Mathematical Methods for Computer Science

UNIVERSITY OF CAMBRIDGE

Computer Laboratory

Computer Science Tripos, Part IB

Michaelmas Term 2005

R. J. Gibbens

Problem sheet (last revised 27 Nov 2005)
Part (b): Markov chains

William Gates Building
15 JJ Thomson Avenue
Cambridge
CB3 0FD

http://www.cl.cam.ac.uk/

Markov chains

1. Suppose that (X_n) is a Markov chain with n-step transition matrix, $P^{(n)}$, and let $\lambda_i^{(n)} = \mathbb{P}(X_n = i)$ be the elements of a row vector $\lambda^{(n)}$ $(n = 0, 1, 2, \ldots)$. Show that

   (a) $P^{(m+n)} = P^{(m)}P^{(n)}$ for $m, n = 0, 1, 2, \ldots$

   (b) $\lambda^{(n)} = \lambda^{(0)}P^{(n)}$ for $n = 0, 1, 2, \ldots$.

2. Suppose that (X_n) is a Markov chain with transition matrix P. Define the relations *"state j is accessible from state i"* and *"states i and j communicate"*. Show that the second relation is an equivalence relation and define the communicating classes as the equivalence classes under this relation. What is meant by the terms *closed class*, *absorbing class* and *irreducible*?

3. Show that
$$P_{ij}(z) = \delta_{ij} + F_{ij}(z)P_{jj}(z)$$
where
$$P_{ij}(z) = \sum_{n=0}^{\infty} p_{ij}^{(n)} z^n, \qquad F_{ij}(z) = \sum_{n=0}^{\infty} f_{ij}^{(n)} z^n$$
and $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. [You should assume that $p_{ij}^{(n)}$ and $f_{ij}^{(n)}$ are as defined in lectures with $p_{ij}^{(0)} = \delta_{ij}$ and $f_{ij}^{(0)} = 0$ for all states i, j.] The original version of this question contained a typo in the first equation.

4. Suppose that (X_n) is a finite state Markov chain and that for some state i and for all states j
$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$$
for some collection of numbers (π_j). Show that $\pi = (\pi_j)$ is a stationary distribution.

5. Consider the Markov chain with transition matrix
$$P = \begin{pmatrix} 0.128 & 0.872 \\ 0.663 & 0.337 \end{pmatrix}$$
for Markov's example of a chain on the two states $\{\text{vowel}, \text{consonant}\}$ for consecutive letters in a passage of text. Find the stationary distribution for this Markov chain. What are the mean recurrence times for the two states?

6. Define what is meant by saying that (X_n) is a reversible Markov chain and write down the local balance conditions. Show that if a vector π is a distribution over the states of the Markov chain that satisfies the local balance conditions then it is a stationary distribution.

7. Consider the Erhenfest model for m balls moving between two containers with transition matrix
$$p_{i,i+1} = 1 - \frac{i}{m}, \qquad p_{i,i-1} = \frac{i}{m}$$
where i $(0 \le i \le m)$ is the number of balls in a given container. Show that the Markov chain is irreducible and periodic with period 2. Derive the stationary distribution.

8. Consider a random walk, (X_n), on the states $i = 0, 1, 2, \ldots$ with transition matrix

$$\begin{aligned} p_{i,i-1} &= p \qquad i = 1, 2, \ldots \\ p_{i,i+1} &= 1 - p \qquad i = 0, 1, \ldots \\ p_{0,0} &= p \end{aligned}$$

where $0 < p < 1$. Show that the Markov chain is irreducible and aperiodic. Find a condition on p to make the Markov chain positive recurrent and find the stationary distribution in this case.

9. Describe PageRank as a Markov chain model for the motion between nodes in a graph. Explain the main mathematical results that underpin PageRank's connection to a notion of web page *"importance"*.