Topics in concurrency 2020
Distributed games

Glynn Winskel

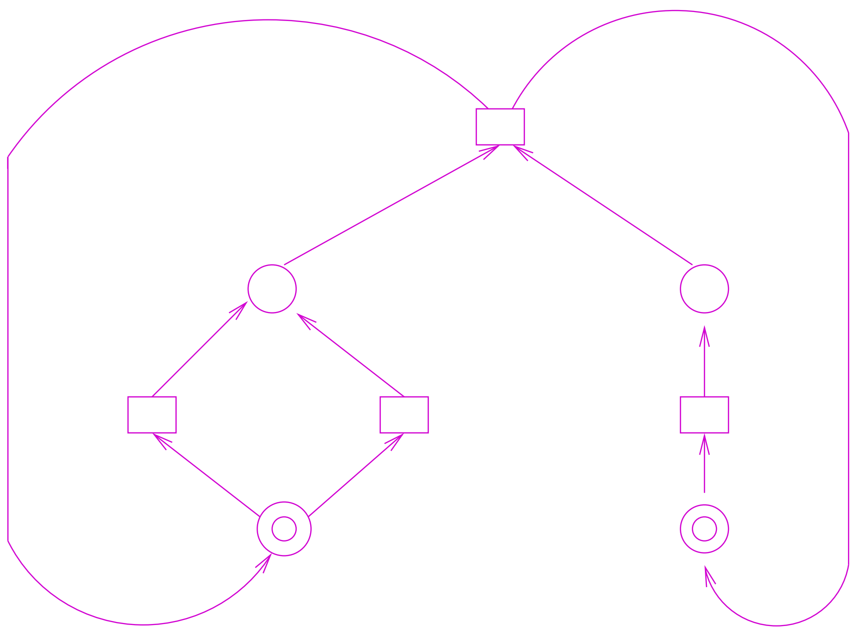
Lectures 10 - 16

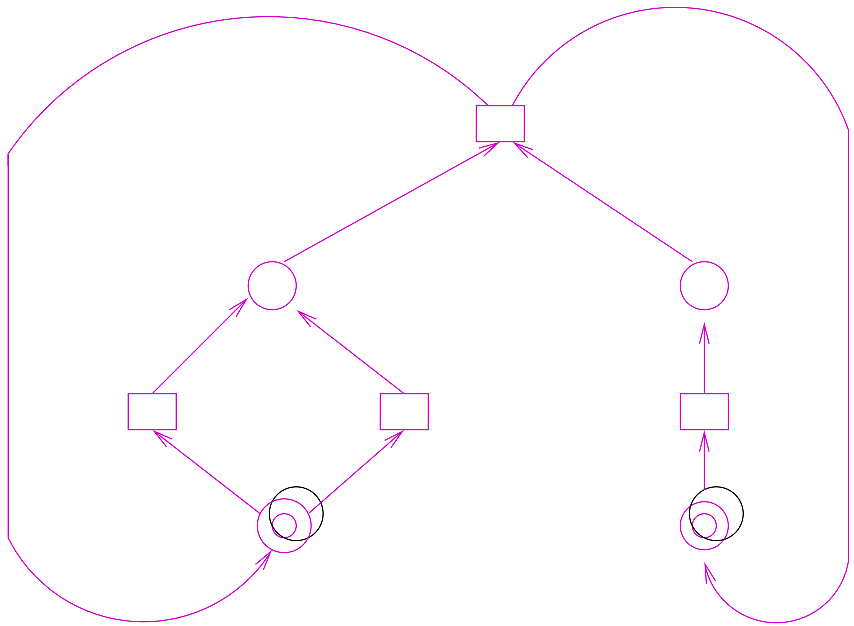
Event structures

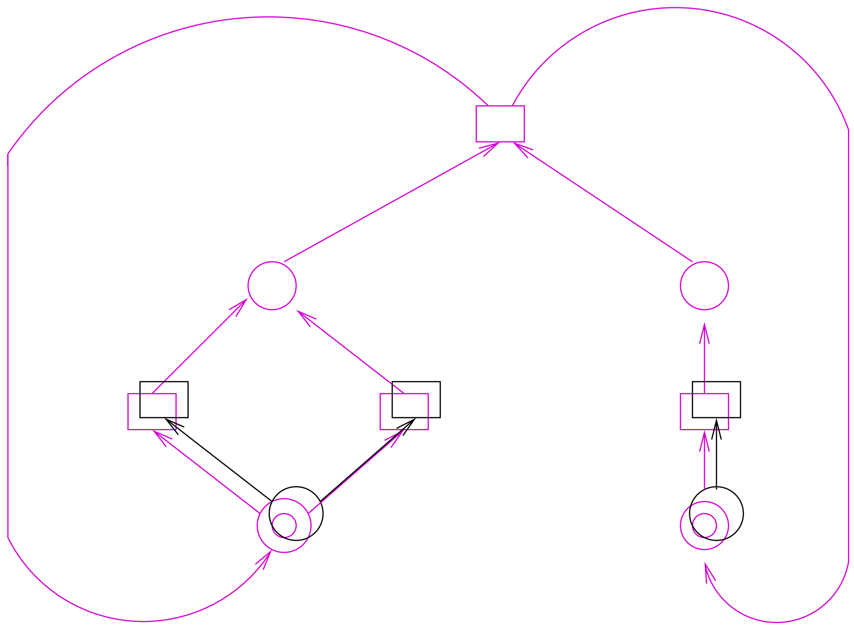
first, of the simplest kind, “prime event structures with binary conflict,” as originally introduced

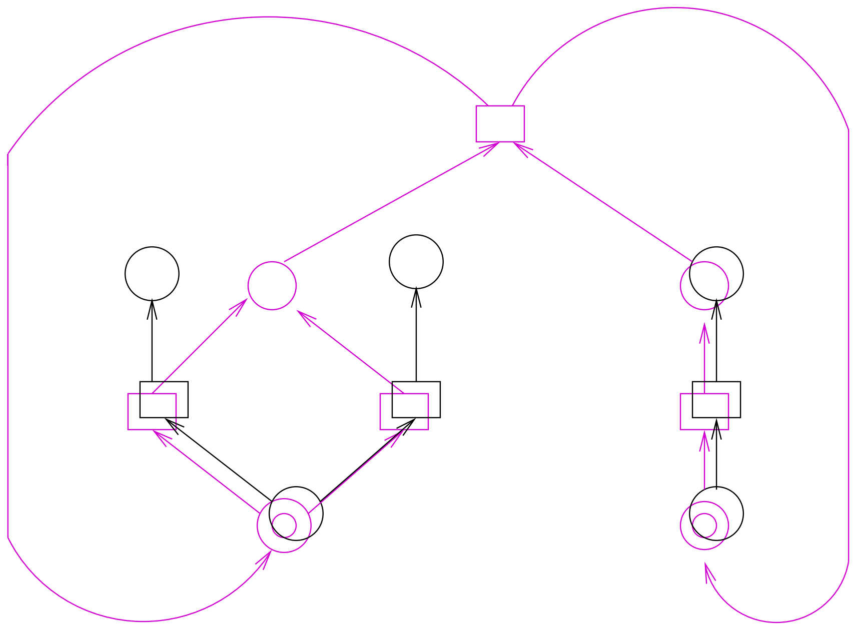
From nets ...

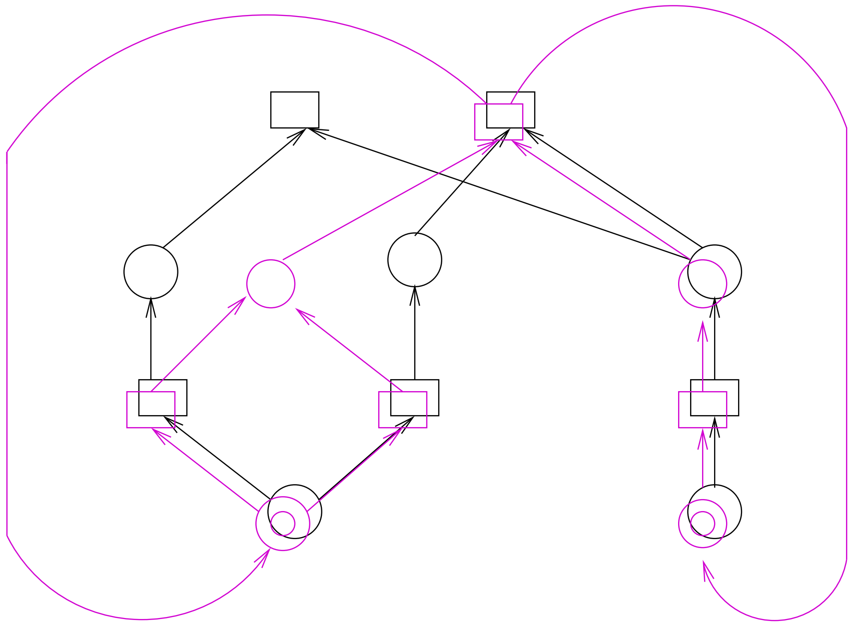
Unfolding a (safe) Petri net:

