) achieves better guarantee on longer clauses
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HYBRID-MAX-CNF(φ, n, m)

1: Let $b \in \{0, 1\}$ be the flip of a fair coin
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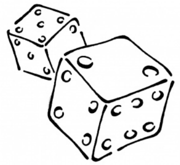

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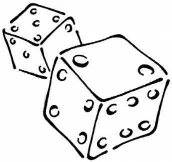

Algorithm sets each variable x_i to TRUE with prob. $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot y_i^*$.
Note, however, that variables are **not** independently assigned!

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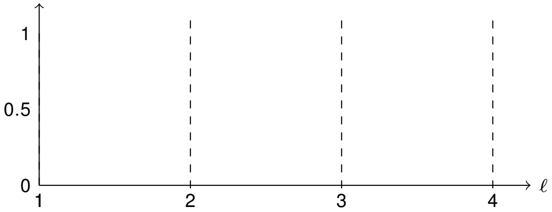

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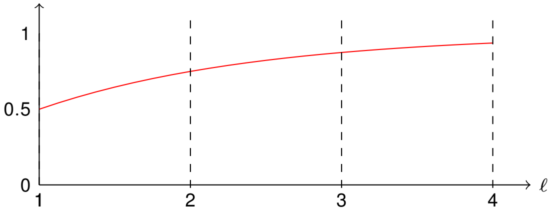

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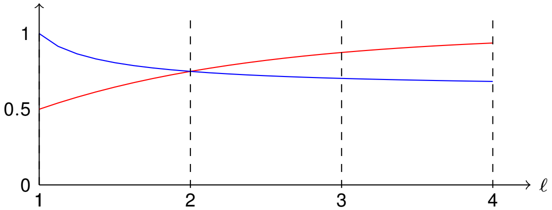

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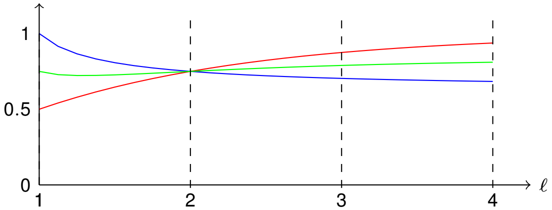

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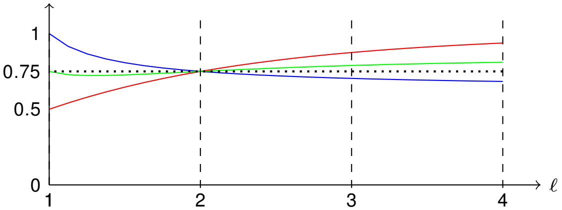

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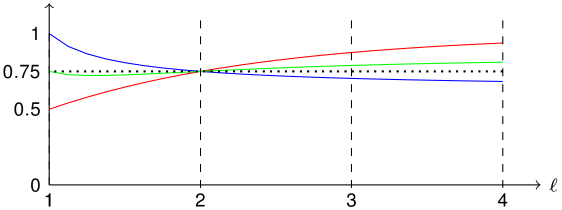
  - Algorithm 2 satisfies it with probability $\beta_\ell \cdot z_i^*$.
  - HYBRID-MAX-CNF(φ, n, m) satisfies it with probability $\frac{1}{2} \cdot \alpha_\ell \cdot z_i^* + \frac{1}{2} \cdot \beta_\ell \cdot z_i^*$.
- Note $\frac{\alpha_\ell + \beta_\ell}{2} = 3/4$ for $\ell \in \{1, 2\}$, and for $\ell \geq 3$, $\frac{\alpha_\ell + \beta_\ell}{2} \geq 3/4$ (see figure)
- $\Rightarrow$ HYBRID-MAX-CNF(φ, n, m) satisfies it with prob. at least $3/4 \cdot z_i^*$ □



MAX-CNF Conclusion

Summary

- Since $\alpha_2 = \beta_2 = 3/4$, we cannot achieve a better approximation ratio than $4/3$ by combining Algorithm 1 & 2 in a different way
- The $4/3$-approximation algorithm can be easily derandomised
  - Idea: use the conditional expectation trick for both Algorithm 1 & 2 and output the better solution
- The $4/3$-approximation algorithm applies unchanged to a weighted version of MAX-CNF, where each clause has a non-negative weight
- Even MAX-2-CNF (every clause has length 2) is NP-hard!

Exercise (easy): Consider any minimsation problem, where x is the optimal cost of the LP relaxation, y is the optimal cost of the IP and z is the solution obtained by rounding up the LP solution. Which of the follwing statements are true?

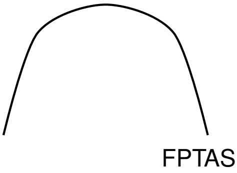
1. $x \leq y \leq z$,
2. $y \leq x \leq z$,
3. $y \leq z \leq x$.

