Advanced Graphics 2006

- Subdivision curves & surfaces

Beware: some slides contain multi-layer animations, which do not print well.

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Modelling smooth 3D surfaces

- Where are smooth 3D surfaces used?
  - Computer Aided Design (CAD)
    - First developed for cars & aeroplanes
    - Adopted for other manufactured objects
  - Computer animation
- What mechanisms exist?
  - Bézier patches
  - NURBS surfaces
  - Subdivision surfaces

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Desirable features

- Need to handle *any* surface
- Need guaranteed continuity
  - C1-continuity
    - Smooth surfaces
  - C2-continuity
    - Smoothly reflecting surfaces
    - Required for some aerodynamics
- Need to allow discontinuities
  - Edges, creases and holes
- Needs to be easy to use

History of 3D modelling 1/3

- Some mechanism was needed for modelling 3D surfaces
- Hermite interpolation was generalised to bivariate patches
  - ...but proved too difficult to use in practice
- Bézier patches
  - Developed for car design around 1960
    - Bézier (Renault), de Casteljau (Citreön), de Boor (GM)

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History of 3D modelling 2/3

- B-spline theory
  - Developed in the 1960s and ’70s, led to:
- NURBS (Non-Uniform Rational B-Splines)
  - More general than Bézier patches
    - Béziers are special cases of NURBS
  - NURBS quickly became the industry standard in CAD
    - ...and remain the industry standard today
  - Adopted by the computer animation industry when it began

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History of 3D modelling 3/3

- Subdivision surfaces
  - Theory developed in 1970s and early ’80s
  - Picked up by computer animation industry in late 1990s
  - Now replaced NURBS in computer animation
    - Solves one of the big problems of NURBS
  - Still under active research for use in CAD
    - Introduces new problems, not present in NURBS, which make it unsuitable for CAD in its present form

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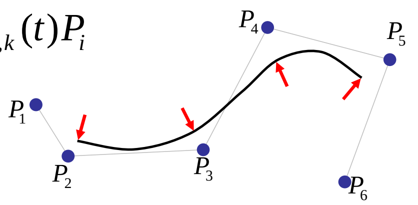
NURBS curve

- A curve is defined parametrically
- Its shape is determined by:
  - control points, P_i
  - and the NURBS basis functions, $N_{i,k}$

$$P(t) = \sum_{i=1}^{n+1} N_{i,k}(t) P_i$$



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Basic properties of NURBS 1/3

$$P(t) = \sum_{i=1}^{n+1} N_{i,k}(t) P_i$$

- The basis functions must sum to 1 to produce a valid new point

$$\sum_{i=1}^{n+1} N_{i,k}(t) = 1, \; t_{\min} \le t \le t_{\max}$$

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Basic properties of NURBS 2/3

$$P(t) = \sum_{i=1}^{n+1} N_{i,k}(t) P_i$$

- The basis functions are calculated from a *knot vector*
  - Just a non-decreasing sequence of real numbers
    - e.g. [0,0,0,1,1,1] or [1,2,3,4,5,6] or [1.2, 3.4, 5.6, 5.6, 7.2, 15.6]
  - See lecture notes or Rogers & Adams for details

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Basic properties of NURBS 3/3

$$P(t) = \sum_{i=1}^{n+1} N_{i,k}(t) P_i$$

- If the basis functions are Cm-continuous at t, then $P(t)$ is guaranteed to be Cm-continuous at t
  - So continuity depends only on the basis functions, $N_{i,k}$
  - Continuity does *not* depend on the locations of the control points
    - you can sometimes get extra continuity by careful positioning of control points

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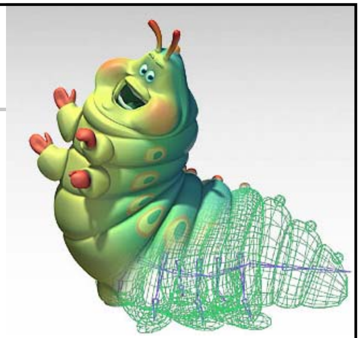
NURBS surface



