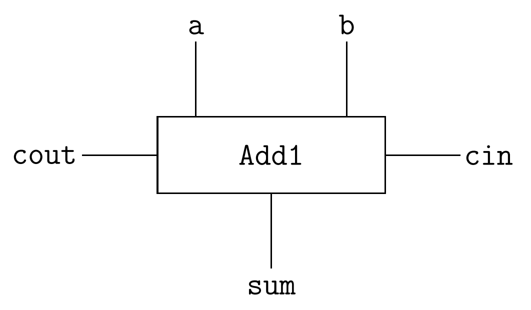
A 1-bit CMOS full adder

- **Here is a diagram of a 1-bit full adder:**



- **Lines `a`, `b`, `cin`, `sum` and `cout` carry the boolean values `T` or `F`.**

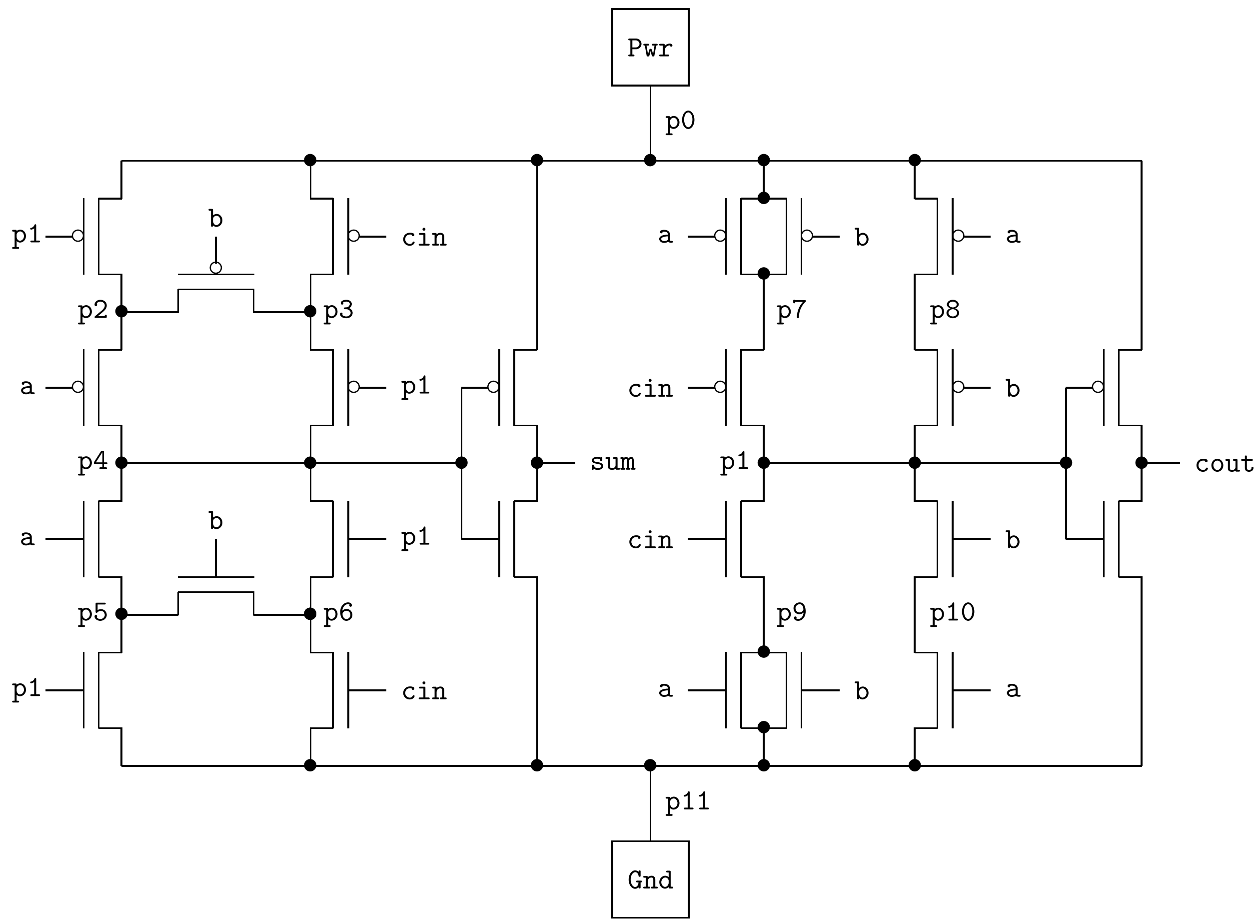
- **Specification of the adder:**

$$\texttt{Add1}(a, b, cin, sum, cout) \quad \equiv$$
