Continuous Mathematics

R. J. Gibbens
Computer Laboratory

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Diploma in Computer Science
University of Cambridge

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Overview

- Review of analysis
  - → Limits, continuity and differentiability;
  - → Power series & transcendental functions;
  - → Taylor series;
  - → Complex variables.
- Fourier series
  - → Introduction & general properties;
  - → Examples.
- Orthogonality
  - → Expansion basis functions;
  - → Orthogonality, inner products & completeness.
- Signals and systems
  - → Fourier transforms;
  - → Wavelets.

Course objectives

- Understand representation in terms of basis functions;
- Be fluent in the use and properties of complex variables;
- Grasp key properties and uses of Fourier analysis and its relation to wavelet analysis.

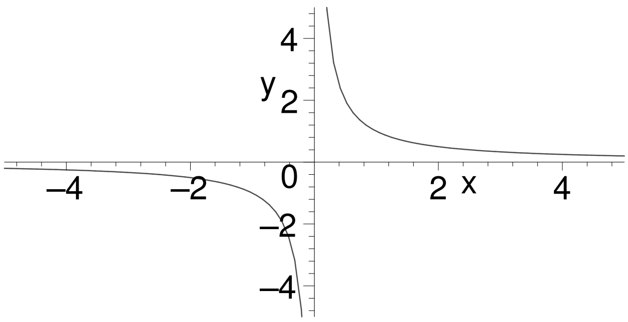
Related courses

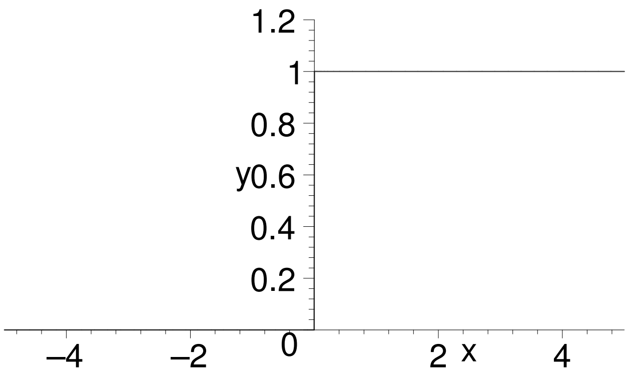
- Computer Graphics & Image Processing;
- Information Theory & Coding.

Limit of a function

If we can make the function $f(x)$ arbitrarily close to ℓ by making x close to a then ℓ is the *limit* of $f(x)$ as $x \to a$, written

$$\boxed{\lim_{x \to a} f(x) = \ell}$$





Rigorous definition: $\forall \epsilon > 0 \; \exists \delta > 0$ such that

$$|f(x) - \ell| < \epsilon \quad \text{when} \quad |x - a| < \delta \,.$$

Continuity

A function, $f(x)$, is said to be *continuous at $x = a$* if the following three conditions hold

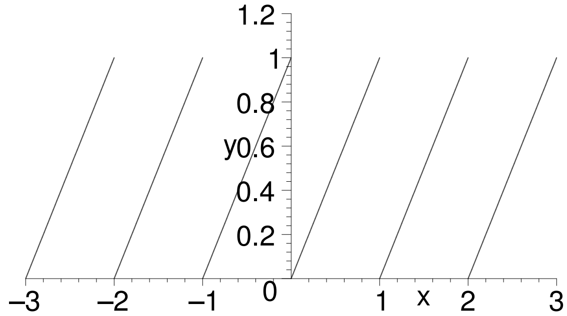
- $\lim_{x \to a} f(x)$ exists;
- $f(x)$ is defined at $x = a$, and
- $\lim_{x \to a} f(x) = f(a)$.

Otherwise, the function is said to be *discontinuous at $x = a$*. A function is *continuous* if it has no points of discontinuity.

At a point of discontinuity, the discontinuity may be *finite* or *infinite*.

Consider the example of

$$f(x) = x - [x] .$$



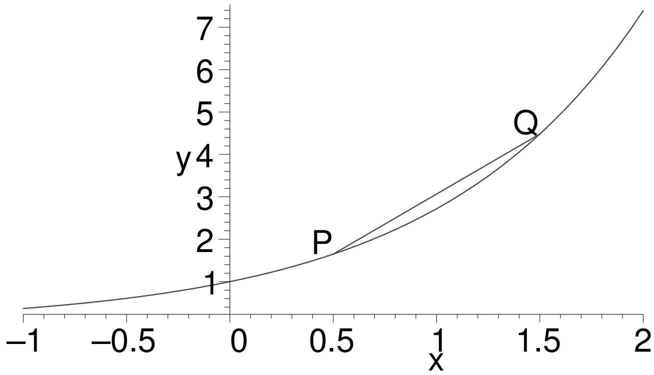
Differentiability

A function, $f(x)$, is *differentiable at x* if
as $\delta x \to 0$

$$\frac{df}{dx} \equiv f'(x) \equiv \frac{f(x+\delta x) - f(x)}{\delta x} \to \text{finite limit}\,.$$

$\frac{df}{dx}$ (or $f'(x)$) is known as the *derivative*. A function is *differentiable* if it's derivative exists everywhere.

The existence of the derivative at a point, x, corresponds to there being a *tangent* to the function, $f(x)$, whose gradient is $f'(x)$.



- product & quotient rules for differentiation;
- differentiability $\Longrightarrow$ continuity.

Convergence of infinite series

If the partial sums of a series

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{r=1}^{n} a_r \, .$$

converge to a finite limit, S, then the infinite series is said to be *convergent*. Otherwise, it is *divergent* (either infinite limit or oscillates).

Example: geometric series

$$\sum_{r=0}^{n} ap^r = S_n = a(1+p+p^2+\cdots+p^n) = a\frac{1-p^{n+1}}{1-p}$$

If $|p| < 1$ then *convergent*
and $S = \lim_{n\to\infty} s_n = \frac{a}{1-p}$. Otherwise, *divergent*.

D'Alembert's ratio test says that

$$\lim_{r\to\infty} |a_{r+1}/a_r| = k < 1 \Longrightarrow \text{convergent}$$
$$\lim_{r\to\infty} |a_{r+1}/a_r| = k > 1 \Longrightarrow \text{divergent}\, .$$

Power series

Power series are infinite series of the form

$$\sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \dots .$$

D'Almbert's ratio test tells us that such power series are convergent if

$$\lim_{r\to\infty} \left| \frac{a_{r+1} x^{r+1}}{a_r x^r} \right| = |x| \lim_{r\to\infty} \left| \frac{a_{r+1}}{a_r} \right| = k < 1 .$$

That is, if x lies in the *interval of convergence*

$$-R < x < R$$

where the *radius of convergence*, R, is given by

$$R = \lim_{r\to\infty} \left| \frac{a_r}{a_{r+1}} \right| .$$

The power series could either converge or diverge if $|x| = R$.