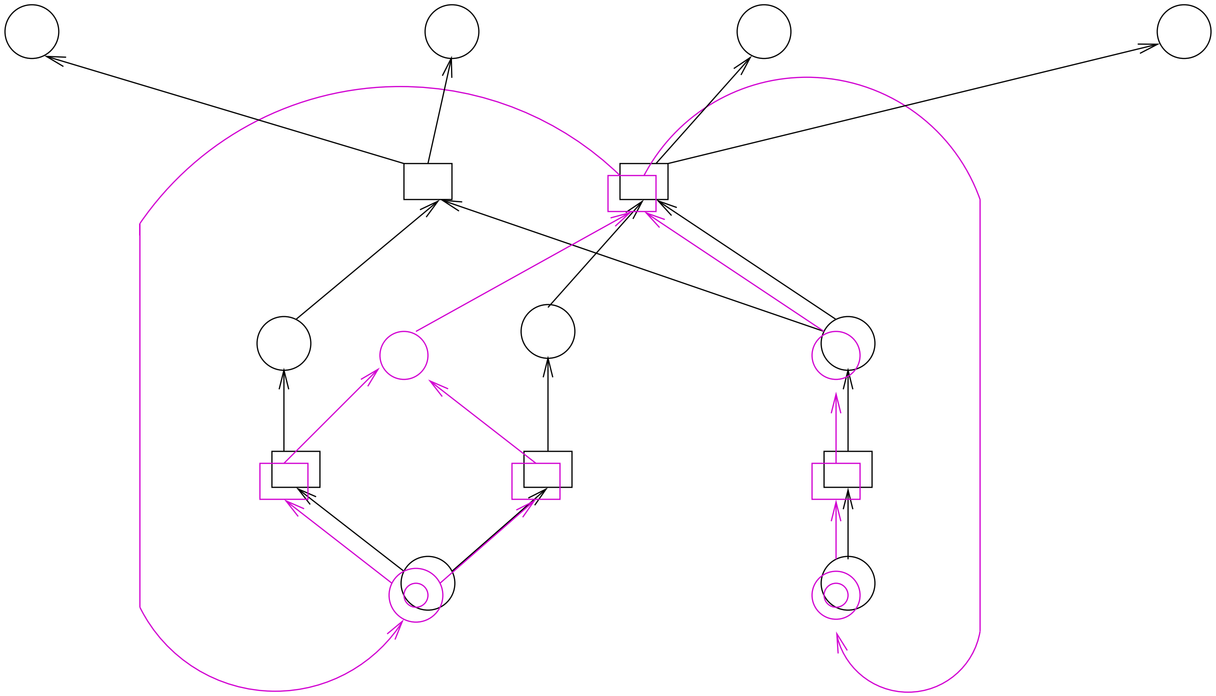


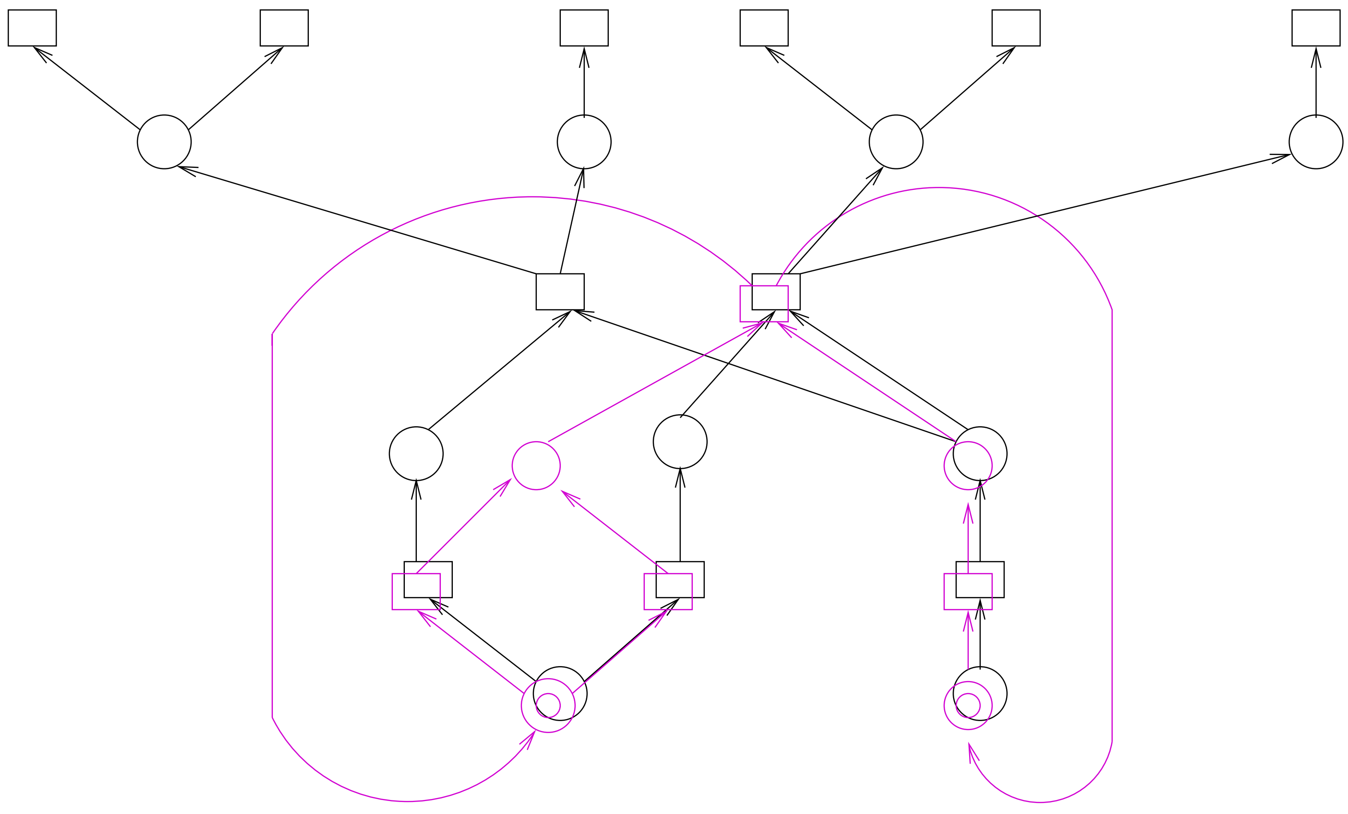


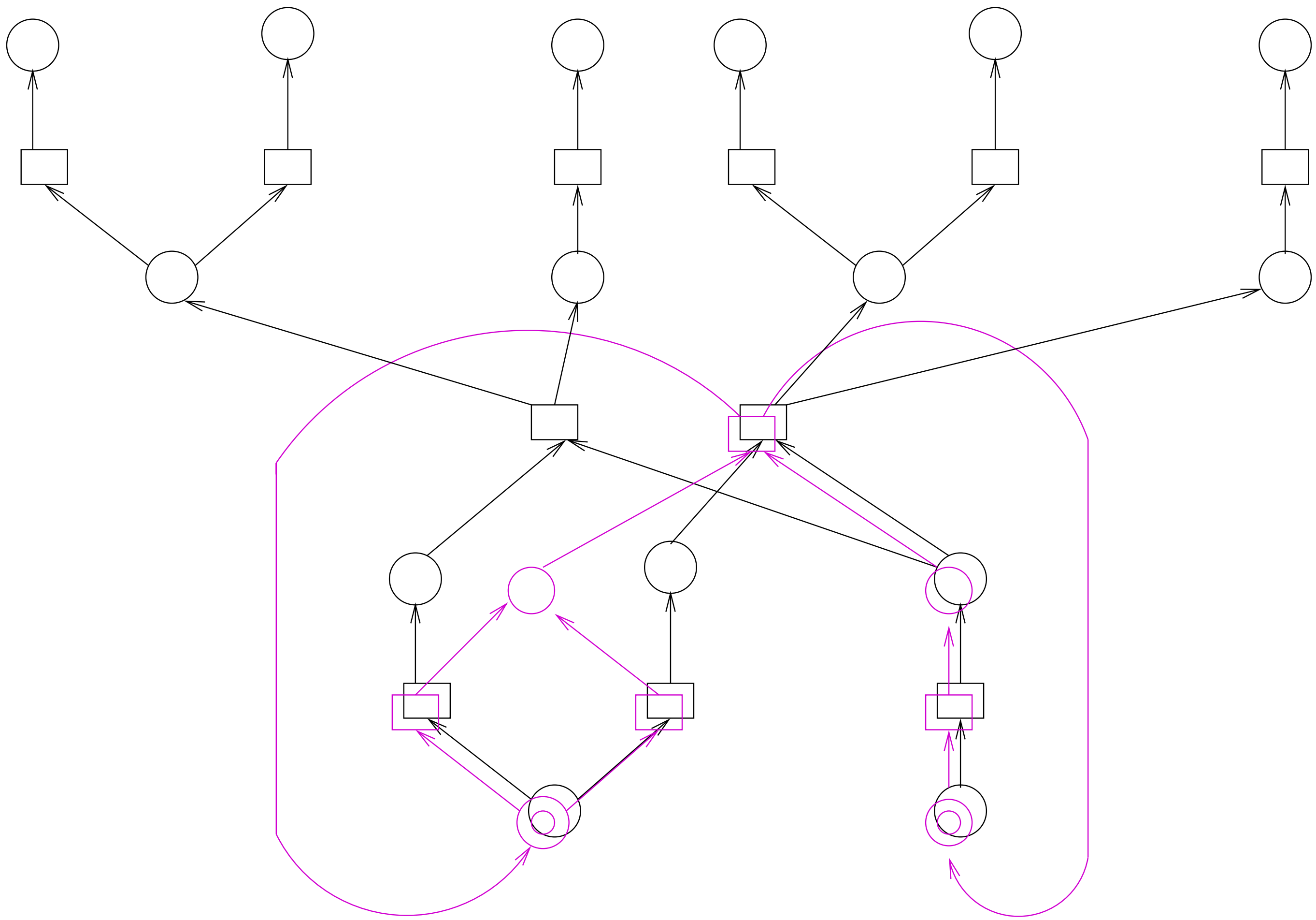


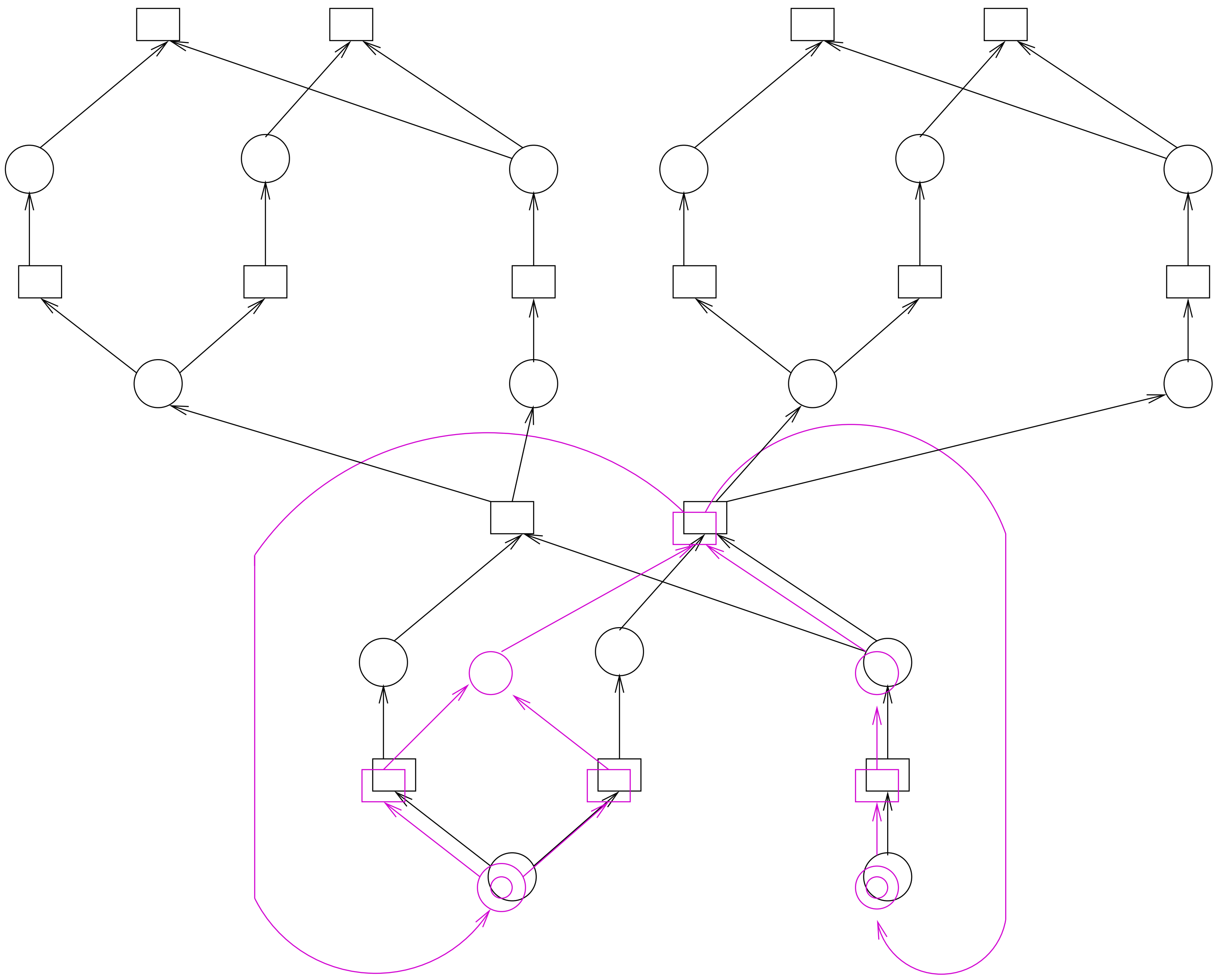


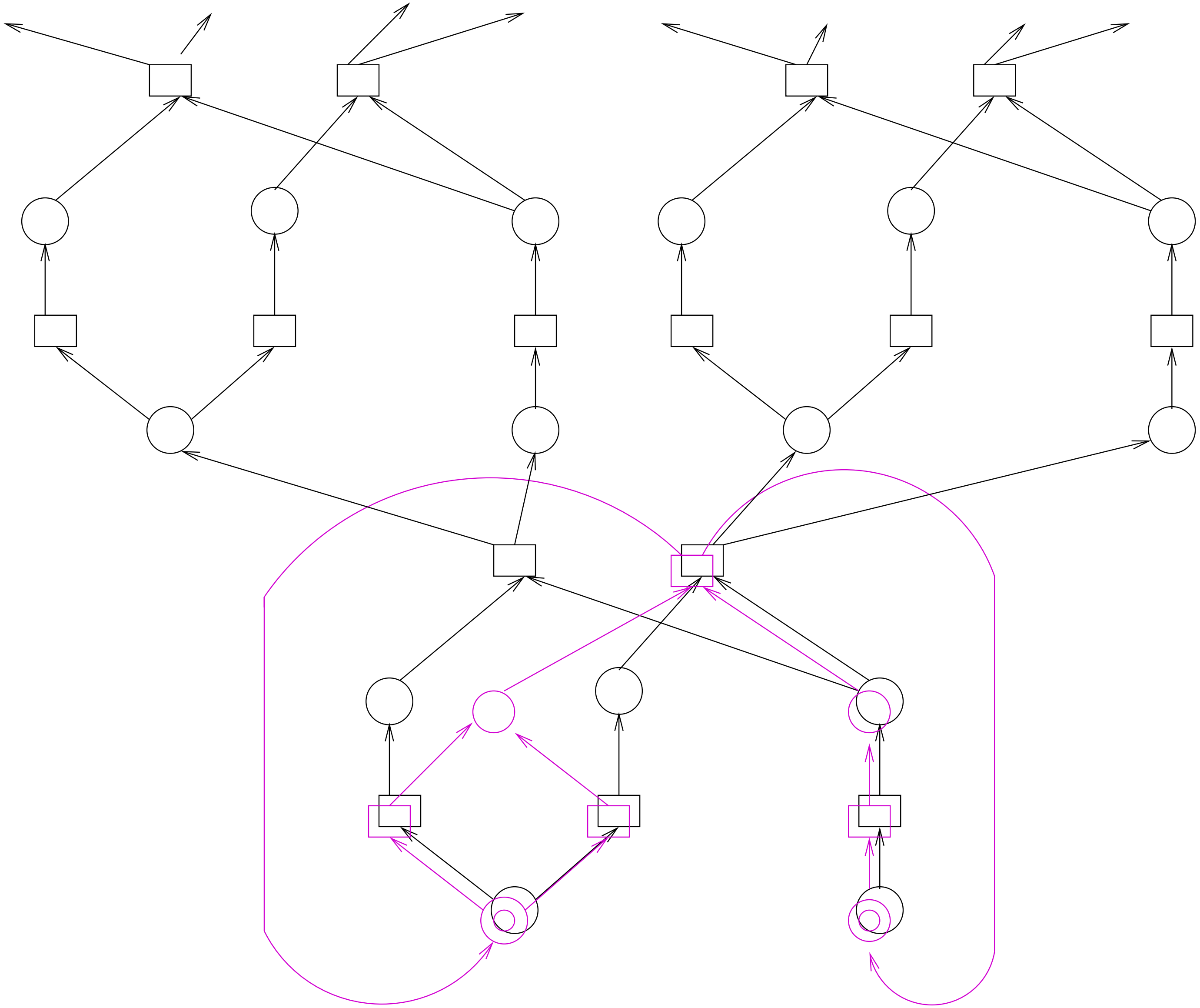


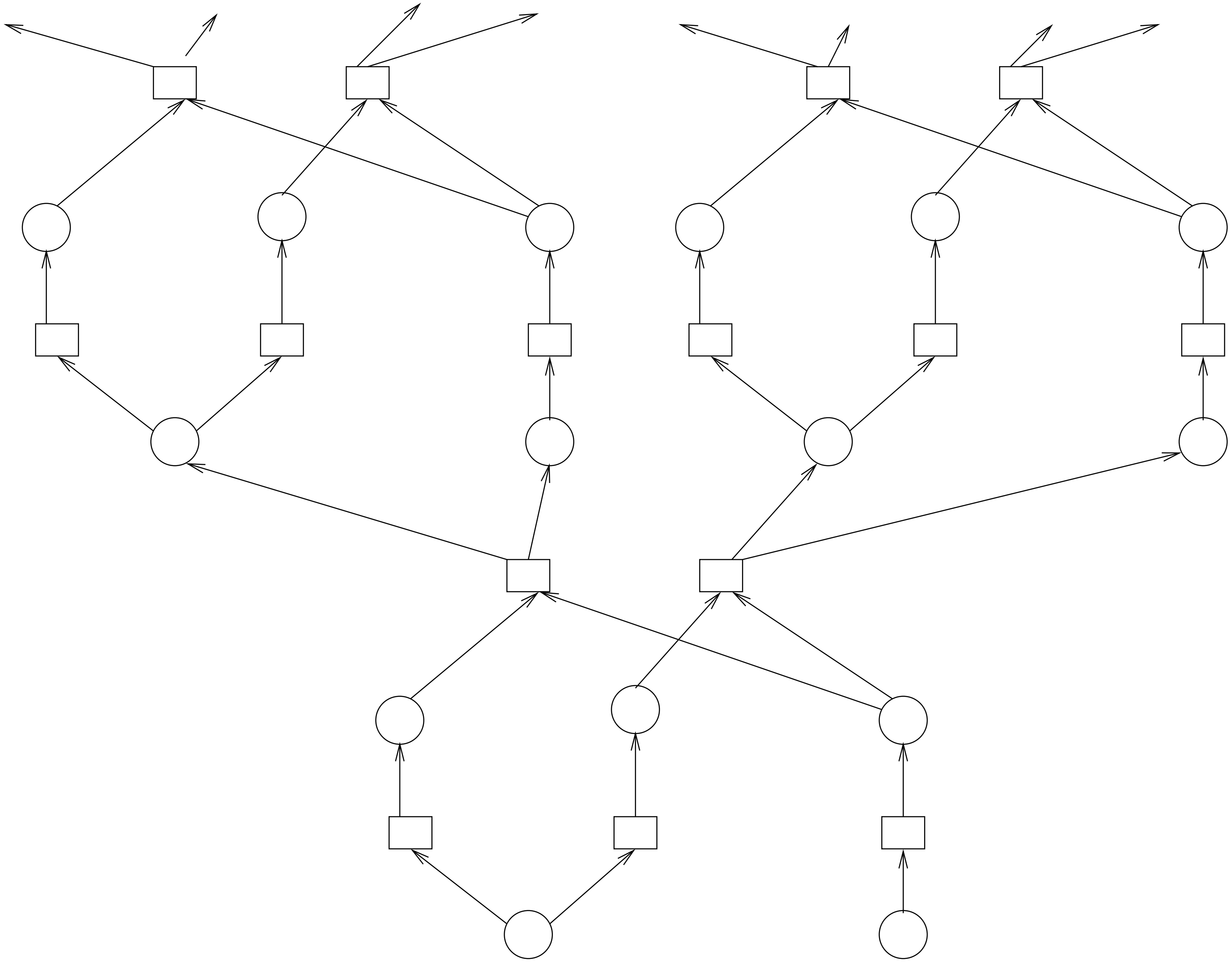


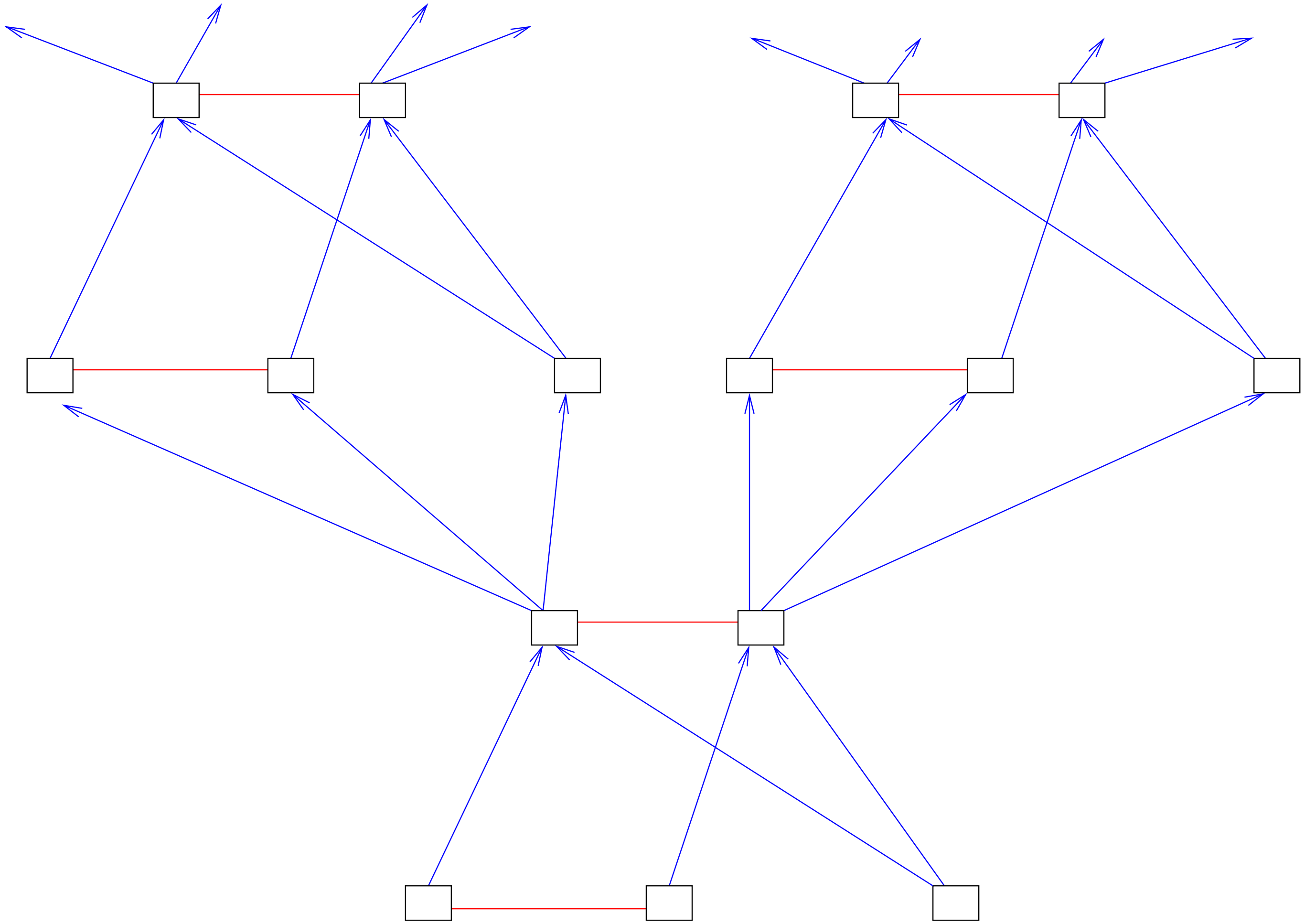












An event structure

Event structures with binary conflict

An *event structure with binary conflict* comprises $E = (\mid E \mid, \leq, \#)$, consisting of