$$(2{\times}\texttt{Bv}(cout) + \texttt{Bv}(sum) = \texttt{Bv}(a) + \texttt{Bv}(b) + \texttt{Bv}(cin))$$

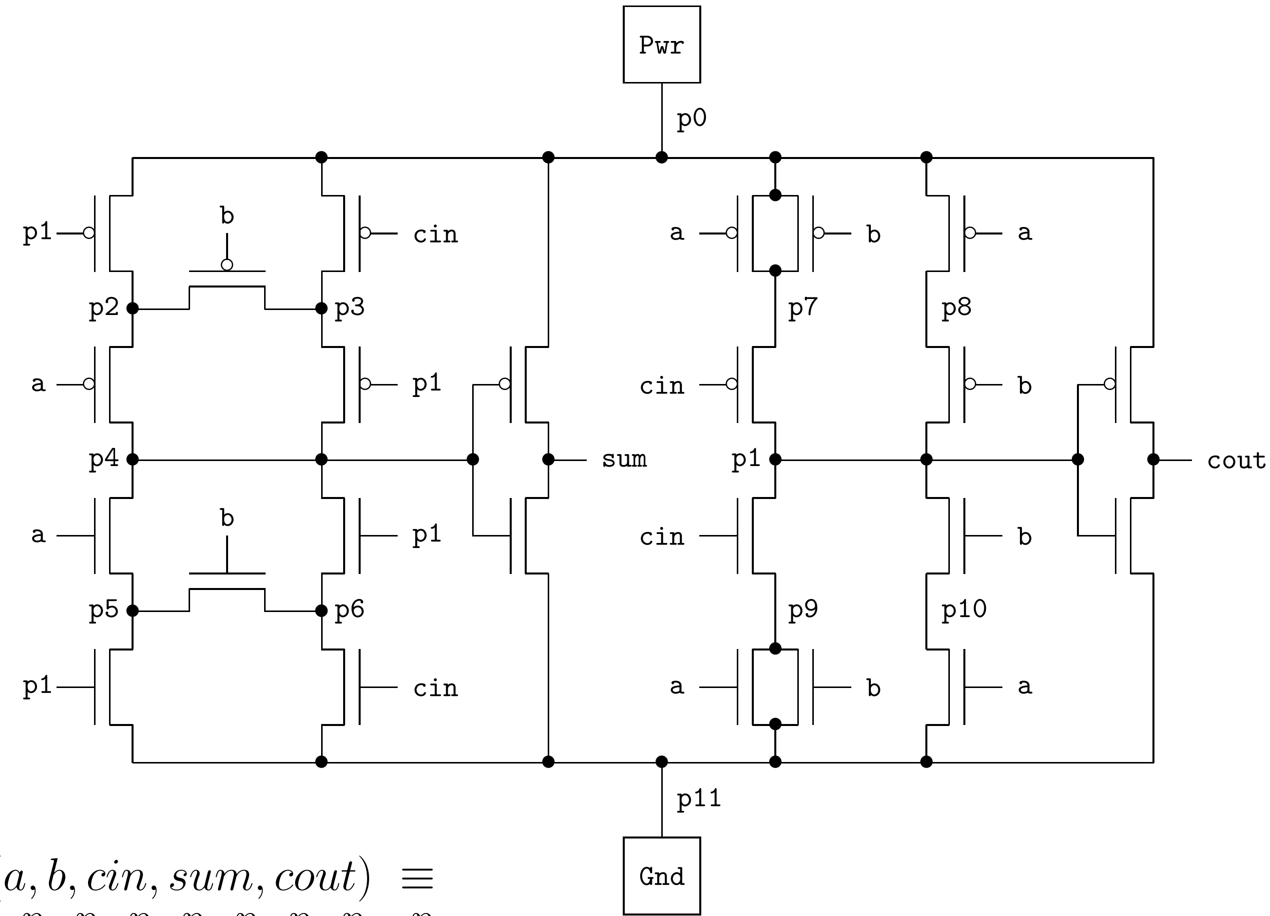
- **A correct implementation has:**
  - **lines `a`, `b`, `cin`, `sum` and `cout`**
  - **constrains a, b, cin, sum and, $cout$ so $\texttt{Add1}(a, b, cin, sum, cout)$**

