Channel Capacity, Sampling Theory and Image, Video & Audio Compression exercises

all numbered exercises are by Cover and Thomas except where noted otherwise

Exercise 8.2:
Maximum likelihood decoding. A source produces independent, equally probable symbols from an alphabet $\{a, b\}$ at a rate of one symbol every 3 seconds. These symbols are transmitted over a binary symmetric channel which is used once each second by encoding the source symbol a as 000 and the source symbol b as 111. If, in the corresponding 3 second interval of the channel output, any of the sequences 000, 001, 010, 100 is received then the output is decoded as a; otherwise the output is decoded as b. Let $\epsilon < \frac{1}{2}$ be the channel crossover probability.

(a) For each possible received 3-bit sequence in the interval corresponding to a given source letter, find the probability that a came out of the source given that received sequence.

(b) Using part (a), show that the above decoding rule minimizes the probability of an incorrect decision.

(c) Find the probability of an incorrect decision.

(d) The source is slowed down to produce one letter every $2n + 1$ seconds, a being encoded by $2n + 1$ "0"s and b being encoded by $2n + 1$ "1"s. What decision rule minimizes the probability of error at the decoder? What is the probability of error as $n \to \infty$? What is the transmission rate as $n \to \infty$?

Exercise 8.3:
An additive noise channel. A channel has additive noise, Z, such that the output Y depends on the input X and the noise Z by the rule $Y = X + Z$. The alphabet for x is $\{0, 1\}$. The alphabet for Z is $\{a, b\}$ where $\Pr\{Z = a\} = \Pr\{Z = b\} = \frac{1}{2}$, $a, b, \in \mathbb{Z}$. Assume that Z is independent of X. What is the channel capacity of this discrete memoryless channel? [Hint: the channel capacity depends on the value of $b - a$.]

Exercise 8.9:
The Z channel. The Z channel has binary input and output alphabets $x, y \in \{0, 1\}$ and transition probabilities $p(x|y)$ given by:

	$x = 0$	$x = 1$
$y = 0$	1	0
$y = 1$	$\frac{1}{2}$	$\frac{1}{2}$

Find the input probability distribution which maximises the mutual information and hence determine the channel capacity.

Exercise 8.11:
Zero-error capacity. A channel with alphabet $\{0, 1, 2, 3, 4\}$ has transition probabilities of the form:

$$p(y|x) = \begin{cases} \frac{1}{2}, & y = (x \pm 1) \mathrm{mod} 5 \\ 0, & \text{otherwise} \end{cases}$$

That is: any symbol is equally likely to transition to the symbol before it or the symbol after it. For example "2" could be received as "1" or "3" with equal probability.

(a) Compute the theoretical channel capacity in bits.

(b) The zero-error capacity of a channel is the number of bits per channel use that can be transmitted with zero probability of error. Clearly, the zero-erro capacity of this five-symbol channel is at least one bit (transmit "0" or "1" with probability $\frac{1}{2}$). Find a block code that shows that the zero-error capacity is greater than 1 bit. [Hint: consider codes of length 2.]

(c) Estimate the value of the zero-error capacity as the number of symbols in a block goes to infinity. [This is not something that you are likely to be able to calculate, so give your best guess along with some sort of justification.]

Exercise A [Kuhn]:
JPEG. Which steps of the JPEG (DCT baseline) algorithm cause a loss of information? Distinguish between accidental loss due to rounding errors and information that is removed for a purpose.

Exercise B [Kuhn]:
JPEG. How can you rotate/mirror an already compressed JPEG image without loosing any further information. Why might the resulting JPEG file not have the exact same filelength?

Exercise C [Kuhn]:
Fax encoding. Decompress this G3-fax encoded pixel sequence, which starts with a white-pixel count: 11010010111101111011000011011100110100 [Hint: see page 2 of
`http://www.cl.cam.ac.uk/Teaching/2002/InfoTheory/mgk/additional-slides-4up.pdf`
for the decoding table]

Exercise D [Dodgson]:
Audio & Video encoding. What is it about audio and video data which allows us to use lossy compression for compressing it?

Exercise E [Dodgson]:
Sampling theory. Without using Fourier transforms, show that, for every $\nu_1 > 0$, there exists a ν_2, $0 \leq \nu_2 \leq \nu_b$, such that a sine wave, $y = \sin(2\pi\nu_1 x)$ of frequency ν_1 will produce exactly the same sample values as a sine wave of frequency ν_2 if sampled at the points $\{x = \frac{n}{2\nu_b}, n \in \mathbb{Z}\}$. Determine a formula for ν_2 in terms of ν_1 and ν_b.