Transcendental functions

The following functions, members of a class of functions known as *transcendental functions*, are defined within their intervals of convergence by these power series representations

$$
\begin{aligned}
\sin x &:= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{all } x \\
\cos x &:= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{all } x \\
\log_e(1+x) &:= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad -1 < x \leq 1 \\
e^x &:= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{all } x \\
\sinh x &:= \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \quad \text{all } x \\
\cosh x &:= \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \quad \text{all } x
\end{aligned}
$$

There are convenient properties for the sum, difference, products & composition of power series.

Taylor series

Subject to some important conditions on $f(x)$

$$f(a+x) = f(a)+\frac{x}{1!}f'(a)+\frac{x^2}{2!}f''(a)+\cdots = \sum_{r=0}^{\infty}\frac{x^r}{r!}f^{(r)}(a)$$

$$f(x) = A_0+A_1(x-a)+A_2(x-a)^2+A_3(x-a)^3+\cdots$$

$$\begin{aligned}
f'(x) &= A_1 + 2A_2(x-a) + 3A_3(x-a)^2 + \cdots \\
f''(x) &= 2A_2 + 3\cdot 2A_3(x-a) + 4\cdot 3A_4(x-a)^2 + \cdots \\
f'''(x) &= 3!A_3 + 4!A_4(x-a) + \cdots \\
f^{(n)}(x) &= n!A_n + (n+1)!A_{n+1}(x-a) + \cdots\,.
\end{aligned}$$

Putting $x = a$ now gives

$$\begin{aligned}
f(a) &= A_0 & f'(a) &= A_1 \\
f''(a) &= 2!A_2 & f'''(a) &= 3!A_3 \\
f^{(n)}(a) &= n!A_n\,.
\end{aligned}$$

Complex variables

Overview

- Imaginary number, i;
- Argand diagram;
- Modulus, argument & complex conjugate;
- Algebra of imaginary numbers;
- Euler's equation:

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Imaginary number, i

We introduce the *imaginary number*, $i = \sqrt{-1}$ so as to solve a wider class of equations. For example,

$$z^2 + 1 = 0 \Longrightarrow z = \pm i$$
$$z^2 - 2z + 2 = 0 \Longrightarrow z = 1 \pm i\,.$$

Note that

$$i^2 = -1\,, \quad i^3 = i^2 i = -i\,, \quad i^4 = (i^2)^2 = 1\,,$$

and

$$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = -i\,, \quad i^{-2} = -1\,, \quad i^{-3} = \frac{1}{i^3} = \frac{1}{-i} = i\,.$$

Complex numbers, z, are written

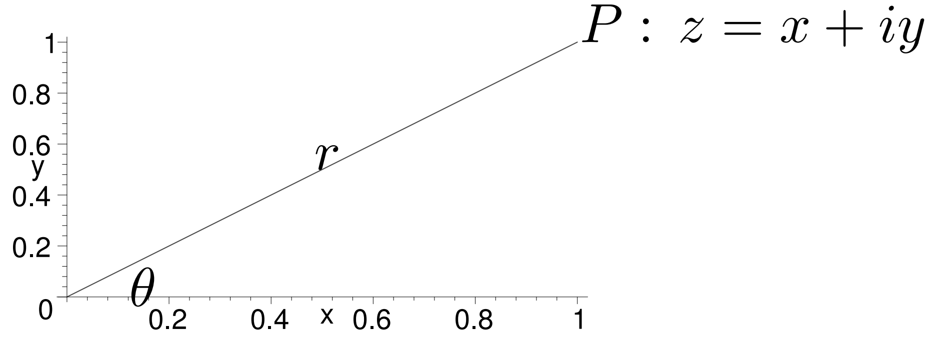
$$z = x + iy \quad \text{or} \quad z = (x, y)$$

Argand diagrams

The idea is to represent any complex number $z = x + iy$ by the point $P : (x, y)$ in the plane, where x is the the *real part* of z and y is the *imaginary part*.

Then using polar coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = x + iy = r(\cos\theta + i\sin\theta)\,.$$



Modulus, argument & complex conjugate

Given

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

define the *modulus* and *argument* of z as

$$|z| = r \qquad \arg(z) = \theta\,.$$

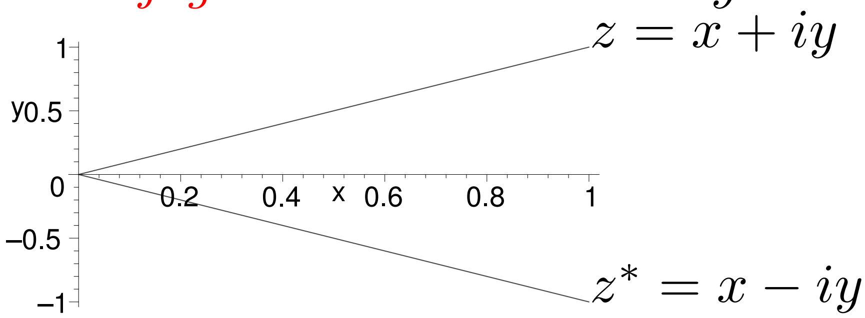
Then

$$|z| = r = \sqrt{x^2 + y^2}$$

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}\,, \qquad \cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)\,.$$

The *complex conjugate* of z is $z^* = x - iy$.



Algebra of complex numbers

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then the *sum* and *difference* of z_1 and z_2 are defined as

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)\,.$$

The *product* of z_1 and z_2 is defined as

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)\,. \end{aligned}$$

The *quotient* of two complex numbers is defined analagously

$$\begin{aligned} \frac{z_2}{z_1} &= \frac{x_2 + iy_2}{x_1 + iy_1} \\ &= \frac{(x_2 + iy_2)(x_1 - iy_1)}{(x_1 + iy_1)(x_1 - iy_1)} \\ &= \frac{x_1 x_2 + y_1 y_2}{x_1^2 + y_1^2} + i\frac{x_1 y_2 - y_1 x_2}{x_1^2 + y_1^2} \end{aligned}$$

provided $x_1^2 + y_1^2 = |z_1|^2 \neq 0$.