Exercise (trickier): Consider a version of the SET-COVER problem, where each element $x \in X$ has to be covered by **at least two** subsets. Design and analyse an efficient approximation algorithm.
Hint: You may use the result that if $X_1, X_2, \ldots, X_n$ are independent Bernoulli random variables with $X := \sum_{i=1}^{n} X_i$, $\mathbf{E}[X] \geq 2$, then

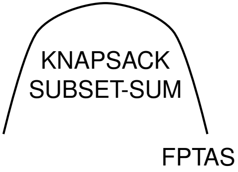
$$\mathbf{Pr}[X \geq 2] \geq 1/4 \cdot (1 - e^{-1}).$$

Outline

Spectrum of Approximations



Spectrum of Approximations



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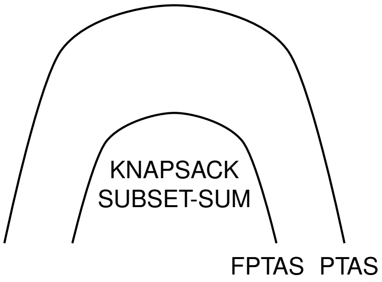

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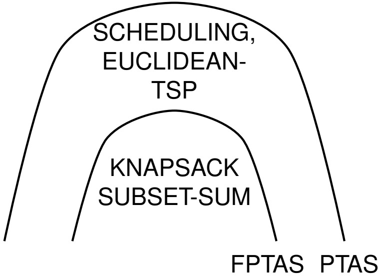

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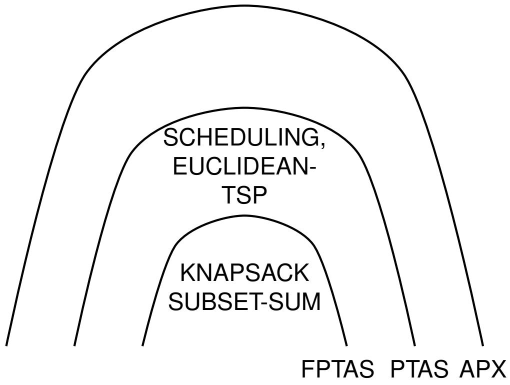

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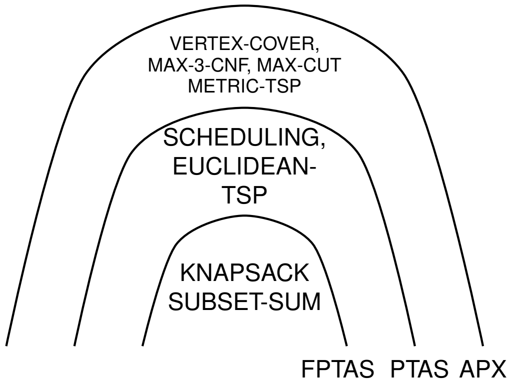

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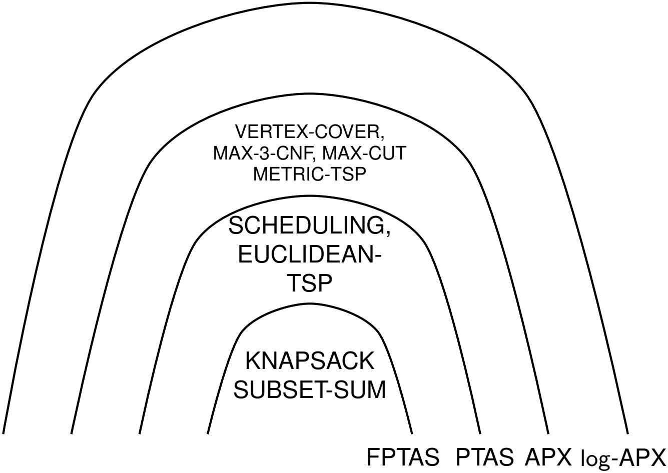

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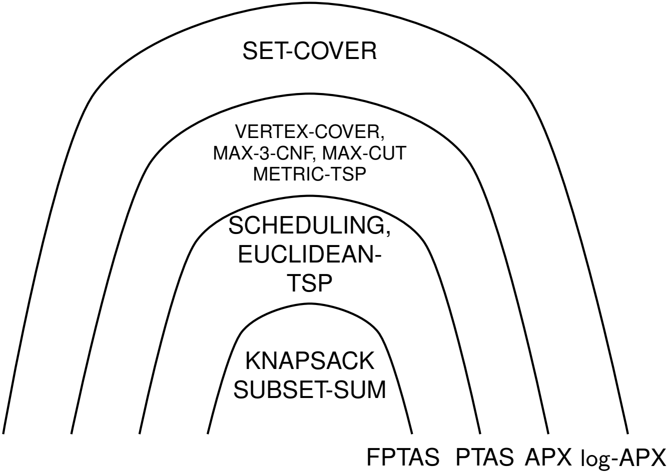