Implementation

- **A CMOS implementation of the adder:**
  - **lines with the same name are connected**
  - **lines** p0, ... , p11 **are internal**
  - **horizontal transistors are bidirectional**



Specification in logic



$\texttt{Add1\_Imp}(a, b, cin, sum, cout) \equiv$
$\exists p_0\ p_1\ p_2\ p_3\ p_4\ p_5\ p_6\ p_7\ p_8\ p_9\ p_{10}\ p_{11}.$

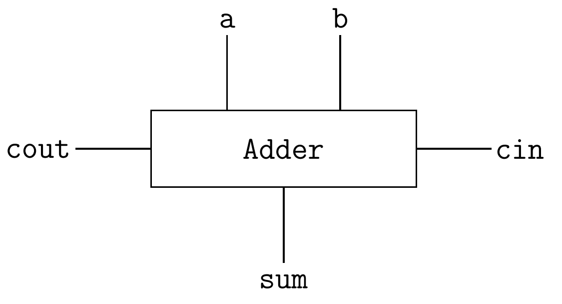
$$\begin{array}{l}
\texttt{Ptran}(p_1, p_0, p_2) \wedge \texttt{Ptran}(cin, p_0, p_3) \wedge \texttt{Ptran}(b, p_2, p_3) \wedge \texttt{Ptran}(a, p_2, p_4) \wedge \\
\texttt{Ptran}(p_1, p_3, p_4) \wedge \texttt{Ntran}(a, p_4, p_5) \wedge \texttt{Ntran}(p_1, p_4, p_6) \wedge \texttt{Ntran}(b, p_5, p_6) \wedge \\
\texttt{Ntran}(p_1, p_5, p_{11}) \wedge \texttt{Ntran}(cin, p_6, p_{11}) \wedge \texttt{Ptran}(a, p_0, p_7) \wedge \texttt{Ptran}(b, p_0, p_7) \wedge \\
\texttt{Ptran}(a, p_0, p_8) \wedge \texttt{Ptran}(cin, p_7, p_1) \wedge \texttt{Ptran}(b, p_8, p_1) \wedge \texttt{Ntran}(cin, p_1, p_9) \wedge \\
\texttt{Ntran}(b, p_1, p_{10}) \wedge \texttt{Ntran}(a, p_9, p_{11}) \wedge \texttt{Ntran}(b, p_9, p_{11}) \wedge \texttt{Ntran}(a, p_{10}, p_{11}) \wedge \\
\texttt{Pwr}(p_0) \wedge \texttt{Ptran}(p_4, p_0, sum) \wedge \texttt{Ntran}(p_4, sum, p_{11}) \wedge \\
\texttt{Gnd}(p_{11}) \wedge \texttt{Ptran}(p_1, p_0, cout) \wedge \texttt{Ntran}(p_1, cout, p_{11})
\end{array}$$

- Verify by Boolean algebra (tedious) or exhaustive enumeration

An n-bit adder

- **n-bit adder computes an n-bit sum and 1-bit carry-out from two n-bit inputs and a 1-bit carry-in**

