Complexity Theory
Lent 2004
Suggested Exercises 4

1. On page 10 of the notes, a number of functions are listed as being constructible. Show that this is the case by giving, for each one, a description of an appropriate Turing machine.

   Prove that if f and g are constructible functions and $f(n) \geq n$, then so are $f(g)$, $f + g$, $f \cdot g$ and 2^f.

2. For any constructible function f, and any language $L \in \mathsf{NTIME}(f(n))$, there is a nondeterministic machine M that accepts L and all of whose computations terminate in time $O(f(n))$ for all inputs of length n. Give a detailed argument for this statement, describing how M might be obtained from a machine accepting L in time $f(n)$.

3. In the lecture, a proof of the Time Hierarchy Theorem was sketched. Give a similar argument for the following Space Hierarchy Theorem:

   > **Space Hierarchy.** For every constructible function f, there is a language in $\mathsf{S}PACE(f(n) \cdot \log f(n))$ that is not in $\mathsf{S}PACE(f(n))$.

4. Show that, if $\mathsf{SPACE}((\log n)^2) \subseteq \mathsf{P}$, then $\mathsf{L} \neq \mathsf{P}$. (Hint: use the Space Hierarchy Theorem from Exercise 3.)

5. $\mathsf{POLYLOGSPACE}$ is the complexity class

   $$\bigcup_k \mathsf{SPACE}((\log n)^k).$$

   (a) Show that, for any k, if $A \in \mathsf{SPACE}((\log n)^k)$ and $B \leq_L A$, then $B \in \mathsf{SPACE}((\log n)^k)$.

   (b) Show that there are no $\mathsf{POLYLOGSPACE}$-complete problems with respect to $\leq_L$. (Hint: use (a) and the space hierarchy theorem).

   (c) Which of the following might be true: $\mathsf{P} \subseteq \mathsf{POLYLOGSPACE}$, $\mathsf{P} \supseteq \mathsf{POLYLOGSPACE}$, $\mathsf{P} = \mathsf{POLYLOGSPACE}$?