- A bivariate generalisation of the univariate NURBS curve

Curve $P(t) = \sum_{i=1}^{n+1} N_{i,k}(t) P_i$

Surface $P(s,t) = \sum_{i=1}^{m+1} \sum_{j=1}^{n+1} N_{i,k}(s) N_{j,k}(t) P_{i,j}$

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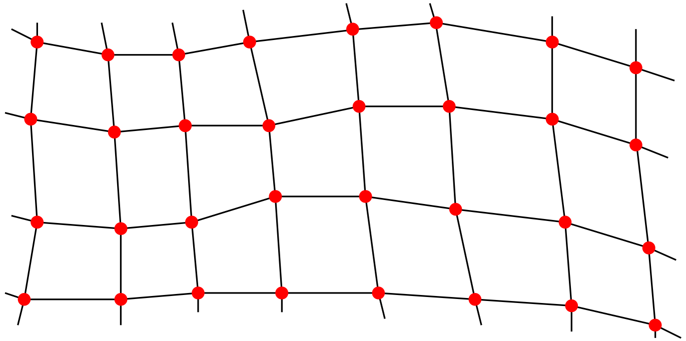
The big constraint...

- NURBS surfaces require a quadrilateral mesh of $(m+1)\times(n+1)$ points



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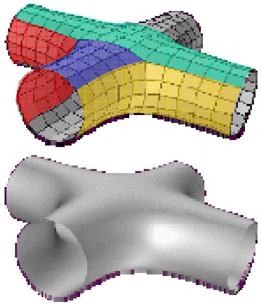
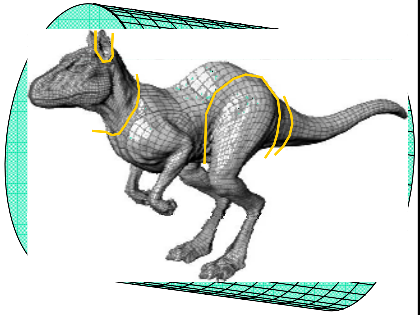
The first problem

- Very few objects are made up of a single rectangular patch, so we need to join patches together





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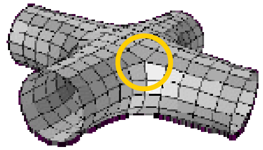
The second problem

- What do we do at special points where other than four patches meet?



  - Either we cannot get C2
    - Which means that curvature is not continuous
  - Or we get C2 be forcing curvature to be zero
    - Which produces a flat spot
  - Or we get C2 using very high degree patches
    - Which are very hard for a designer to control

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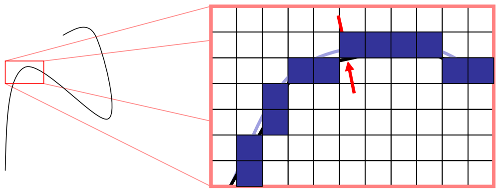
Drawing a NURBS curve

- NURBS curves and surfaces are always drawn on a pixelated surface
- NURBS curves can be approximated by straight lines
  - So long as each straight line deviates from the curve by less than half a pixel



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Drawing a NURBS surface

- NURBS surfaces are sub-divided and drawn as a series of planar polygons
- Each polygon is only one or two pixels in area on the screen
- Shading algorithms are used to ensure that the surfaces appear to be smoothly curved

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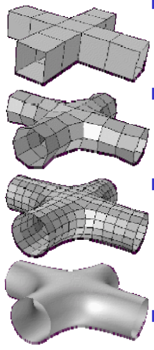
Subdivision surfaces



- Do away with the explicit parametric representation
- Base a curve or surface solely on its control points and their connectivity
- Provide a simple mechanism which produces a larger, more refined set of control points from the current set
- Iterate refinement until the appropriate level of detail is achieved

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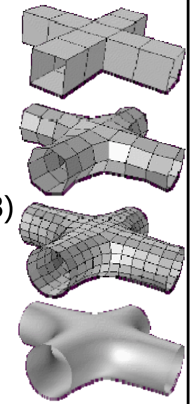
History of subdivision schemes



- A univariate (curve) scheme was described by de Rahm in 1947
- Rediscovered by Chaikin in 1974
- Extended to bivariate (surfaces)
  - Doo-Sabin bi-quadratic patches (1978)
  - Catmull-Clark bi-cubic patches (1978)
- Flurry of mathematical work in the early 1980s
  - Dyn & Levin at Tel Aviv University