Euler's equation

Using the power series representations for $e^z, \sin z$ and $\cos z$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

So, putting $z = i\theta$, gives

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Fourier series

Taylor series allow us to represent certain functions as power series. Such functions must be *continuous* and *infinitely differentiable* within the interval of convergence.

We now consider how functions that may be neither differentiable nor continuous at certain points can be represented by a trigonometric series of the form

$$\frac{a_0}{2} + \sum_{r=1}^{\infty} (a_r \cos rx + b_r \sin rx)$$

where a_0, a_r and b_r are constants for $r = 1, 2, \ldots$.

Since this trigonometric series is unchanged by replacing x by $x + 2k\pi$, where k is an integer, we can only use it to represent a certain restricted class of functions.

Periodic functions

We say that a function, $f(x)$, is *periodic* with period p if

$$f(x+p) = f(x)$$

for all x. So, it suffices if we know the value of $f(x)$ for all values in the interval $0 < x \leq p$ of length p.

Examples

- $f(x) = \cos(x)$ has period $p = 2\pi$
- $f(x) = \cos(2x)$ has period $p = \pi$
- $f(x) = \cos\left(\frac{\pi}{T}x\right)$ has period $p = 2T$

Fourier coefficients

Suppose that $f(x)$ is any integrable function defined on the interval $-\pi < x \leq \pi$. Let,

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx\,, \\
a_r &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos rx\,dx\,, \qquad r = 1, 2, 3, \ldots\,, \\
b_r &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin rx\,dx\,, \qquad r = 1, 2, 3, \ldots
\end{aligned}
$$

then the resulting series is called the *Fourier series* and the coefficients are the *Fourier coefficients*

$$\frac{a_0}{2} + \sum_{r=1}^{\infty} \left(a_r \cos rx + b_r \sin rx\right)\,.$$

Question: when does the Fourier series for $f(x)$ converge to $f(x)$?

To answer this question we need to define *Dirichlet's conditions*.

Dirichlet's conditions

Suppose $f(x)$ is defined arbitrarily in the interval $-\pi < x \leq \pi$ and extended to other values of x by the periodicity condition $f(x + 2k\pi) = f(x)$ where k is an integer.

The function $f(x)$ satisfies the *Dirichlet's conditions*, if in $-\pi < x \leq \pi$,

- $f(x)$ is continuous except for a *finite number* of points of finite discontinuities, and
- has only a *finite number* of maxima and minima.

(Such functions are also called *piecewise regular*.)

Fourier's theorem

Fourier was able to show that a function, $f(x)$, satisfying the Dirichlet conditions has a Fourier series which converges to $f(x)$ at all points in the interval where $f(x)$ is *continuous*.

Moreover, at a point of discontinuity (necessarily finite), say at $x = x_0$, the Fourier series converges to the value

$$\frac{1}{2} \lim_{\delta \to 0} \{ f(x_0 + \delta) + f(x_0 - \delta) \} \ ,$$

which is just the mean of the two limiting values of $f(x)$ as x approaches x_0 from the right and left-hand sides.

Derivation of Fourier coefficients

If r and s are positive integers or zero then it follows by simple integration that

$$\int_{-\pi}^{\pi} \cos rx \cos sx \, dx = \begin{cases} 0 & \text{for } r \neq s \\ 2\pi & \text{for } r = s = 0 \\ \pi & \text{for } r = s > 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin rx \sin sx \, dx = \begin{cases} 0 & \text{for } r \neq s \\ 0 & \text{for } r = s = 0 \\ \pi & \text{for } r = s > 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin rx \cos sx \, dx = 0 \qquad \text{for all } r \text{ and } s$$

$$\int_{-\pi}^{\pi} \cos rx \, dx = \begin{cases} 0 & \text{for } r > 0 \\ 2\pi & \text{for } r = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin rx \, dx = 0 \qquad \text{for all } r\,.$$

Derivation of Fourier coefficients, ctd

Using these results and multiplying both sides of

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} (a_r \cos rx + b_r \sin rx)$$

by $\cos sx$ and integrating from $x = -\pi$ to π yields expressions for a_0 and a_r $(r = 1, 2, 3, \ldots)$.

Similarly, multiplying both sides by $\sin sx$ and integrating from $x = -\pi$ to π yields expressions for b_r $(r = 1, 2, 3, \ldots)$.

Note that we can merge the expressions for a_0 and a_r into the single formula

$$a_r = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos rx \, dx \,,$$

as r takes values $r = 0, 1, 2, \ldots$.

Even and odd functions

A function $f(x)$ is said to be *even* if for all x

$$\boxed{f(x) = f(-x)}$$

So, for example, $f(x) = \cos(x)$ is an even function.

Similarly, a function $f(x)$ is said to be *odd* if for all x

$$\boxed{f(x) = -f(-x)}$$

So, for example, $f(x) = \sin(x)$ is an odd function.

Even functions: cosine series

Suppose that $f(x)$ in the interval $-\pi < x \leq \pi$ is an *even* function then for all $r = 1, 2, 3, \ldots$

$$
\begin{aligned}
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x)\, dx\,, \\
a_r &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos rx\, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos rx\, dx\,, \\
b_r &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin rx\, dx = 0\,.
\end{aligned}
$$

Hence, if $f(x)$ is an even function its Fourier series reduces to a series where all the sine terms vanish.

Odd functions: sine series

Alternatively, suppose that $f(x)$ in the interval $-\pi < x \leq \pi$ is an *odd* function then for all $r = 1, 2, 3, \ldots$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = 0\,,$$

$$a_r = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos rx\, dx = 0\,,$$

$$b_r = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin rx\, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin rx\, dx\,.$$

Hence, if $f(x)$ is an odd function its Fourier series reduces to a series where all the cosine terms vanish.

Change of interval (period)

Instead of using the interval $-\pi < x \le \pi$ and a period of 2π consider the more general situation of an interval $-T < x \le T$ with period $2T$.

So suppose that $f(x)$ satisifes the Dirichlet conditions in $-T < x \le T$ and is defined outside this interval by the periodicity condition $f(x + 2Tk) = f(x)$, where k is an integer. Then, writing $z = \pi x/T$ gives

$$f(x) = f\left(\frac{Tz}{\pi}\right) = F(z)\,,$$

where now $F(z)$ is a periodic function of z of period 2π.