- **Diagram:**



- `cin` **and** `cout` **carry single bits, i.e. Booleans**

- `a`, `b` **and** `sum` **carry n-bit words**

- `Adder` n **specifies an n+1-bit adder !!!**

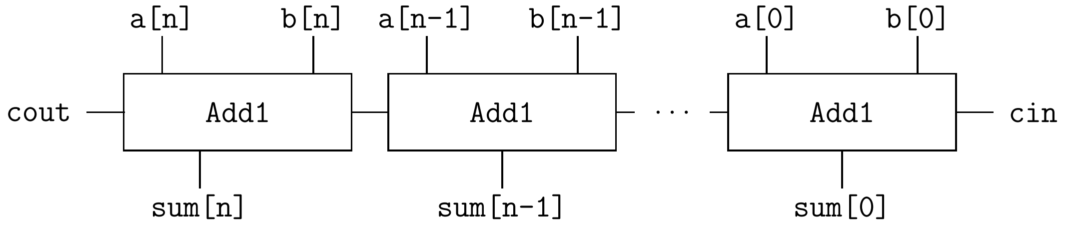
- **Example:** `Adder`(3) **specifies a 4-bit adder**

Specification

- **The definition of `Adder` is:**

$$\texttt{Adder}(n)(a, b, cin, sum, cout) \;\equiv$$
$$(2^{n+1} \times \texttt{Bv}(cout) + \texttt{V}(sum\texttt{[}n:0\texttt{]}) =$$
$$\texttt{V}(a\texttt{[}n:0\texttt{]}) + \texttt{V}(b\texttt{[}n:0\texttt{]}) + \texttt{Bv}(cin))$$

- **Diagram of implementation:**



- **By primitive recursion:**

$$\texttt{Adder\_Imp}(0)(a, b, cin, sum, cout) \;\equiv$$
$$\texttt{Add1}(a\texttt{[}0\texttt{]}, b\texttt{[}0\texttt{]}, cin, sum\texttt{[}0\texttt{]}, cout)$$

$$\texttt{Adder\_Imp}(n{+}1)(a, b, cin, sum, cout) \;\equiv$$
$$\exists c.\ \texttt{Adder\_Imp}(n)(a, b, cin, sum, c) \;\wedge$$
$$\texttt{Add1}(a\texttt{[}n{+}1\texttt{]}, b\texttt{[}n{+}1\texttt{]}, c, sum\texttt{[}n{+}1\texttt{]}, cout)$$

Verification:

- **Prove by induction on n that for all n:**

  $\texttt{Adder\_Imp}(n)(a, b, cin, sum, cout)$
  $\Rightarrow$
  $\texttt{Adder}(n)(a, b, cin, sum, cout)$

- **Basis:**

  $\texttt{Adder\_Imp}(0)(a, b, cin, sum, cout)$
  $\Rightarrow$
  $\texttt{Adder}(0)(a, b, cin, sum, cout)$

- **Expanding definitions of `Adder_Imp` and `Adder`:**

  $\texttt{Add1}(a\texttt{[}0\texttt{]}, b\texttt{[}0\texttt{]}, cin, sum\texttt{[}0\texttt{]}, cout)$
  $\Rightarrow$
  $(2^{0+1} \times \texttt{Bv}(cout) + \texttt{V}(sum\texttt{[}0:0\texttt{]}) = \texttt{V}(a\texttt{[}0:0\texttt{]}) + \texttt{V}(b\texttt{[}0:0\texttt{]}) + \texttt{Bv}(cin))$

- **Expanding definition of `Add1` and simplifying:**

  $(2 \times \texttt{Bv}(cout) + \texttt{Bv}(sum\texttt{[}0\texttt{]}) = \texttt{Bv}(a\texttt{[}0\texttt{]}) + \texttt{Bv}(b\texttt{[}0\texttt{]}) + \texttt{Bv}(cin))$
  $\Rightarrow$
  $(2 \times \texttt{Bv}(cout) + \texttt{V}(sum\texttt{[}0:0\texttt{]}) = \texttt{V}(a\texttt{[}0:0\texttt{]}) + \texttt{V}(b\texttt{[}0:0\texttt{]}) + \texttt{Bv}(cin))$