- a set $\mid E \mid$, of *events*

- partially ordered by $\leq$, the *causal dependency relation*, and

- a binary, irreflexive, symmetric relation $\#$ on E, the *conflict relation*,

which satisfy

$$\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,$$

$$\text{if } e \geq e_0 \# e'_0 \leq e', \text{ then } e \# e'$$

Say e, e' are *concurrent* if $\neg(e \# e') \ \& \ e \not\leq e' \ \& \ e' \not\leq e$.

States of an event structure

The *configurations*, $\mathcal{C}^\infty(E)$, of an event structure E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall e, e' \in x.\ \neg(e \# e')$ and

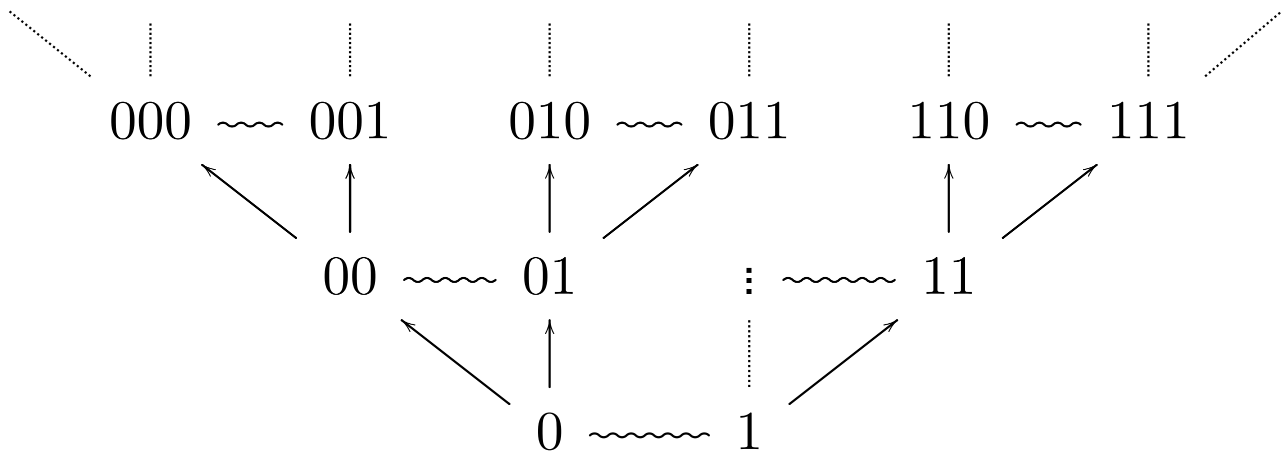
Down-closed: $\forall e, e'.\ e' \leq e \in x \Rightarrow e' \in x$.

For an event e the set $[e] =_{\mathrm{def}} \{e' \in E \mid e' \leq e\}$ is a configuration describing the whole causal history of the event e.

$x \subseteq x'$, $i.e.$ x is a sub-configuration of x', means that x is a sub-history of x'.