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Use of subdivision schemes

- Pixar picked up the ideas and tested them in Geri's Game (1997)
- ...then discarded its NURBS based software in favour of subdivision schemes

- NURBS
  - Toy Story 1995
  - A Bug's Life 1998
- Subdivision surfaces
  - Toy Story II 1999
  - Monsters Inc. 2001
  - Finding Nemo 2003

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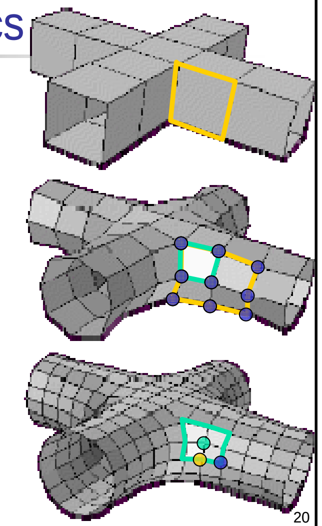
Subdivision basics



- An example: Catmull-Clark subdivision
  - Introduce new points
    - At face-centres
    - At mid-edges
  - Adjust positions of original points
  - Repeat until sufficiently detailed

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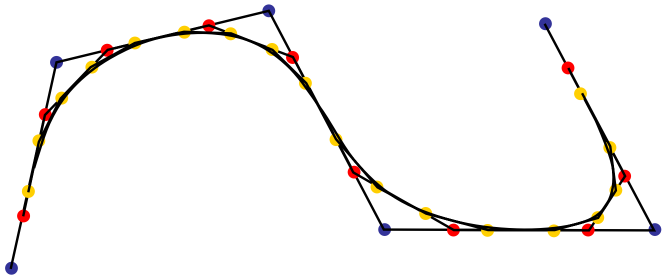
Chaikin curve subdivision

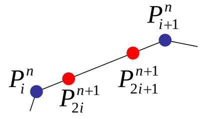
- Underlies Doo-Sabin surface subdivision
- C1-continuous in the limit
- Essentially just a ¼-¾ rule



The maths of Chaikin

$$P_{2i}^{n+1} = \frac{3}{4} P_i^n + \frac{1}{4} P_{i+1}^n$$

$$P_{2i+1}^{n+1} = \frac{1}{4} P_i^n + \frac{3}{4} P_{i+1}^n$$



$$h = [\ldots,0,0,\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{1}{4},0,0,\ldots]$$

$$P^n = [\ldots, P_0^n, P_1^n, P_2^n, \ldots]$$

$$\overleftrightarrow{P}^n = [\ldots, P_0^n, 0, P_1^n, 0, P_2^n, 0, \ldots]$$

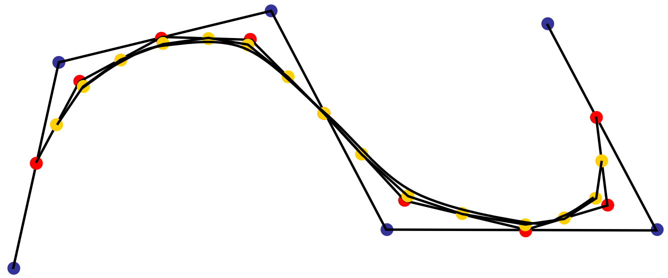
$$P^{n+1} = h * \overleftrightarrow{P}^n$$

The limit curve

- It can be shown that the limit curve of the Chaikin scheme is the uniform quadratic B-spline, which is guaranteed to be C1
- When drawing curves in computer graphics, we draw a set of straight lines, so only need to subdivide until each segment is about a pixel long and we have a good enough approximation to the curve

C2 approximating scheme

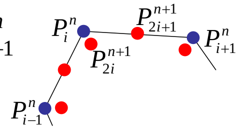
- Underlies Catmull-Clark surface subdivision
- Can be described as: “Insert a midpoint and adjust the old control points”