- **Follows by** $\texttt{V}(w\texttt{[}0:0\texttt{]}) = \texttt{Bv}(w\texttt{[}0\texttt{]})$

Induction step

- **Step:**

$$(\texttt{Adder\_Imp}(n)(a,b,cin,sum,cout) \Rightarrow \texttt{Adder}(n)(a,b,cin,sum,cout))$$
$$\Rightarrow$$
$$(\texttt{Adder\_Imp}(n{+}1)(a,b,cin,sum,cout) \Rightarrow \texttt{Adder}(n{+}1)(a,b,cin,sum,cout))$$

- **Assume:**

$$(\texttt{Adder\_Imp}(n)(a,b,cin,sum,cout) \Rightarrow \texttt{Adder}(n)(a,b,cin,sum,cout))$$

- **Then show:**

$$\begin{aligned}
&\texttt{Adder_Imp}(n{+}1)(a,b,cin,sum,cout)\\
&= \exists c.\ \texttt{Adder_Imp}(n)(a,b,cin,sum,c) \wedge \\
&\qquad \texttt{Add1}(a\texttt{[}n{+}1\texttt{]}, b\texttt{[}n{+}1\texttt{]}, c, sum\texttt{[}n{+}1\texttt{]}, cout)\\
&\Rightarrow \exists c.\ \texttt{Adder}(n)(a,b,cin,sum,c) \wedge \\
&\qquad \texttt{Add1}(a\texttt{[}n{+}1\texttt{]}, b\texttt{[}n{+}1\texttt{]}, c, sum\texttt{[}n{+}1\texttt{]}, cout)\\
&= \exists c.\ (2^{n+1}\texttt{Bv}(c) + \texttt{V}(sum\texttt{[}n:0\texttt{]}) = \texttt{V}(a\texttt{[}n:0\texttt{]}) + \texttt{V}(b\texttt{[}n:0\texttt{]}) + \texttt{Bv}(cin))\\
&\qquad \wedge\\
&\qquad (2\texttt{Bv}(cout) + \texttt{Bv}(sum\texttt{[}n{+}1\texttt{]}) = \texttt{Bv}(a\texttt{[}n{+}1\texttt{]}) + \texttt{Bv}(b\texttt{[}n{+}1\texttt{]}) + \texttt{Bv}(c))
\end{aligned}$$

Step continued

If:

$(A = B) \wedge (C = D)$

then it follows that $(\Rightarrow)$

$(A + 2^{n+1}C) = (B + 2^{n+1}D)$

hence:

$$\begin{aligned}
\exists c.\ & \overbrace{(2^{n+1}\texttt{Bv}(c) + \texttt{V}(sum\texttt{[}n:0\texttt{]})}^{A} = \overbrace{\texttt{V}(a\texttt{[}n:0\texttt{]}) + \texttt{V}(b\texttt{[}n:0\texttt{]}) + \texttt{Bv}(cin)}^{B} \\
& \wedge \\
& \underbrace{(2\texttt{Bv}(cout) + \texttt{Bv}(sum\texttt{[}n{+}1\texttt{]})}_{C} = \underbrace{\texttt{Bv}(a\texttt{[}n{+}1\texttt{]}) + \texttt{Bv}(b\texttt{[}n{+}1\texttt{]}) + \texttt{Bv}(c)}_{D}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \exists c.\ & \overbrace{2^{n+1}\texttt{Bv}(c) + \texttt{V}(sum\texttt{[}n:0\texttt{]})}^{A} + \overbrace{2^{n+1}2\texttt{Bv}(cout) + 2^{n+1}\texttt{Bv}(sum\texttt{[}n{+}1\texttt{]})}^{2^{n+1}C} \\
& = \overbrace{\texttt{V}(a\texttt{[}n:0\texttt{]}) + \texttt{V}(b\texttt{[}n:0\texttt{]}) + \texttt{Bv}(cin)}^{B} \\
& \quad + \underbrace{2^{n+1}\texttt{Bv}(a\texttt{[}n{+}1\texttt{]}) + 2^{n+1}\texttt{Bv}(b\texttt{[}n{+}1\texttt{]}) + 2^{n+1}\texttt{Bv}(c)}_{2^{n+1}D}
\end{aligned}$$