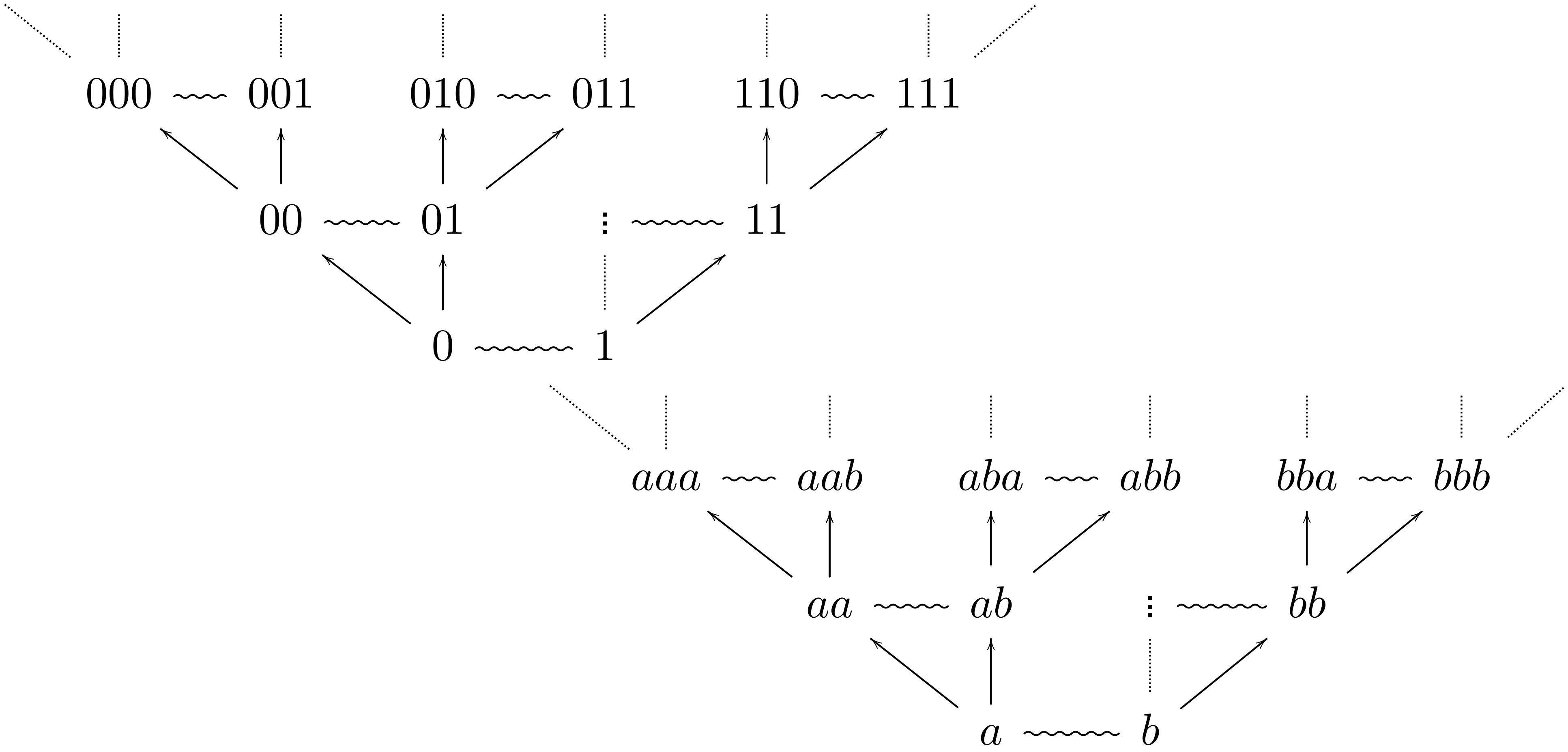
$(\mathcal{C}^\infty(E), \subseteq)$ is a domain. Write $\mathcal{C}(E)$ for the set of *finite* configurations.

Example: Streams as event structures

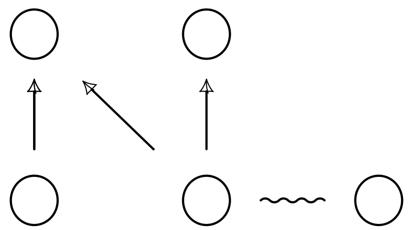


∼∼∼ conflict (inconsistency) —→ immediate causal dependency

Simple parallel composition



Another example

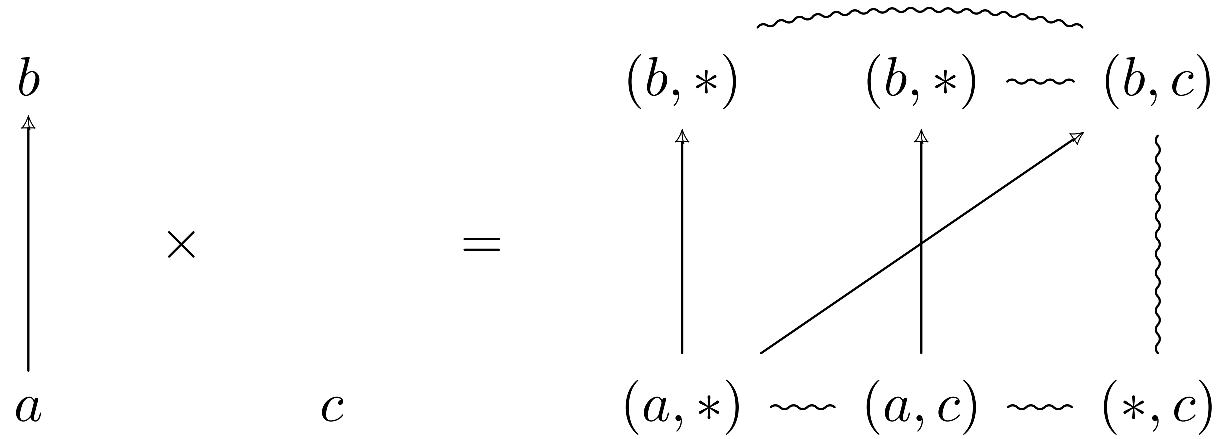


CCS operations on event structures?

$a.a.\mathbf{nil} \| \overline{a}.\mathbf{nil}$?

Product of event structures—an example

CCS parallel composition is derived from the product of event structures, $e.g.$



The duplication of events with common images under the projections, as in the two events carrying $(b,*)$ *can be troublesome!*
$\rightsquigarrow$ stable families, rigid families, ...

Convenient to use more general event structures based on consistency rather than binary conflict

Often in examples event structures have a binary conflict and we can take advantage of this in diagrams

Event structures

An *event structure* comprises $(E, \leq, \mathrm{Con})$, consisting of

- a set E, of *events*

- partially ordered by $\leq$, the *causal dependency relation*, and

- a nonempty family Con of finite subsets of E, the *consistency relation*,

which satisfy

$$\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,$$

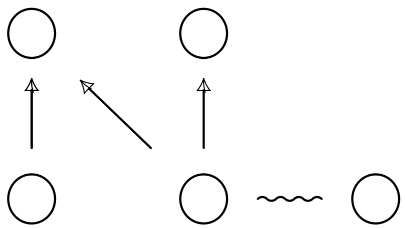
$$\{e\} \in \mathrm{Con} \text{ for all } e \in E,$$

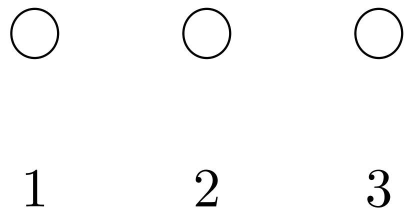
$$Y \subseteq X \in \mathrm{Con} \Rightarrow Y \in \mathrm{Con}, \quad \text{and}$$

$$X \in \mathrm{Con} \;\&\; e \leq e' \in X \Rightarrow X \cup \{e\} \in \mathrm{Con}.$$

Say e, e' are *concurrent* if $\{e, e'\} \in \mathrm{Con} \;\&\; e \not\leq e' \;\&\; e' \not\leq e$.

Examples





$$\text{Con} = \{\ \emptyset,\ \{1\}, \{2\}, \{3\},\ \{1,2\}, \{1,3\}, \{2,3\}\ \}$$

The example on the right shows that the event structures defined by consistency are more general than those with binary conflict

Configurations of an event structure

The *configurations*, $\mathcal{C}^\infty(E)$, of an event structure E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq_{\mathrm{fin}} x.\ X \in \mathrm{Con}$ and

Down-closed: $\forall e, e'.\ e' \leq e \in x \Rightarrow e' \in x$.

For an event e the set $[e] =_{\mathrm{def}} \{e' \in E \mid e' \leq e\}$ is a configuration describing the whole causal history of the event e.

$x \subseteq x'$, *i.e.* x is a sub-configuration of x', means that x is a sub-history of x'.

If E is countable, $(\mathcal{C}^\infty(E), \subseteq)$ is a Berry dI domain (and all such so obtained).
Finite configurations: $\mathcal{C}(E)$.

Maps of event structures

A **map** of event structures $f : E \to E'$ is a partial function $f : E \rightharpoonup E'$ such that for all $x \in \mathcal{C}(E)$

$$fx \in \mathcal{C}(E') \text{ and } e_1, e_2 \in x \ \& \ f(e_1) = f(e_2) \Rightarrow e_1 = e_2 \, .$$

Note that when f is total it restricts to a bijection $x \cong fx$, for any $x \in \mathcal{C}(E)$.

Maps *preserve concurrency*, and *locally reflect causal dependency:*

$$e_1, e_2 \in x \ \& \ f(e_1) \leq f(e_2) \text{ (both defined) } \Rightarrow e_1 \leq e_2 \, .$$

A total map is **rigid** when it preserves causal dependency.

Computation paths

A **computation path**: a partial order $p = (| p |, \leq_p)$ for which the set $\{e' \in | p | \ \mid \ e' \leq_p e\}$ is finite for all $e \in | p |$.

Say a path p is **prime** if it has a top element $top(p)$.