The maths of the C2 scheme

$$P_{2i}^{n+1} = \boxed{\tfrac{1}{8}} P_{i-1}^{n} + \boxed{\tfrac{6}{8}} P_{i}^{n} + \boxed{\tfrac{1}{8}} P_{i+1}^{n}$$

$$P_{2i+1}^{n+1} = \boxed{\tfrac{4}{8}} P_{i}^{n} + \boxed{\tfrac{4}{8}} P_{i+1}^{n}$$



$$\boxed{h = [\ldots, 0, 0, \tfrac{1}{8}, \tfrac{4}{8}, \tfrac{6}{8}, \tfrac{4}{8}, \tfrac{1}{8}, 0, 0, \ldots]}$$

$$P^n = [\ldots, P_0^n, P_1^n, P_2^n, \ldots]$$

$$\overleftrightarrow{P^n} = [\ldots, P_0^n, 0, P_1^n, 0, P_2^n, 0, \ldots]$$

$$P^{n+1} = h * \overleftrightarrow{P^n}$$

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Why this notation?

- Easy to analyse
- Allows use of the z-transform

$$h = [\ldots, h_0, h_1, h_2, \ldots]$$
$$\Downarrow$$
$$h(z) = \ldots + h_0 z^0 + h_1 z^1 + h_2 z^2 + \ldots$$

vector
$\Downarrow$
polynomial

$$P^{n+1} = h * \overleftrightarrow{P^n}$$
$$\Downarrow$$
$$P^{n+1}(z) = h(z) \times P^n(z^2)$$

convolution
$\Downarrow$
multiplication

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The analysis tools

- Generating function formalism
  - Use the z-transform on the kernel, h
  - Provides sufficient conditions for continuity
    - Essentially checks that the differences between adjacent points decrease fast enough at each refinement step to produce a smooth curve
- There is also a matrix formalism
  - Analyse stationary points
  - Provides necessary conditions for continuity
- For details see our research papers ☺

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Useful subdivision kernels

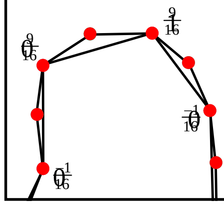
$$h = \tfrac{1}{4}[1,3,3,1]$$

- C1, approximating, limit curve is quadratic B-spline

$$h = \tfrac{1}{8}[1,4,6,4,1]$$

- C2, approximating, limit curve is cubic B-spline

$$h = \tfrac{1}{16}[-1,0,9,16,9,0,-1]$$



- C1, interpolating, "four-point scheme"
- There is also a C2 interpolating six-point scheme

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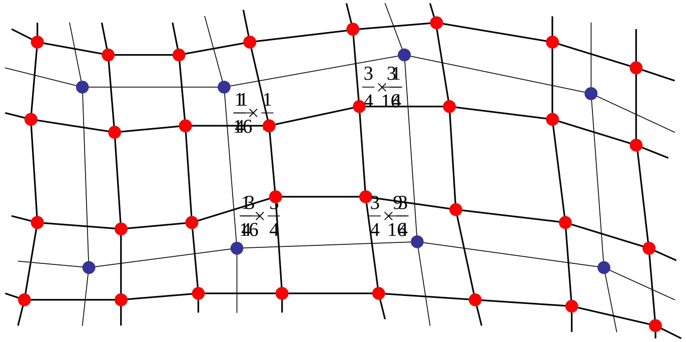
From Chaikin to Doo-Sabin

- Doo-Sabin scheme is bivariate generalisation of Chaikin ¼-¾ scheme



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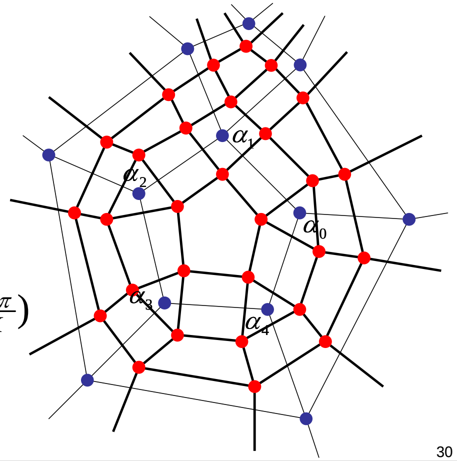
Extraordinary polygons

