Continuous Mathematics

Computer Science Tripos, Part IB, Part II (General)
Diploma in Computer Science
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Problem sheet

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1. Given $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ what are the real and imaginary parts of $z_3 = z_1z_2$?

2. Given $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ what is the modulus, $|z_1|$, of $z_1$ and what is the modulus of $z_2 = z_1z_2$?

3. Given $z_2 = x_2 + iy_2$ what is $\arg(z_2)$, the argument of $z_2$? Is it unique?

4. Express $z_1 = x_1 + iy_1$ in complex polar form using the modulus and argument of $z_1$.

5. Suppose that $|z_1| = |z_2| = 1$. Using an Argand diagram, explain how computing their product $z_3 = z_1z_2$ amounts to a rotation in the complex plane. Why is the multiplication of these complex variables reduced an addition? What is the value of $|z_3|$?

6. Given $z = \exp(2\pi i/5)$, what is the value of $z^5$? Explain your result using an Argand diagram.

7. Consider the complex exponential function $f(x) = \exp(2\pi i\omega x)$. What are the real and imaginary parts of $f(x)$ as functions of $x$?

8. For the imaginary number $i = \sqrt{-1}$, consider the quantity $\sqrt{i}$. Express $\sqrt{i}$ as a complex exponential. In what quadrant of the complex plane does it lie? What are the real and imaginary parts of $\sqrt{i}$? What is the modulus of $\sqrt{i}$?

9. Given $f(x) = \cos(1/x)$, does $\lim_{x \to 0} f(x)$ exist? What happens if instead $f(x) = x\cos(1/x)$?

10. Show that “continuity at $x = a$” does not imply “differentiable at $x = a$” by constructing a suitable counterexample.

11. Write down the Taylor’s series approximation to the value of a function $f(b)$ given only the function and its first three derivatives evaluated at $x = a$, namely, $f(a)$, $f’(a)$, $f”(a)$ and $f'''(a)$. You may assume that these derivatives exist and that $f$ and each of its derivatives is a continuous function.

12. Give an expression for computing $f(t)$ if we know only its projections $< f(t), \Psi_j(t) >$ onto this set of basis functions $\{\Psi_j(t)\}$. Explain what is happening.

13. What will be the Fourier Transform of the $m$th derivative of $f(x)$ with respect to $x$ in terms of the Fourier Transform, $F(\mu)$, of $f(x)$: $\left( \frac{d}{dx} \right)^m f(x)$?

14. What happens to the Fourier Transform after shifting $f(x)$ by a distance $\alpha$: $f(x - \alpha)$?

15. What happens to the Fourier Transform after dilating $f(x)$ by a factor $\alpha$: $f(x/\alpha)$?

16. What is the principal computational advantage of using orthogonal functions, over non-orthogonal ones, when representing a set of data as a linear combination of a universal set of basis functions?

If $\Psi_k(x)$ belongs to a set of orthonormal basis functions, and $f(x)$ is a function or a set of data that we wish to represent in terms of these basis functions, what is the basic computational operation we need to perform involving $\Psi_k(x)$ and $f(x)$?

17. Any real-valued function $f(x)$ can be represented as the sum of one function $f_e(x)$ that has even symmetry (it is unchanged after being flipped around the origin $x = 0$) so that $f_e(x) = f_e(-x)$, plus one function $f_o(x)$ that has odd symmetry, so that $f_o(x) = -f_o(-x)$. Such a decomposition of any function $f(x)$ into $f_e(x) + f_o(x)$ is illustrated by

$$f_e(x) = \frac{1}{2}f(x) + \frac{1}{2}f(-x)$$

$$f_o(x) = \frac{1}{2}f(x) - \frac{1}{2}f(-x).$$

Use this type of decomposition to explain why the Fourier transform of any real-valued function has Hermitian symmetry: its real-part has even symmetry, and its imaginary-part has odd symmetry. Comment on how this redundancy can be exploited to simplify computation of Fourier transforms of real-valued, as opposed to complex-valued, data.
18. Newton’s definition of a derivative in his formulation of The Calculus captures the notion of integer-order differentiation, e.g. the first or second derivative, etc. But in scientific computing we sometimes need a notion of fractional-order derivatives, as for example in fluid mechanics.

Explain how “Fractional Differentiation” (derivatives of non-integer order) can be given precise quantitative meaning through Fourier analysis.

Suppose that a continuous function \( f(x) \) has Fourier Transform \( F(\mu) \). Outline an algorithm (as a sequence of mathematical steps, not an actual program) for computing the \( 1.5^{th} \) derivative of some function \( f(x) \)

\[
\frac{d^{1.5} f(x)}{dx^{1.5}}
\]

19. Given the definition of the Fourier transform and its inverse show that if \( \alpha \) and \( A \) are non-zero constants then

\[
\hat{F}(\mu) = A \int_{-\infty}^{\infty} f(x)e^{-i\alpha \mu x} \, dx
\]

implies that

\[
f(x) = \frac{|\alpha|}{2\pi A} \int_{-\infty}^{\infty} \hat{F}(\mu)e^{i\alpha \mu x} \, d\mu
\]

In order to see what is going on start with the case \( \alpha = 1 \) and \( A = 1/2\pi \).

20. Comment on the strengths and weaknesses of the Fourier analysis approach compared with an approach using wavelets.