Change of interval (period), ctd

Hence in $-\pi < z \leq \pi$

$$F(z) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left(a_r \cos rz + b_r \sin rz\right) \; ,$$

where

$$a_r = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z) \cos rz \, dz \, , \quad (r = 0, 1, 2, \ldots) \, ,$$

$$b_r = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z) \sin rz \, dz \, , \quad (r = 1, 2, 3, \ldots) \, .$$

Thus, putting $z = \pi x/T$, gives in $-T < x \leq T$

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left(a_r \cos \frac{\pi x r}{T} + b_r \sin \frac{\pi x r}{T} \right)$$

where

$$a_r = \frac{1}{T} \int_{-T}^{T} f(x) \cos \frac{\pi x r}{T} \, dx \, , \quad (r = 0, 1, 2, , \ldots) \, ,$$

$$b_r = \frac{1}{T} \int_{-T}^{T} f(x) \sin \frac{\pi x r}{T} \, dx \, , \quad (r = 1, 2, 3, \ldots) \, .$$

Compact complex representation

Using the Euler relation $e^{i\theta} = \cos\theta + i\sin\theta$

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} (a_r \cos rx + b_r \sin rx)$$

can be written as

$$f(x) = \sum_{r=-\infty}^{\infty} c_r e^{irx}$$

where for all $r > 0$

$$c_r = \frac{1}{2}(a_r - ib_r) \qquad \text{and} \qquad c_{-r} = \frac{1}{2}(a_r + ib_r)$$

and

$$c_0 = \frac{1}{2} a_0 \,.$$

Also, for $r = 0, \pm 1, \pm 2, \ldots,$

$$c_r = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-irx} \, dx \,.$$

Compact complex representation, ctd

Note that the coefficients, c_r, are complex numbers and that for each positive integer $r = 1, 2, 3, \ldots$ we have two coefficients: c_r and c_{-r}.

Assuming $f(x)$ is a real-valued function, these two coefficients are complex conjugates, that is,

$$c_{-r} = c_r^* \qquad \text{for } r > 0\,.$$

In summary, we have obtained a *representation* of $f(x)$ in terms of a (doubly) infinite series of complex exponential functions

$$f(x) = \sum_{r=-\infty}^{\infty} c_r e^{irx}$$

where

$$c_r = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-irx}\, dx\,.$$

Examples

Consider two examples

- square wave

$$f(x) = \begin{cases} -1 & -\pi < x < 0\,, \\ 1 & 0 < x \leq \pi\,. \end{cases}$$

  with period 2π, and

- sawtooth wave

$$f(x) = x \qquad -1 < x \leq 1$$

  with period 2.

The square wave

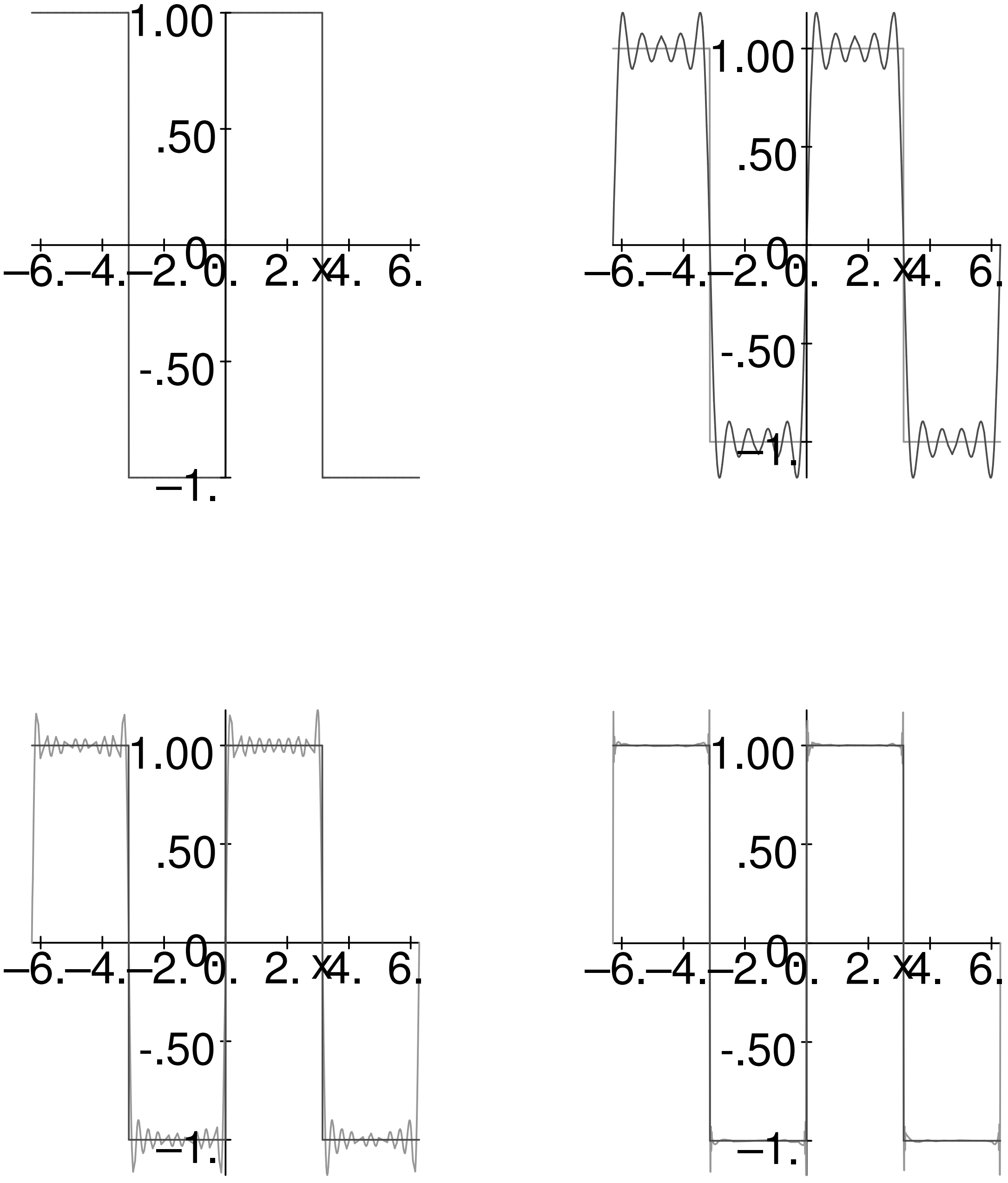
The square wave is an *odd* function so $a_r = 0$ for all $r \geq 0$. The cosine terms can be evaluated by

$$\begin{aligned} b_r &= \frac{2}{\pi} \int_0^{\pi} \sin rx \, dx \\ &= \frac{2}{r\pi}(1 - \cos r\pi) \\ &= \begin{cases} \frac{4}{r\pi} & r \text{ odd}, \\ 0 & r \text{ even}. \end{cases} \end{aligned}$$