Rigid inclusion between paths $p = (| p |, \leq_p)$ and $q = (| q |, \leq_q)$:

$$p \hookrightarrow q \text{ iff } | p | \subseteq | q | \ \& \ \forall e \in | p |, e' \in | q |. \ e' \leq_p e \iff e' \leq_q e \, .$$

Rigid families

A **rigid family** is a non-empty set of *finite* paths $\mathcal{R}$ for which

$$p \hookrightarrow q \in \mathcal{R} \Rightarrow p \in \mathcal{R} \,.$$

Example. $\mathcal{C}(E)$ with configurations inheriting order from event structure E, ordered by inclusion.

But rigid families are more general ...

Event structures from rigid families

A rigid family $\mathcal{R}$ determines an event structure $\mathrm{Pr}(\mathcal{R})$ whose order of finite configurations is isomorphic to $(\mathcal{R}, \hookrightarrow)$.

The *event structure* $\mathrm{Pr}(\mathcal{R})$ has *events* the subset P of prime paths of $\mathcal{R}$;
causal dependency given by rigid inclusion; and
consistency by compatibilty w.r.t. rigid inclusion.

There is an order isomorphism $\varphi_{\mathcal{R}} : (\mathcal{R}, \hookrightarrow) \cong (\mathcal{C}(\mathrm{Pr}(\mathcal{R})), \subseteq)$ given by

$$\varphi_{\mathcal{R}}(q) = \{p \in P \mid p \hookrightarrow q\}$$

for $q \in \mathcal{R}$. Its inverse:

$$\theta_{\mathcal{R}}(x) = \bigcup x \,,$$

on $x \in \mathcal{C}(\mathrm{Pr}(\mathcal{R}))$.

The product of event structures A and B

The set $| A | \times_* | B |$ is defined as
$\{(a,*) \mid a \in | A |\} \cup \{(a,b) \mid a \in | A |,\ b \in | B |\} \cup \{(*,b) \mid b \in | B |\}$.
It has partial projections π_1, π_2.

$A \times B =_{\mathrm{def}} \Pr(\mathcal{R})$ where the rigid family $\mathcal{R}$ satisfies: $p \in \mathcal{R}$ iff

(i) $| p | \subseteq | A | \times_* | B |$;

(ii) $\pi_1 | p | \in \mathcal{C}(A)$ and $\pi_2 | p | \in \mathcal{C}(B)$ and the projections are locally injective on $| p |$ *i.e.*, $\forall c, c' \in | p |.\ \pi_1(c) = \pi_1(c') \Rightarrow c = c'$ and
$\forall c, c' \in | p |.\ \pi_2(c) = \pi_2(c') \Rightarrow c = c'$;

(iii) $\leq_p$ is the least transitive relation such that
$c \leq_p c'$ if $\pi_1(c) \leq_A \pi_1(c')$ or $\pi_2(c) \leq_B \pi_2(c')$.

Augmentations

Let E be an event structure with configuration x.

A path $p = (| p |, \leq_p)$ is an **augmentation** of x iff $| p | = x$ and

$$\forall e \in | p |, e' \in | E |.\ e' \leq_E e \Rightarrow e' \leq_p e\,.$$

Define a partial operation on augmentations

$$\wedge : \mathrm{Aug}(E) \times \mathrm{Aug}(E) \rightharpoonup \mathrm{Aug}(E)$$

by taking

$$p \wedge q = \begin{cases} (| p |, \leq) & \text{if } | p | = | q | \ \&\ \leq = (\leq_p \cup \leq q)^* \text{ is a finitary po,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Games and strategies as event structures

Event structures with polarity

An event structure with polarity comprises (A, pol) where A is an event structure with a polarity function $pol_A : A \to \{+, -, 0\}$ ascribing a polarity $+$ (Player), $-$ (Opponent) or 0 (neutral) to its events.

A **game** shall be represented by an event structure with polarity in which no moves are neutral. (We can add winning conditions, payoff, ... later.)

Notation. For configurations x and y, write $x \subseteq^- y$ to mean inclusion in which all the intervening events are moves of *Opponent*. Write $x \subseteq^+ y$ for inclusion in which the intervening events are *neutral or moves of Player*.

Operations on games

The **dual**, $A^\perp$, of a game A, comprises the same underlying event structure as A but with a reversal of polarities.

A strategy in A will be a strategy for Player; a strategy for Opponent — a counterstrategy — a strategy in $A^\perp$*.*

The **simple parallel composition** of games, and generally event structures with polarity A and B, simply juxtaposes them: $A \| B$.

A strategy from a game A to a game B will be a strategy in $A^\perp \| B$*.*

Plays in a game A

A **play** (of Player) in A, an event structure with polarity, is an augmentation $p = (|\, p\,|, \leq_p)$ of $|\, p\,| \in \mathcal{C}^\infty(A)$ which is *courteous*:

$$\forall a, a' \in |\, p\,|.\ a' \rightarrowtriangle_p a \ \&\ pol_A(a') = + \text{ or } pol_A(a) = - \Rightarrow a' \rightarrowtriangle_A a\,.$$

Write $\mathrm{Plays}(A)$ for the set of plays in A.

If A is a game, the only augmentations allowed of a play p additional to the immediate causal dependency of A are those of the form $\boxminus \rightarrowtriangle_p \boxplus$.

Strategies

Let A be an event structure with polarity.

A **bare strategy** in A is a rigid family $\sigma \subseteq \mathrm{Plays}(A)$ which is

receptive, $p \in \sigma \ \& \ |\, p \,| \subseteq^- x \in \mathcal{C}(A) \quad \Rightarrow \quad \exists q \in \sigma.\ p \hookrightarrow q \in \sigma \ \& \ |\, q \,| = x$.

(Note q is unique by courtesy.)

Write $\sigma : A$.

When A is a game, say σ is a **strategy**.

Strategies as maps of event structures

Proposition. If $\sigma : A$ then $top : \mathrm{Pr}(\sigma) \to A$ is a total map of event structures which preserves polarity and satisfies

- *courtesy*, $s' \rightarrowtriangle s$ and $pol(s') = +$ or $pol(s) = -$ in $\mathrm{Pr}(\sigma)$ implies $f_\sigma(s') \rightarrowtriangle_A f_\sigma(s)$ in A, and

- *receptivity*, $f_\sigma x \subseteq^- y$ in $\mathcal{C}(A)$, for $x \in \mathcal{C}(\mathrm{Pr}(\sigma))$, implies there is a unique $x' \in \mathcal{C}(\mathrm{Pr}(\sigma))$ such that $f_\sigma x' = y$.

When A is a game such total maps satisfying courtesy and receptivity are the concurrent strategies of Rideau and W, precisely those total maps preserving polarity for which the copycat strategy is identity w.r.t. composition of strategies. The strategies of this course correspond to the "rigid images" of R-W's concurrent strategies; for which configurations map injectively to augmentations of A.

From maps to strategies

Let $f : S \to A$ be a total map of event structures which preserves polarity. Define $\sigma(f)$ to be its *rigid image*, the rigid family

$$\sigma(f) = \{(fx, \leq_x^f) \mid x \in \mathcal{C}(S)\}$$

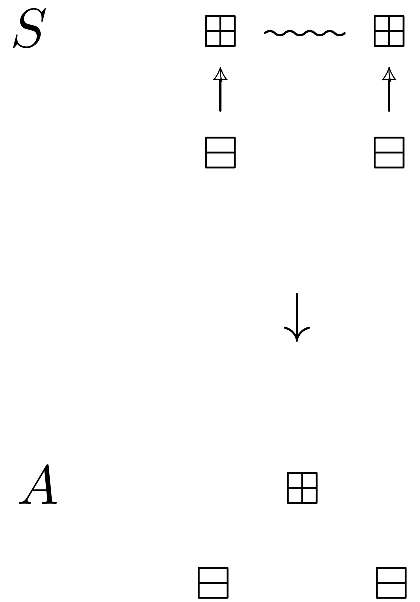
where

$$a' \leq_x^f a \iff \exists s', s \in x.\ a' = f(s')\ \&\ a = f(s)\ \&\ s' \leq_S s\,,$$

for $x \in \mathcal{C}(S)$.

Proposition.
The rigid image is a strategy $\sigma(f) : A$ if f is receptive and courteous.
(The converse may fail: we may have $\sigma(f) : A$ with f not receptive.)

Example of a strategy presented as a map



configurations of S = *"states of play"*

configurations of A = *"positions of the game"*

The strategy: answer either move of Opponent by the Player move.

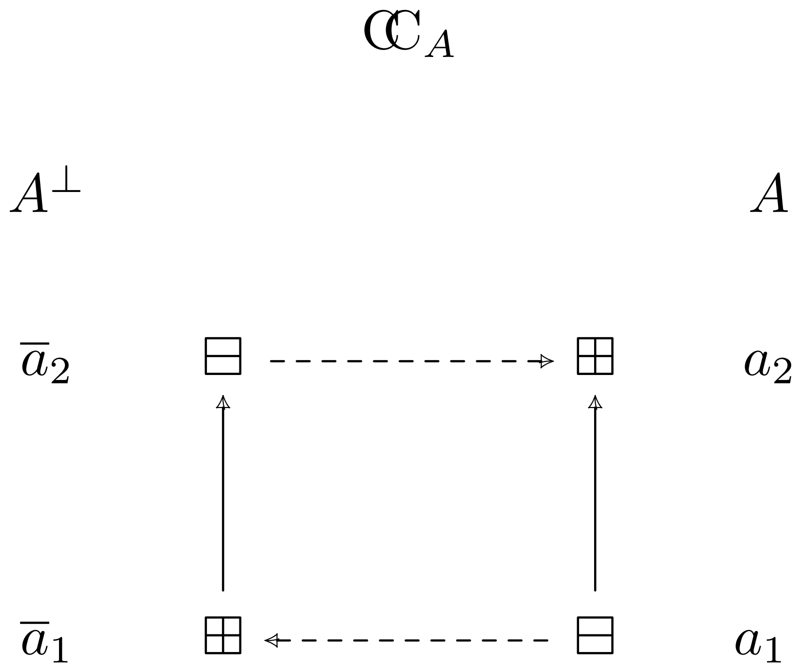
Strategies between games

Let A and B be games.

