$$b_1 b_2 b_3 b_4 b_5 b_6 b_7$$
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$$b_1 b_2 b_3 b_4 b_5 b_6 b_7$$

Then considering the information capacity per symbol:

$$C = \max(H(Y) - H(Y|X))$$

$$= \frac{1}{7} \left( 7 - \sum_j \sum_k p(y_k | x_j) \log \left( \frac{1}{p(y_k | x_j)} \right) p(x_j) \right)$$

$$= \frac{1}{7} \left( 7 + \sum_j 8 \left( \frac{1}{8} \log \frac{1}{8} \frac{1}{N} \right) \right)$$

$$= \frac{1}{7} \left( 7 + N \left( \frac{8}{8} \log \frac{1}{8} \frac{1}{N} \right) \right)$$

$$= \frac{4}{7}$$

The capacity of the channel is $4/7$ information bits per binary digit of the channel coding. Can we find a mechanism to encode 4 information bits in 7 channel bits subject to the error property described above?

The (7/4) Hamming Code provides a systematic code to perform this - a systematic code is one in which the obvious binary encoding of the source symbols is present in the channel encoded form. For our source which emits at each time interval 1 of 16 symbols, we take the binary representation of this and copy it to bits $b_3, b_5, b_6$ and $b_7$ of the encoded block; the remaining bits are given by $b_4, b_2, b_1$, and syndromes by $s_4, s_2, s_1$:

$$b_4 = b_5 \oplus b_6 \oplus b_7 \text{ and,}$$
$$s_4 = b_4 \oplus b_5 \oplus b_6 \oplus b_7$$
$$b_2 = b_3 \oplus b_6 \oplus b_7 \text{ and,}$$
$$s_2 = b_2 \oplus b_3 \oplus b_6 \oplus b_7$$
$$b_1 = b_3 \oplus b_5 \oplus b_7 \text{ and,}$$
$$s_1 = b_1 \oplus b_3 \oplus b_5 \oplus b_7$$

On reception if the binary number $s_4 s_2 s_1 = 0$ then there is no error, else $s_4 s_2 s_1$ is the bit in error.

This Hamming code uses 3 bits to correct 7 ($= 2^3 - 1$) error patterns and transfer 4 useful bits. In general a Hamming code uses $m$ bits to correct $2^m - 1$ error patterns and transfer $2^m - 1 - m$ useful bits. The Hamming codes are called perfect as they use $m$ bits to correct $2^m - 1$ errors.