$$\begin{aligned}
= \ & \exists c.\ (\texttt{V}(sum\texttt{[}n{+}1:0\texttt{]}) + 2^{n+2}\texttt{Bv}(cout) = \texttt{V}(a\texttt{[}n{+}1:0\texttt{]}) + \texttt{V}(b\texttt{[}n{+}1:0\texttt{]}) + \texttt{Bv}(cin)) \\
= \ & (\texttt{V}(sum\texttt{[}n{+}1:0\texttt{]}) + 2^{n+2}\texttt{Bv}(cout) = \texttt{V}(a\texttt{[}n{+}1:0\texttt{]}) + \texttt{V}(b\texttt{[}n{+}1:0\texttt{]}) + \texttt{Bv}(cin)) \\
= \ & \texttt{Adder}(n{+}1)(a, b, cin, sum, cout))
\end{aligned}$$

Sequential Devices

- **Pure combinational adder:**

  $$\begin{array}{l}\texttt{Adder}(n)(a,b,cin,sum,cout) \;\equiv \\ \quad (2^{n+1}\times\texttt{Bv}(cout) + \texttt{V}(sum\texttt{[}n:0\texttt{]}) = \\ \qquad \texttt{V}(a\texttt{[}n:0\texttt{]}) + \texttt{V}(b\texttt{[}n:0\texttt{]}) + \texttt{Bv}(cin))\end{array}$$

- a, b **and** sum **range over words**

- cin **and** $cout$ **range over bits (Booleans)**

- **Zero-delay adder:**

  $$\begin{array}{l}\texttt{Combinational\_Adder}(n)(a,b,cin,sum,cout) \;\equiv \\ \quad \forall t.\ \texttt{Adder}(n)(a(t),b(t),cin(t),sum(t),cout(t))\end{array}$$

- a, b **and** sum **range over functions from time to words**

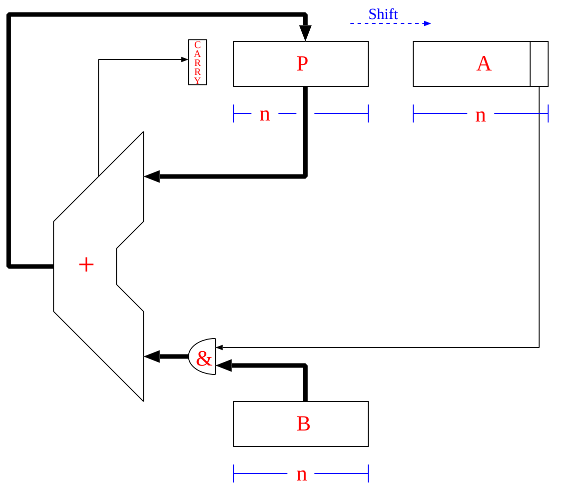
- cin **and** $cout$ **range over functions from time to bits**

- **Unit-delay adder:**

  $$\begin{array}{l}\texttt{Unit\_Delay\_Adder}(n)(a,b,cin,sum,cout) \;\equiv \\ \quad \forall t.\ \texttt{Adder}(n)(a(t),b(t),cin(t),sum(t{+}1),cout(t{+}1))\end{array}$$

Textbook add-shift multiplier

- **A standard add-shift multiplier:**



- **This can be verified directly**

- **Verification can be done directly in HOL or using Hoare Logic**

- **HOL proof by induction on word size**
  - **essence the of proofs (the invariant) are the same**
  - **compare sections 1.8 and 2.7 of notes (only if you enjoy messy details)**