Hence, the Fourier series expansion of the square wave function in $-\pi < x \leq \pi$ is

$$f(x) = \frac{4}{\pi} \left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots \right).$$

The square wave, ctd



The sawtooth wave

Again we have an *odd* function and, using the change of interval relations,

$$\begin{aligned} b_r &= \int_{-1}^{1} x \sin \pi r x \, dx \\ &= -\frac{2}{\pi r}(-1)^r \,. \end{aligned}$$

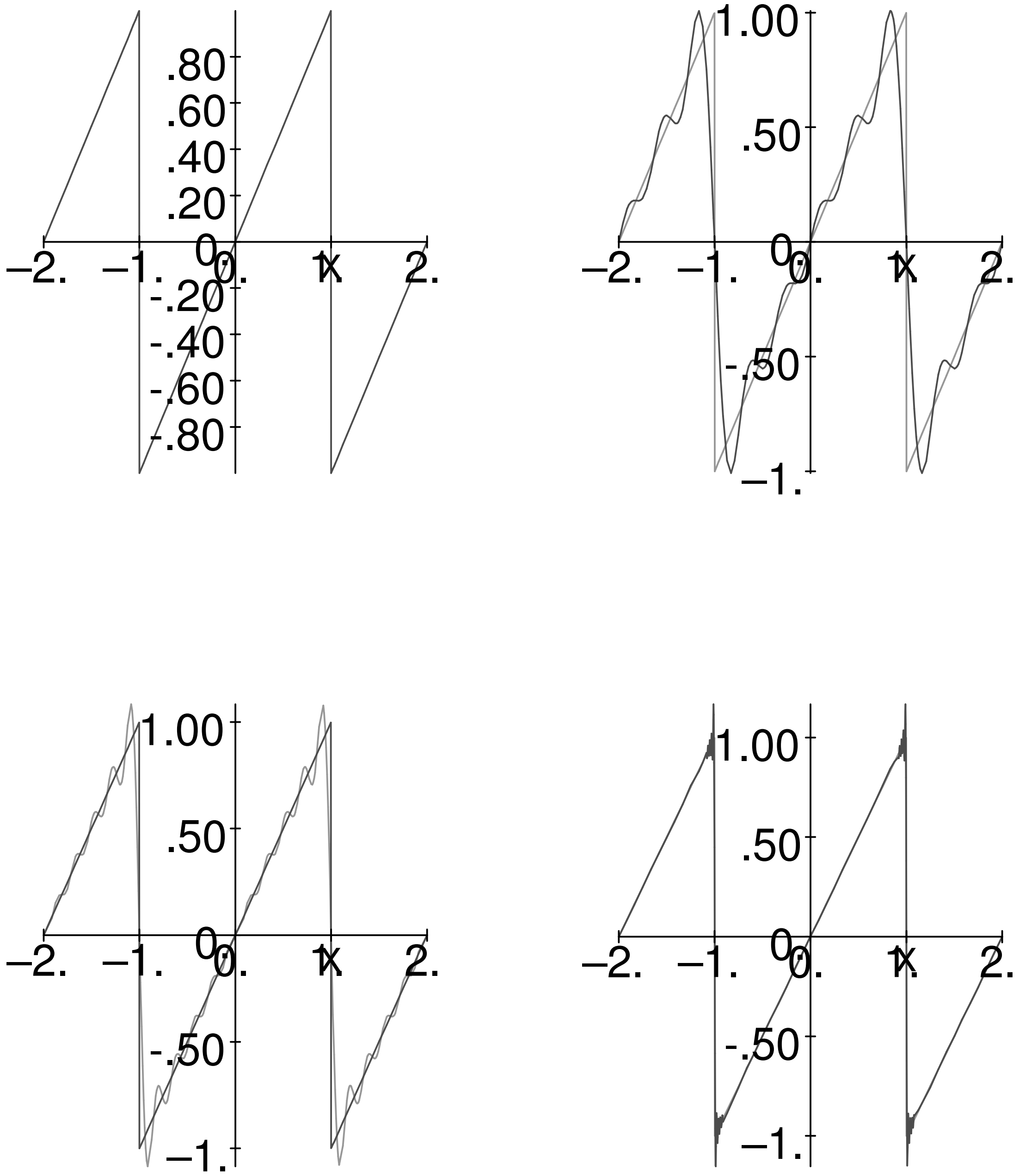
Hence in the interval $-1 < x < 1$

$$f(x) = x = \frac{2}{\pi}\left(\sin \pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x - \cdots\right) .$$

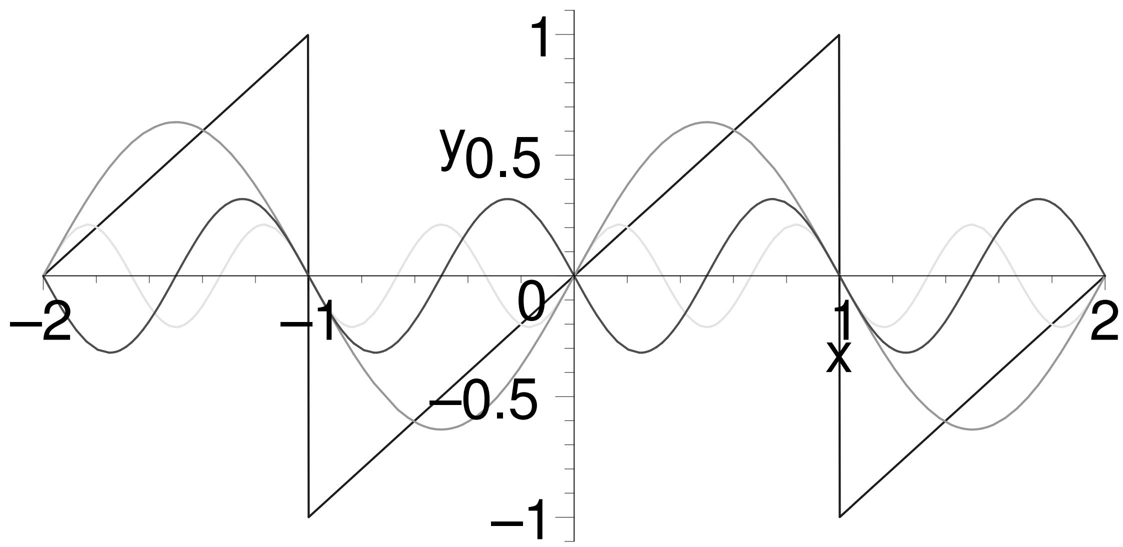
At $x = \pm 1$, finite discontinuities occur. Hence at these points the series does not represent x but converges to the value

$$\frac{1}{2}\lim_{\delta \to 0}\{f(x_0+\delta) + f(x_0-\delta)\} = \frac{1}{2}\{1 + (-1)\} = 0 \,.$$

