Space Complexity

We've already seen the definition $\mathsf{SPACE}(f(n))$: the languages accepted by a machine which uses $O(f(n))$ tape cells on inputs of length n. *Counting only work space*

$\mathsf{NSPACE}(f(n))$ is the class of languages accepted by a *nondeterministic* Turing machine using at most $f(n)$ work space.

As we are only counting work space, it makes sense to consider bounding functions f that are less than linear.

Classes

$\mathsf{L} = \mathsf{SPACE}(\log n)$

$\mathsf{NL} = \mathsf{NSPACE}(\log n)$

$\mathsf{PSPACE} = \bigcup_{k=1}^{\infty} \mathsf{SPACE}(n^k)$

The class of languages decidable in polynomial space.

$\mathsf{NPSPACE} = \bigcup_{k=1}^{\infty} \mathsf{NSPACE}(n^k)$

Also, define

co-NL – the languages whose complements are in NL.

co-NPSPACE – the languages whose complements are in NPSPACE.

Inclusions

We have the following inclusions:

$$\mathsf{L} \subseteq \mathsf{NL} \subseteq \mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSPACE} \subseteq \mathsf{NPSPACE} \subseteq \mathsf{EXP}$$

where $\mathsf{EXP} = \bigcup_{k=1}^{\infty} \mathsf{TIME}(2^{n^k})$

Moreover,

$$\mathsf{L} \subseteq \mathsf{NL} \cap \text{co-}\mathsf{NL}$$

$$\mathsf{P} \subseteq \mathsf{NP} \cap \text{co-}\mathsf{NP}$$

$$\mathsf{PSPACE} \subseteq \mathsf{NPSPACE} \cap \text{co-}\mathsf{NPSPACE}$$

Establishing Inclusions

To establish the known inclusions between the main complexity classes, we prove the following.

- $\mathsf{SPACE}(f(n)) \subseteq \mathsf{NSPACE}(f(n))$;
- $\mathsf{TIME}(f(n)) \subseteq \mathsf{NTIME}(f(n))$;
- $\mathsf{NTIME}(f(n)) \subseteq \mathsf{SPACE}(f(n))$;
- $\mathsf{NSPACE}(f(n)) \subseteq \mathsf{TIME}(k^{\log n + f(n)})$;

The first two are straightforward from definitions.

The third is an easy simulation.

The last requires some more work.

Reachability

Recall the Reachability problem: given a *directed* graph $G = (V, E)$ and two nodes $a, b \in V$, determine whether there is a path from a to b in G.

A simple search algorithm solves it:

1. mark node a, leaving other nodes unmarked, and initialise set S to $\{a\}$;
2. while S is not empty, choose node i in S: remove i from S and for all j such that there is an edge (i, j) and j is unmarked, mark j and add j to S;
3. if b is marked, accept else reject.

NL Reachability

We can construct an algorithm to show that the Reachability problem is in NL:

1. write the index of node a in the work space;
2. if i is the index currently written on the work space:
   (a) if $i = b$ then accept, else
   guess an index j ($\log n$ bits) and write it on the work space.
   (b) if (i, j) is not an edge, reject, else replace i by j and return to (2).

We can use the $O(n^2)$ algorithm for Reachability to show that:

$\mathsf{NSPACE}(f(n)) \subseteq \mathsf{TIME}(k^{\log n + f(n)})$

for some constant k.

Let M be a nondeterministic machine working in space bounds $f(n)$.

For any input x of length n, there is a constant c (depending on the number of states and alphabet of M) such that the total number of possible configurations of M within space bounds $f(n)$ is bounded by $n \cdot c^{f(n)}$.

Here, $c^{f(n)}$ represents the number of different possible contents of the work space, and n different head positions on the input.

Configuration Graph

Define the *configuration graph* of M, x to be the graph whose nodes are the possible configurations, and there is an edge from i to j if, and only if, $i \rightarrow_M j$.

Then, M accepts x if, and only if, some accepting configuration is reachable from the starting configuration $(s, \triangleright, x, \triangleright, \varepsilon)$ in the configuration graph of M, x.

Using the $O(n^2)$ algorithm for Reachability, we get that M can be simulated by a deterministic machine operating in time

$$c'(nc^{f(n)})^2 = c'c^{2(\log n + f(n))} = k^{(\log n + f(n))}$$

In particular, this establishes that NL $\subseteq$ P and NPSPACE $\subseteq$ EXP.

Savitch's Theorem

Further simulation results for nondeterministic space are obtained by other algorithms for Reachability.

We can show that Reachability can be solved by a *deterministic* algorithm in $O((\log n)^2)$ space.

Consider the following recursive algorithm for determining whether there is a path from a to b of length at most n (for n a power of 2):

$O((\log n)^2)$ space Reachability algorithm:

Path(a, b, i)

if $i = 1$ and (a, b) is not an edge reject
else if (a, b) is an edge or $a = b$ accept
else, for each node x, check:

1. is there a path $a - x$ of length $i/2$; and
2. is there a path $x - b$ of length $i/2$?

if such an x is found, then accept, else reject.

The maximum depth of recursion is $\log n$, and the number of bits of information kept at each stage is $3 \log n$.

Savitch's Theorem - 2

The space efficient algorithm for reachability used on the configuration graph of a nondeterministic machine shows:

$$\mathsf{NSPACE}(f(n)) \subseteq \mathsf{SPACE}(f(n)^2)$$

for $f(n) \geq \log n$.

This yields

$$\mathsf{PSPACE} = \mathsf{NPSPACE} = \text{co-}\mathsf{NPSPACE}.$$

Complementation

A still more clever algorithm for Reachability has been used to show that nondeterministic space classes are closed under complementation:

If $f(n) \geq \log n$, then

$$\mathsf{NSPACE}(f(n)) = \mathsf{co\text{-}NSPACE}(f(n))$$

In particular

$$\mathsf{NL} = \mathsf{co\text{-}NL}.$$