A strategy **from** A **to** B is a strategy $\sigma : A^{\perp} \| B$.

Composition - the idea:
Given $\sigma : A^{\perp} \| B$ and $\tau : B^{\perp} \| C$ we compose them by letting them play against each other in the shared game B.

Example: copycat strategy from A to A



Copycat in general

In copycat, $\text{\c{c}}_A : A^\perp \| A$, $\text{\c{c}}_A = \{(x, \leq_{\mathbb{C}_A} \restriction x) \mid x \in \mathcal{C}(\mathbb{C}_A)\}$ where

$\mathbb{C}_A$ has the same events and polarity as $A^\perp \parallel A$ but with causal dependency $\leq_{\mathbb{C}_A}$ given as the transitive closure of the relation

$$\leq_{A^\perp \| A} \cup \{(\overline{c}, c) \mid c \in A^\perp \parallel A \;\;\&\;\; pol_{A^\perp \| A}(c) = +\}$$

where $\overline{c} \leftrightarrow c$ is the natural correspondence between $A^\perp$ and A. A finite subset is consistent iff its down-closure is consistent in $A^\perp \| A$. Then,

$$x \in \mathcal{C}(\mathbb{C}_A) \text{ iff } x \in \mathcal{C}(A^\perp \parallel A) \;\&\; \forall c \in x.\; pol_{A^\perp \| A}(c) = + \;\Rightarrow\; \overline{c} \in x\,.$$

The Scott order

Defining a partial order — *the Scott order* — on configurations of A

$$y \sqsubseteq_A x \;\text{ iff }\; y \supseteq^- \cdot \subseteq^+ \cdot \supseteq^- \cdots \supseteq^- \cdot \subseteq^+ x$$

we obtain a factorization system $((\mathcal{C}(A), \sqsubseteq_A), \supseteq^-, \subseteq^+)$, *i.e.* $\exists! z.$

$$\begin{array}{ccc} & & x \\ & \sqsubseteq & \cup\!\!\mid^{+} \\ y & \supseteq^- & z\,. \end{array}$$

Proposition $z \in \mathcal{C}(\mathbb{C}_A)$ iff $z_2 \sqsubseteq_A z_1$.

The Scott order on configurations, and elements of rigid families, plays a surprising "undercover" role in the development of distributed/concurrent games.

Interaction of strategies — simple case

Let A be a game. Let $\sigma : A$ be a strategy and $\tau : A^{\perp}$ a counterstrategy.

Their **interaction** is given by

$$\tau \circledast \sigma = \{p \wedge q \mid p \in \sigma \;\&\; q \in \tau \;\&\; p \wedge q \text{ is defined}\}\,.$$

Proposition. The interaction is a bare strategy $\tau \circledast \sigma : A^0$ in which all moves are neutral.

Interaction in general

Let A, B, C be games.

$$\circledast : \mathrm{Plays}(B^{\perp} \| C) \times \mathrm{Plays}(A^{\perp} \| B) \rightharpoonup \mathrm{Plays}(A^{\perp} \| B^{0} \| C)$$

acts so

$$q \circledast p =_{\mathrm{def}} (p \| y_C) \wedge (x_{A^{\perp}} \| q)$$

where

$$|\, p \,| = x_{A^{\perp}} \| x_B \quad \text{and} \quad |\, q \,| = y_{B^{\perp}} \| y_C \,.$$

Let $\sigma : A^{\perp} \| B$ and $\tau : B^{\perp} \| C$. Define their **interaction** by

$$\tau \circledast \sigma = \{ q \circledast p \ \mid \ p \in \sigma \ \& \ q \in \tau \ \& \ q \circledast p \text{ is defined} \} : A^{\perp} \| B^{0} \| C \,.$$

Composition of strategies

Let A, B, C be games. Define the **projection**

$$(\_)\!\downarrow : \mathrm{Plays}(A^\perp \| B^0 \| C) \to \mathrm{Plays}(A^\perp \| C)\,,$$

of a play p in $A^\perp \| B^0 \| C$,with $\mid p \mid = x_{A^\perp} \| x_B \| x_C$, to a play $p\!\downarrow$ in $A^\perp \| C$, to be the restriction of the order on p to the set $x_{A^\perp} \| x_C$.

For plays $p \in \mathrm{Plays}(A^\perp \| B)$ and $q \in \mathrm{Plays}(B^\perp \| C)$, define

$$q \odot p =_{\mathrm{def}} (q \circledast p)\!\downarrow\ .$$

Let $\sigma : A^\perp \| B$ and $\tau : B^\perp \| C$. Define their **composition** by

$$\tau \odot \sigma = \{(q \odot p)\!\downarrow \mid\ p \in \sigma \ \&\ q \in \tau \ \&\ q \odot p \text{ is defined}\}\,.$$

A category of games and strategies

Theorem. Composition of strategies is associative and has identity the copycat strategy. *I.e.*, taking objects to be games and arrows from a game A to a game B to be strategies in the game $A^\perp \| B$, with composition the composition of strategies, yields a category.

In fact, the category is *cpo-enriched*: inclusion between strategies is respected by composition and forms a cpo with bottom.

Often write $\sigma : A \mathrel{+\!\!\!\to} B$ when $\sigma : A^\perp \| B$.

If $\sigma : A \mathrel{+\!\!\!\to} B$ and $\tau : B \mathrel{+\!\!\!\to} C$ then $\tau \odot \sigma : A \mathrel{+\!\!\!\to} C$.

Deterministic strategies

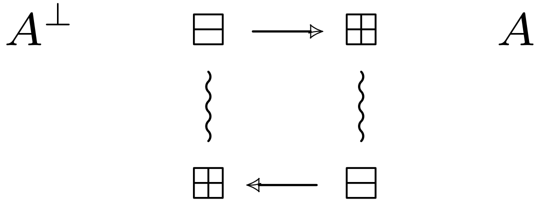
Let A be an event structure with polarity. A bare strategy $\sigma : A$ is *deterministic* iff

$$p \hookrightarrow^{+} q \;\&\; p \hookrightarrow r \text{ in } \sigma \Rightarrow \exists s \in \sigma.\, q \hookrightarrow s \;\&\; r \hookrightarrow s\,.$$

The interaction of deterministic bare strategies is deterministic.
The composition of deterministic strategies is deterministic.

Nondeterministic copycats

Take A to consist of two events, one $+$ve and one $-$ve event, inconsistent with each other $\boxplus \rightsquigarrow \boxminus$. The construction $\mathbb{CC}_A$:



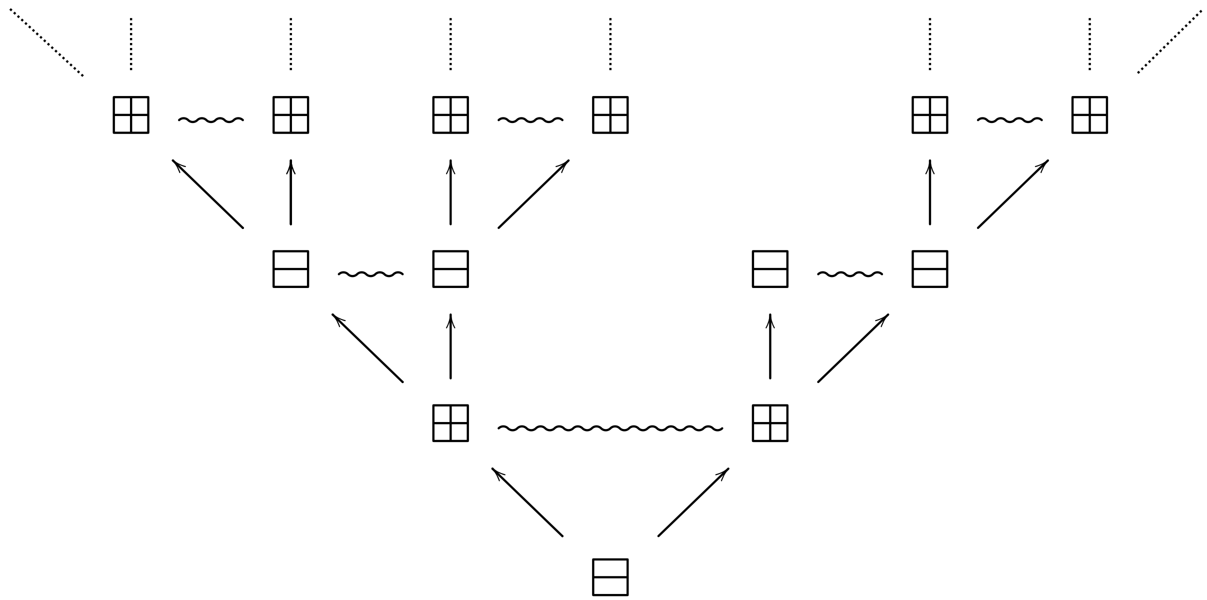
Recall æ_A is built from $\mathcal{C}(\mathbb{CC}_A)$. To see æ_A is not deterministic, take x to be the singleton set consisting *e.g.* of the $-$ve event on the left and s, s' to be the $+$ve and $-$ve events on the right.