The sawtooth wave, ctd



Superpositions of waves



Examples of series representation: Taylor's series

Recall, that *Taylor's series* allows us to represent certain functions as

$$f(x) = \sum_{r=0}^{\infty} \frac{(x-a)^r}{r!} f^{(r)}(a)$$

or, equivalently, for some constants c_r, as the power series

$$\sum_{r=0}^{\infty} c_r x^r \,.$$

For example,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Examples of series representation: Fourier series

Fourier series allows us to represent certain periodic functions by the series

$$\sum_{r=-\infty}^{\infty} c_r e^{irx}$$

that is, using Euler's equation, in terms of the functions

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \cos 3x, \sin 3x, \ldots$$

For example, the periodic sawtooth function in the interval $-1 < x < 1$ has the Fourier series

$$\frac{2}{\pi}\left(\sin \pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x - \cdots\right).$$

Expansion basis functions: general approach

The Taylor and Fourier series are just two examples of a general approach where we seek to represent a given function $f(x)$ in terms of linear combinations

$$f(x) = \sum_k c_k \Psi_k(x)$$

of certain other functions, $\Psi_k(x)$, called the *expansion basis functions*.

For example, with the Fourier series the basis functions are the complex exponentials

$$\Psi_k(x) = e^{i\mu_k x}$$

where $k = 0, \pm 1, \pm 2, \ldots$ and the *frequency* of the kth basis function $\Psi_k(x)$ is $\mu_k = k$.

Note that changing the period from 2π to $2T$ has the effect of changing the frequency of the kth basis function to $\mu_k = \pi k/T$.

Basis functions: application to systems analysis

This approach proves to be very useful because it allows us to choose some universal set of functions (the basis functions) and then represent many other functions in terms of just a set of numerical coefficients (the constants, c_k).

In the analysis of systems a major benefit of doing this is that knowledge of how members of the chosen universal set of basis functions behave in the system gives us knowledge about how arbitrary input functions will be treated by the system.

For periodic functions we only need to derive the response when the input is in one of the relatively simple forms: $\cos(rx)$ or $\sin(rx)$, for integer choices of r.

Definition of orthogonality

A set of basis functions is called *orthogonal* if they satisfy the rule that the integral of the conjugate product of any two distinct basis functions equals zero, that is,

$$\int_{-T}^{T} \Psi_k^*(x)\Psi_j(x)\,dx = 0 \quad (k \neq j)$$

where the integral is taken over one period, $-T < x \leq T$.

We will later consider the case of *aperiodic* functions corresponding to letting $T \to \infty$ so that the orthogonality condition then becomes

$$\int_{-\infty}^{\infty} \Psi_k^*(x)\Psi_j(x)\,dx = 0 \quad (k \neq j)\,.$$

Inner products

We call such integrals, *inner products*, and use the angle bracket notation

$$< \Psi_k(x),\, \Psi_j(x) > := \int_{-\infty}^{\infty} \Psi_k^*(x)\Psi_j(x)\, dx\,.$$

The analogous notion for vectors is the *scalar product*,

$$\mathbf{x} \cdot \mathbf{y}$$

between two vectors $\mathbf{x}$ and $\mathbf{y}$ — a notion which is closely related to notions of *length*, *distance* and *projection* in Euclidean space.

Orthogonality example

Consider $\Psi_k(x) = e^{ikx}$ then $\Psi_k^*(x) = e^{-ikx}$ and so the inner product $< \Psi_k(x), \Psi_j(x) >$ is given by

$$\int_{-\pi}^{\pi} \Psi_k^*(x)\Psi_j(x)\,dx$$
$$= \int_{-\pi}^{\pi} e^{-ikx}e^{ijx}\,dx$$
$$= \int_{-\pi}^{\pi} e^{i(j-k)x}\,dx$$
$$= \int_{-\pi}^{\pi} (\cos((j-k)x) + i\sin((j-k)x)\;dx$$
$$= \begin{cases} 0 & (k \neq j) \\ 2\pi & (k = j) \end{cases}$$

So, we see that the basis functions e^{ikx} are indeed orthogonal with period 2π.

Orthonormal basis functions

If, in addition to orthogonality, the set of basis functions satisfy the property that the inner product of every basis function with itself is equal to one, that is,

$$< \Psi_j(x), \, \Psi_j(x) > = \int_{-\infty}^{\infty} \Psi_j^*(x) \Psi_j(x) \, dx = 1$$

then the set of basis functions is said to be *orthonormal*.

That is,

$$< \Psi_k(x), \, \Psi_j(x) > = \begin{cases} 0 & (k \neq j) \\ 1 & (k = j) \, . \end{cases}$$

Given a set of orthogonal basis functions we can make them orthonormal just by scaling each of them by a suitable factor.

Derivation of expansion coefficients

Consider a function represented in terms of orthonormal basis functions

$$f(x) = \sum_k c_k \Psi_k(x)\,.$$

Question: How do we determine the c_k?

Consider, for any k,

$$\begin{aligned}
< \Psi_k(x),\, f(x) > &= \int_{-T}^{T} \Psi_k^*(x) f(x)\, dx \\
&= \int_{-T}^{T} \Psi_k^*(x) \sum_j c_j \Psi_j(x)\, dx \\
&= \sum_j c_j \int_{-T}^{T} \Psi_k^*(x) \Psi_j(x)\, dx \\
&= \sum_j c_j < \Psi_k(x),\, \Psi_j(x) > \\
&= c_k
\end{aligned}$$

Completeness of basis functions

We say that a set of basis functions is *complete* when all functions of interest can be represented by an expansion of the form

$$\sum_k c_k \Psi(x)\,.$$

In other words, the space of basis functions spans the required set of functions.

More rigorously, we can say that a set of basis functions is complete if no nontrivial function of interest, $f(x)$, is orthogonal to *all* the basis functions $\Psi_k(x)$. That is,

$$< f(x),\, \Psi_k(x) >= 0 \qquad \text{for all } k$$

implies that f is the trivial function $f(x) = 0$ for all x.

Interpretations

Suppose that the functions $\Psi_k(x)$ form orthonormal basis functions.