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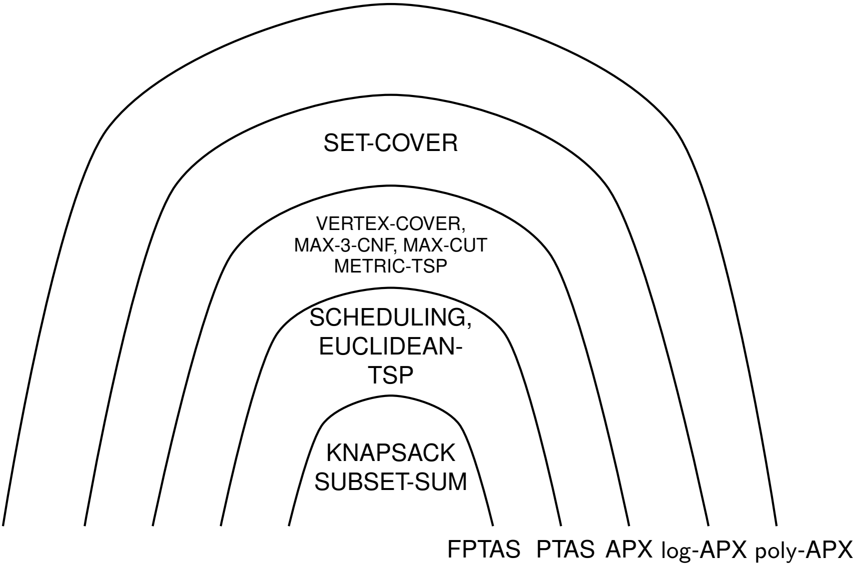
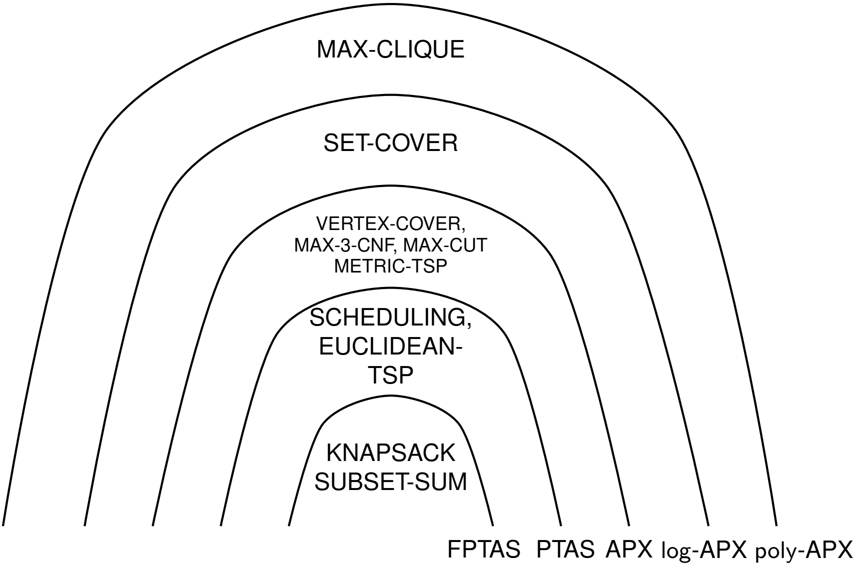


Spectrum of Approximations



Topics Covered

I. Sorting and Counting Networks
- 0/1-Sorting Principle, Bitonic Sorting, Batcher's Sorting Network

  Bonus Material: A Glimpse at the AKS network
- Balancing Networks, Counting Network Construction, Counting vs. Sorting

II. Linear Programming
- Geometry of Linear Programs, Applications of Linear Programming
- Simplex Algorithm, Finding a Feasible Initial Solution
- Fundamental Theorem of Linear Programming

III. Approximation Algorithms: Covering Problems
- Intro to Approximation Algorithms, Definition of PTAS and FPTAS
- (Unweighted) Vertex-Cover: 2-approx. based on Greedy
- (Unweighted) Set-Cover: $O(\log n)$-approx. based on Greedy

IV. Approximation Algorithms via Exact Algorithms
- Subset-Sum: FPTAS based on Trimming and Dynamic Programming
- Scheduling: 2-approx. based on Simple Greedy, $4/3$-approx. using LPT

  Bonus Material: A PTAS for Machine Scheduling based on Rounding and Dynamic Programming

V. The Travelling Salesman Problem
- Inapproximability of the General TSP problem
- Metric TSP: 2-approx. based on MST, $3/2$-approx. based on MST + matching

VI. Approximation Algorithms: Rounding and Randomisation
- MAX3-CNF: $8/7$-approx. based on Guessing, Derandomisation with Greedy
- (Weighted) Vertex-Cover: 2-approx. based on Deterministic Rounding
- (Weighted) Set-Cover: $O(\log n)$-approx. based on Randomised Rounding
- MAX-CNF: $4/3$-approx. based on Guessing + Randomised Rounding

Thank you and Best Wishes for the Exam!