- Need special co-efficients for these

$$\alpha_0 = \frac{1}{4} + \frac{5}{4K}$$

$$\alpha_i = \frac{1}{4K}\left(3 + 2\cos\frac{2i\pi}{K}\right)$$

(Doo-Sabin)



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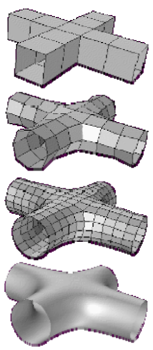
Catmull-Clark subdivision



- Catmull-Clark is based on the $^1/_8$[1,4,6,4,1] univariate scheme

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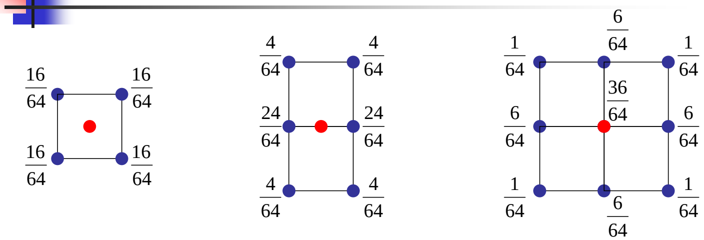
Catmull-Clark rules



face **edge** **vertex**

- This is easy: the rules are simply the tensor product of the univariate $^1/_8$[1,4,6,4,1] rules.

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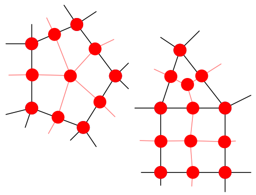
Catmull-Clark special cases

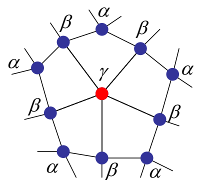
- This is more difficult: we need to find co-efficients which maintain continuity
  - It is only possible to get C1 continuity at these extraordinary points.



$$\alpha = \frac{1}{4n^2}$$

$$\beta = \frac{3}{2n^2}$$

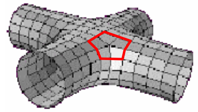
$$\gamma = 1 - \frac{7}{4n}$$



Extraordinary polygons: disappear after one step

Extraordinary vertices: remain in the mesh

Subdivision *vs* NURBS



- Extraordinary points
  - Subdivision handles them easily
  - NURBS requires the use of other types of surface to fill in the holes
- Memory requirements
  - Subdivision needs a lot (many MB)
  - NURBS is very compact
- Artifacts
  - Some artifacts present in both
  - Subdivision has extra artifacts

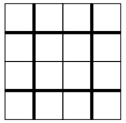
The future

- Computers now have enough memory to handle subdivision easily
- Subdivision now standard for computer animation
- NURBS still standard for CAD
- Subdivision will eventually replaced NURBS for CAD if we can sort out the artifact problems

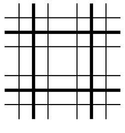
Our work at Cambridge

- Univariate schemes that are not binary
  - Ternary (×3) schemes
  - Sesquiary (×1½) schemes
- Towards a bestiary of bivariate schemes
  - Classification & analysis of all schemes
  - Identification & analysis of new schemes (especially ternary)
- Geometrically-sensitive subdivision
  - Modifying existing schemes to take account of geometric relationships

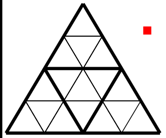
Principal subdivision schemes



- Catmull-Clark
  - C2, approximating
    - only C1 at extra-ordinary points
- Kobbelt (four-point)
  - C1, interpolating



- Doo-Sabin
  - C1, approximating



- Loop
  - C2, approximating
    - only C1 at extra-ordinary points
- Butterfly
  - C1, interpolating

These are the five subdivision schemes which were thought to be the only useful ones

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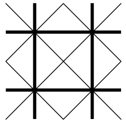
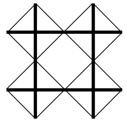
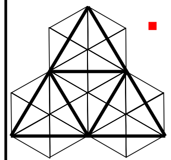
√2 and √3 schemes



- Velho-Zorin
  - C4, approximating
    - only C1 at extraordinary points



- Reif-Peters
  - C1, approximating
  - The simplest possible scheme: there are no special cases!



- √3 (Kobbelt)
  - C2, approximating
    - only C1 at extraordinary points

These were discovered in the late 1990s

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