- One way to think of the coefficients, c_k, is that they measure the projection of the some function $f(x)$ along the *"coordinate"* given by basis function $\Psi_k(x)$.
- The set of basis functions of the form e^{ikx} $(k = 0, \pm 1, \pm 2, \ldots)$ can be shown to be complete for the space of piecewise regular functions with period 2π.
- Each function e^{ikx} gives one "coordinate" which together are sufficient to represent any such piecewise regular function.

Fourier transforms: motivation

Fourier series allow us to represent a class of periodic functions on the interval $-T < x \leq T$.

We now look at the representation of *aperiodic* functions over the infinite range $-\infty < x < \infty$.

Consider the Fourier series

$$f(x) = \sum_k c_k e^{i\mu_k x}$$

and, informally, take the limit to add up over all possible frequencies $-\infty < \mu < \infty$ to yield the integral representation for $f(x)$

$$f(x) = \int_{-\infty}^{\infty} F(\mu) e^{i\mu x} \, d\mu \, .$$

Fourier transform and its inverse

The orthogonality relations for complex exponentials then gives us that

$$F(\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\mu x} \, dx \, .$$

The function $F(\mu)$ is known as the *Fourier transform* of $f(x)$.

Whereas, the expression

$$f(x) = \int_{-\infty}^{\infty} F(\mu) e^{i\mu x} \, d\mu$$

allows us to invert the transform, $F(\mu)$, and return to the original function $f(x)$.

Beware: there are other conventions for the choice of basis functions which lead to minor variants of these definitions, the position of the 2π's is critical!

Properties of the Fourier transform

The Fourier transform $F(\mu)$ of a function $f(x)$ tells us about the contribution to f made by the complex exponential $e^{i\mu x}$ at frequency μ.

The transformation can be thought of as taking us from the time (or space) domain, x, to a representation in the frequency domain, μ.

There are a number of helpful properties of Fourier transforms according to the following operations of

- shift;
- scale (or dilation);
- differentiation;
- convolution.

Shift property

Suppose we shift the original function $f(x)$ by some displacement α. What happens to the Fourier transform?

Taking the Fourier transform of $f(x-\alpha)$ gives

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-\alpha)e^{-i\mu x}\,dx \\ &\qquad = \frac{e^{-i\mu\alpha}}{2\pi}\int_{-\infty}^{\infty} f(u)e^{-i\mu u}\,du \\ &\qquad = e^{-i\mu\alpha}F(\mu) \end{aligned}$$

Thus the Fourier transform of the shifted function $f(x-\alpha)$ is obtained by multiplication of the original Fourier transform by the factor $e^{-i\mu\alpha}$.

Scale property

The scale (or dilation) property tells us what happens when we consider $f(\alpha x)$ instead of $f(x)$ (for $\alpha \neq 0$).

The Fourier transform of $f(\alpha x)$ is given by

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(\alpha x) e^{-i\mu x} \, dx$$

$$= \begin{cases} \frac{1}{2\pi\alpha} \int_{-\infty}^{\infty} f(u) e^{-i(\mu/\alpha)u} \, du & (\alpha > 0) \\ \frac{1}{2\pi\alpha} \int_{\infty}^{-\infty} f(u) e^{-i(\mu/\alpha)u} \, du & (\alpha < 0) \end{cases}$$

$$= \frac{1}{|\alpha|} F\left(\frac{\mu}{\alpha}\right) .$$

Differentiation property

Suppose $f(x)$ is a differentiable function. What does the Fourier transform of $f'(x)$ look like?

Informally, we have that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} f'(x)e^{-i\mu x}\,dx \\
&\qquad = \frac{1}{2\pi}\left[e^{-i\mu x}f(x)\right]_{x=-\infty}^{x=\infty} + \frac{i\mu}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\mu x}\,dx \\
&\qquad = i\mu F(\mu)\,.
\end{aligned}$$

Using integration by parts as well as assuming a regularity condition that $f(x) \to 0$ as $x \to \pm\infty$.

For higher order derivatives we have that

$$\left(\frac{d}{dx}\right)^m f(x) \quad \text{has Fourier transform} \quad (i\mu)^m F(\mu)$$

where m is the order of the derivative.

Convolution

The *convolution* of two functions $f(x)$ and $g(x)$ is given by

$$h(x) = (f * g)(x) = \frac{1}{2\pi} \int_{\alpha=-\infty}^{\alpha=\infty} f(\alpha) g(x-\alpha)\, d\alpha\,.$$

Thus the convolution is a way of combining two functions which in a sense uses one to *"blur"* the other, making all possible relative shifts between two functions when computing the integral of their product.

Convolution is an extremely important operation in systems theory because it describes how any linear system $f(t)$ acts on any input $g(t)$ to generate the corresponding output $h(t)$. The output is just given by the convolution of the input with the characteristic system response function, so that,

$$h(t) = f(t) * g(t)\,.$$

Fourier transforms and convolutions

Consider the Fourier transform of $f * g$ given by

$$\frac{1}{2\pi} \int\limits_{x=-\infty}^{x=\infty} \frac{1}{2\pi} \int\limits_{\alpha=-\infty}^{\alpha=\infty} f(\alpha) g(x-\alpha) e^{-i\mu x}\, d\alpha dx$$

$$= \frac{1}{2\pi} \int\limits_{\alpha=-\infty}^{\alpha=\infty} \int\limits_{x=-\infty}^{x=\infty} f(\alpha) e^{-i\mu\alpha} \frac{g(x-\alpha) e^{-i\mu(x-\alpha)}}{2\pi}\, dx d\alpha$$

$$= \frac{1}{2\pi} \int\limits_{\alpha=-\infty}^{\alpha=\infty} f(\alpha) e^{-i\mu\alpha} G(\mu)\, d\alpha$$

$$= F(\mu) G(\mu)\,.$$

Here $f(x)$ and $g(x)$ have Fourier transforms $F(\mu)$ and $G(\mu)$, respectively.

Wavelets

Wavelets are a further method of representing functions which have received much interest in applied fields over the last several decades.

The approach fits into the general scheme of expansion using basis functions. Here we expand the functions $f(x)$ in terms of a doubly-infinite series

$$f(x) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} d_{jk} \Psi_{jk}(x)$$

where $\Psi_{jk}(x)$ are the basis functions.

The basis functions arise from *shifting* and *scaling* operations applied to a single function, $\Psi(x)$, known as the *mother wavelet*.

