Cantor's diagonal argument revisited

Let $X$ be a set. There is no bijection $\Theta : X \rightarrow \mathcal{P}(X)$.

Proof by contradiction. Suppose $\Theta : X \rightarrow \mathcal{P}(X)$ is a bijection.

Define $Y = \{ x \in X \mid x \notin \Theta(x) \}$.

As $\Theta$ is a bijection, there is $y \in X$ such that $\Theta(y) = Y$. But ...
Ch. 4 Inductive definitions

Where induction principles come from

Rule induction
for rules
\[
\begin{array}{c}
\vdots \\
y \vdash x \\
\end{array}
\]
Boolean propositions from rules

\[ A, B, \ldots ::= a, b, c, \ldots \mid \text{true} \mid \text{false} \mid A \land B \mid A \lor B \mid \lnot A \]
\[
a, b, c, \ldots \in \text{Var}
\]

\[ \frac{a \in \text{Var}}{a} \quad \frac{\text{true}}{\text{false}} \]

\[ \frac{A \quad B}{A \land B} \quad \frac{A \quad B}{A \lor B} \quad \frac{A}{\lnot A} \]

A derivation:

\[ \frac{a}{\lnot a} \quad \frac{b \quad \text{true}}{b \lor \text{true}} \]

\[ \frac{\lnot a \land (b \lor \text{true})}{\lnot a \land (b \lor \text{true})} \]
Non-negative integers $\mathbb{N}_0$ from rules

- $0 \in \mathbb{N}_0$
- If $n \in \mathbb{N}_0$, then $n+1 \in \mathbb{N}_0$

\[
\begin{array}{c}
0 \\
\hline
n \\
\hline
n + 1
\end{array}
\]

$\mathbb{N}_0$ is the least set closed under the rules.
Strings $\Sigma^*$

$\Sigma$ is a set of symbols, the alphabet

empty string $\varepsilon \in \Sigma^*$

concatenation

If $x \in \Sigma^*$ and $a \in \Sigma$, then $ax \in \Sigma^*$

\[
\begin{array}{c}
\varepsilon \\
\hline
a \\
\hline
ax \\
\hline
a \in \Sigma
\end{array}
\]
An instance of a rule:

\[ x_1, x_2, \ldots, x_i, \ldots \xrightarrow{*} \text{premise} \]

\[ y \text{ conclusion} \]

A pair \((X/y)\) where

\[ X = \{x_i, x_2, \ldots, x_i, \ldots\}. \]

When \(X\) is finite, the rule is finitary.

NB: Can have \(X = \emptyset\).
A set $Q$ is $R$-closed iff
\[ \forall (x/y) \in R, \ x \in Q \Rightarrow y \in Q. \]

Define a set inductively defined by $R$:
\[ I_R = \bigcap \{ Q \mid Q \text{ is } R\text{-closed} \} \]

**Proposition 4.3 P.55**

(i) $I_R$ is $R$-closed.

(ii) $\forall Q$ is $R$-closed $\Rightarrow I_R \subseteq Q$.

**Rule induction:**
\[
\forall x \in I_R, \ P(x) \quad \text{if for all rules } (X/y) \in R, \ s.t. \ X \subseteq I_R
\]
\[
(\forall x \in X, \ P(x)) \Rightarrow P(y).
\]
Transitive closure of a relation

Let $R \subseteq U \times U$.

Its transitive closure $R^+ \subseteq U \times U$ is given by:

$$(a, b) \in R \quad (a, b) \in R \quad (b, c) \in R \quad \Rightarrow \quad (a, c) \in R^+$$

$$R^+ = \left\{ (a, b) \in U \times U \mid \text{there is an } R\text{-chain from } a \text{ to } b \right\}$$

\[ a = a, R a_2, R a_3, \ldots, R a_n = b \]