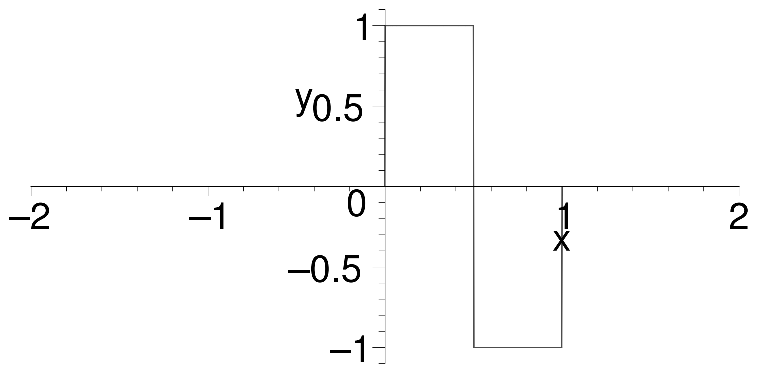
The basis functions are given for integers j and k by

$$\boxed{\Psi_{jk}(x) = \Psi(2^j x - k)}$$

The Haar wavelet

A common example is the *Haar wavelet* whose mother function is both *localised* and *oscillatory* defined by

$$\Psi(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2}\,, \\ -1 & \text{if } \frac{1}{2} \le x < 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$



Wavelet dilations and translations

The Haar mother wavelet oscillates and has a width (or scale) of one. The *dyadic dilates* of $\Psi(x)$, namely,

$$\ldots, \Psi(2^{-2}x), \Psi(2^{-1}x), \Psi(x), \Psi(2x), \Psi(2^2x), \ldots$$

have widths

$$\ldots, 2^2, 2^1, 1, 2^{-1}, 2^{-2}, \ldots$$

respectively. Since the dilate $\Psi(2^jx)$ has width 2^{-j}, its translates

$$\Psi(2^jx-k) = \Psi(2^j(x-k2^{-j})), \quad k = 0, \pm 1, \pm 2, \ldots$$

will cover the whole x-axis. The collection of coefficients d_{jk} are termed the *discrete wavelet transform* of the function $f(x)$.

Interpretation of d_{jk}

How should we intrepret the values d_{jk}?

Since the Haar basis function $\Psi(2^j x - k)$ vanishes except when

$$0 \le 2^j x - k < 1\,, \quad \text{that is} \quad k2^{-j} \le x < (k+1)2^{-j}$$

we see that d_{jk} gives us information about the bahaviour of f near the point $x = k2^{-j}$ measured on the scale of 2^{-j}.

For example, the coefficients $d_{-10,k}$, $k = 0, \pm 1, \pm 2, \ldots$ correspond to variations of f that take place over intervals of length $2^{10} = 1024$ while the coefficients $d_{10,k}$ $k = 0, \pm 1, \pm 2, \ldots$ correspond to fluctuations of f over intervals of length 2^{-10}.

These observations help explain how the discrete wavelet transform can be an exceptionally efficient scheme for representing functions.

Comparison with Fourier analysis

Some of the practical motivations underlying the use of expansion basis functions such as Fourier analysis or wavelet analysis are

- improved understanding,
- denoising signals, and
- data compression.

By representation of signals or functions in other forms these tasks become easier.

The approach taken with Fourier analysis represents signals in terms of trigonometric functions and as such is particularly suited to situations where the signal is relatively smooth and is not of limited extent.

Properties of naturally arising data

Much naturally arising data has been found to be better represented using wavelets which are better able to cope with discontinuities and where the signal is of local extent. Generally, the efficiency of the representation depends on the types of signal involved. If your signal contains

- discontinuities (in both the signal and its derivatives), or
- varying frequency behaviour

then wavelets are likely to represent the signal more efficiently than is possible with Fourier analysis.

Other classes of wavelets

One of the most useful features of wavelets is the ease with which a scientist can select the basis functions adapted for the given problem.

In fact, the Haar mother wavelet is perhaps the simplest of a very wide class of possible wavelet systems used in practice today.

Many applied fields have started to make use of wavelets including astronomy, acoustics, signal and image processing, neurophysiology, music, magnetic resonance imaging, speach discrimination, optics, fractals, turbulence, earthquake prediction, radar, human vision, etc.

Sampled data

Suppose we have sampled N uniformly spaced function evaluations, $f(0), f(1), \ldots, f(N-1)$, say. The complex exponentials $e^{i\mu n}$, as functions of n, have period N if

$$e^{i\mu(n+N)} = e^{i\mu n}\,.$$

So that,

$$e^{i\mu N} = 1$$

that is, $\mu = k2\pi/N$, for integer k.

Furthermore, when m is an integer

$$e^{i2\pi kn/N} = e^{i2\pi(k+mN)n/N}$$

so the parameters

$$\mu = \frac{k2\pi}{N}, \frac{(k \pm N)2\pi}{N}, \frac{(k \pm 2N)2\pi}{N}, \ldots$$

all give the same function.

Sampled data, ctd

So there are just N distinct complex exponentials with period N, namely,

$$e^{ik2\pi/N}, \quad k = 0, 1, \ldots, N-1$$

Hence, we can write each $f(n)$ as

$$f(n) = \sum_{k=0}^{N-1} F(k) e^{ik2\pi/N} \quad n = 0, 1, \ldots, N-1$$

The corresponding expression for the coefficients $F(k)$ turns out to be

$$F(k) = \frac{1}{N} \sum_{n=0}^{N-1} f(n) e^{-ik2\pi n/N} \quad k = 0, 1, \ldots, N-1$$

The quantities, $F(k)$, are known as the *Discrete Fourier Transform* (DFT) of f.

Application to image compression

The basic steps of the JPEG compression algorithm can be summarized as follows.

Take blocks of 8×8 pixels (each pixel is encoded as 24 bits) and then use a discrete Fourier transform (in fact, the related discrete cosine transform is used) to produce $8 \times 8 = 64$ separate frequency coefficients, per block.

Next use quantization factors (involving 64 tunable parameters) to compress each frequency coefficient block by block.

Store the resulting (integer) values with runlength encoding, etc.

This is a lossy form of compression but the idea is that when we invert the transformation and re-assemble the image the eye is not sensitive to the losses.

FBI fingerprint database

Each of the FBI's fingerprint cards when digitized amounts to about 10Mb of data. They have 200 million such cards!

These cards must be both stored as well as transmitted so compression is very important.

A wavelet-based approach called *Wavelet Scalar Quantization* compression was developed by Los Alamos Labs and NIST in the US.

Reference

`http://www.c3.lanal.gov/~brislawn/FBI/FBI.html`