Lemma. Let A be an event structure with polarity. The copycat strategy $\text{\ae}_A$ is deterministic iff A satisfies

$$\begin{aligned} \forall x \in \mathcal{C}(A), a, a' \in |\, A \,|.\ & x \cup \{a\} \in \mathcal{C}(A) \ \&\ x \cup \{a'\} \in \mathcal{C}(A)\ \& \\ & pol_A(a) = + \ \&\ pol_A(a') = - \\ & \Rightarrow x \cup \{a, a'\} \in \mathcal{C}(A)\,. \qquad \text{(\textbf{Race-free})} \end{aligned}$$

$\rightsquigarrow$ **A subcategory of race-free games and deterministic strategies.**

Example: a tree-like game



~~~ conflict (inconsistency) ⟶ immediate causal dependency

⊞ Player move ⊟ Opponent move
~~~

Special cases

Simple games *"Simple games"* of game semantics arise when we restrict $\mathbf{Games}$ to objects and deterministic strategies which are 'tree-like'—alternating polarities, with conflicting branches, beginning with opponent moves.

Stable spans and stable functions The sub-bicategory of $\mathbf{Games}$ where the events of games are purely $+$ve is equivalent to the bicategory of 'stable spans' used in nondeterministic dataflow; feedback is given by trace.
When deterministic we obtain a sub-bicategory equivalent to Berry's **dI-domains and stable functions**.

Closure operators A deterministic strategy in A determines a closure operator on $\mathcal{C}^\infty(A)^\top$ of Abramsky and Melliès.

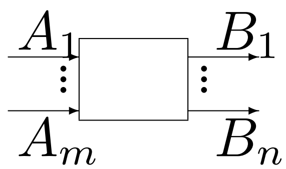
Receptive ingenuous strategies Deterministic concurrent strategies coincide with the *receptive* ingenuous strategies of and Melliès and Mimram.

A language for strategies

Types: Games A, B, C, ... with operations $A^{\perp}$, $A \| B$, sums $\sum_{i \in I} A_i$, recursively-defined types, ...

A **term**

$$x_1 : A_1, \cdots, x_m : A_m \vdash \ t \ \dashv y_1 : B_1, \cdots, y_n : B_n \ ,$$



denotes a strategy from $A_1 \| \cdots \| A_m$ to $B_1 \| \cdots \| B_n$.

Idea: t denotes a strategy as a map $S \to \vec{A}^{\perp} \| \vec{B}$ and through this its rigid image, a strategy $\sigma : \vec{A}^{\perp} \| \vec{B}$. Supports duality of strategies: $\sigma : A^{\perp} \| B \cong (B^{\perp})^{\perp} \| A^{\perp}$

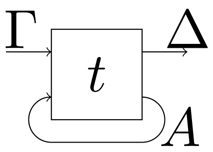
The term t describes witnesses, finite configurations of S, to a relation between finite configurations $\vec{x}$ of $\vec{A}$ and $\vec{y}$ of $\vec{B}$.

Copycat $x : A \vdash\ y \sqsubseteq_A x\ \dashv y : A$ and other terms "wiring" causal dependencies.

Composition $\dfrac{\Gamma \vdash t \dashv \Delta \qquad \Delta \vdash u \dashv \mathrm{H}}{\Gamma \vdash \exists \Delta. [\, t \parallel u \,] \dashv \mathrm{H}}$

Sum $\Sigma_{i \in I} t_i$ **Conjunction** $t_1 \wedge t_2$

Feedback



Recursion $\mu\, x : A.\, t$ **Duplication** $x : A \vdash \delta_A \dashv y : A, z : A$ provided ...

The above is linear; require games with symmetry and (co)monads for nonlinearity. Supports an operational semantics via bare strategies. Extends with probabilities ...

From strategies to probabilistic strategies

A strategy:

$$\sigma : A$$

Aim

(1) To endow σ, or equivalently $\mathrm{Pr}(\sigma)$, with probability, while

(2) taking account of the fact that in a strategy Player can't be aware of the probabilities assigned by Opponent. (*E.g.* in 'Matching pennies')

Causal independence between Player and Opponent moves will entail their probabilistic independence. Equivalently, probabilistic dependence of Player on Opponent moves will presuppose their causal dependence.

Probabilistic event structures

A **probabilistic event structure** comprises an event structure $E = (E, \leq, \mathrm{Con})$ together with a *(normalized) continuous valuation*, *i.e.* a function w from the Scott open subsets of configurations $\mathcal{C}^\infty(E)$ to $[0, 1]$ which is

(normalized) $w(\mathcal{C}^\infty(E)) = 1$ **(strict)** $w(\emptyset) = 0$

(monotone) $U \subseteq V \Rightarrow w(U) \leq w(V)$

(modular) $w(U \cup V) + w(U \cap V) = w(U) + w(V)$

(continuous) $w(\bigcup_{i \in I} U_i) = \sup_{i \in I} w(U_i)$ for *directed* unions $\bigcup_{i \in I} U_i$.

Intuition: $w(U)$ is the probability of the result being in U.
A cts valuation extends to a probability measure on Borel sets of configurations.

A workable characterization: A **probabilistic event structure** comprises an event structure E with a *configuration-valuation* $v : \mathcal{C}(E) \to [0,1]$ which satisfies

(normalized) $v(\emptyset) = 1$ and

(non −ve drop) $d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0,$ for all $n \in \omega$, and $y \subseteq x_1, \cdots, x_n$ in $\mathcal{C}(E)$.

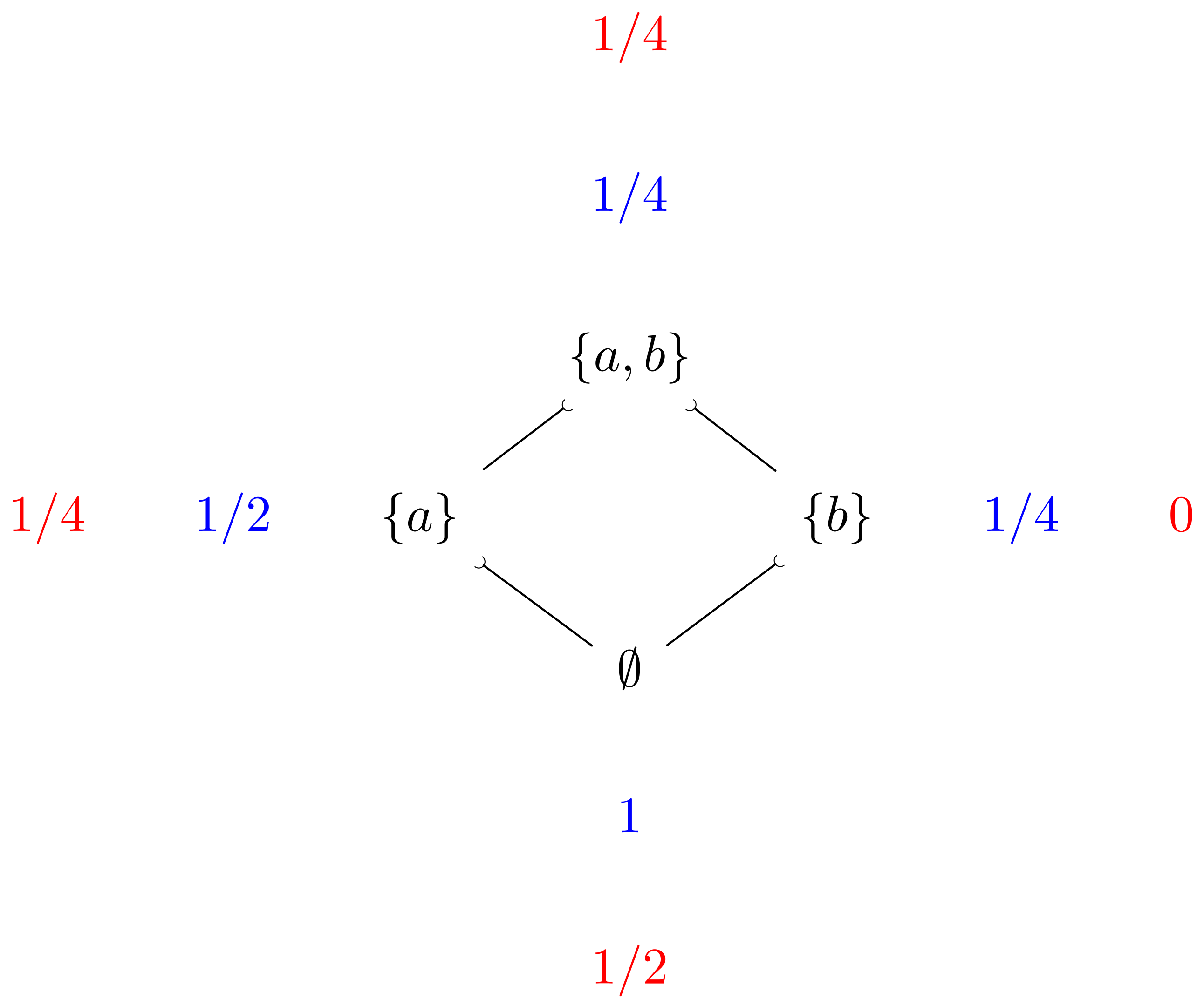
For $y \subseteq x_1, \cdots, x_n$ in $\mathcal{C}(E)$,

$$d_v^{(n)}[y; x_1, \cdots, x_n] =_{\mathrm{def}} v(y) - \sum_I (-1)^{|I|+1} v(\bigcup_{i \in I} x_i)\,,$$

—the index I ranges over $\emptyset \neq I \subseteq \{1, \cdots, n\}$ s.t. $\{x_i \mid i \in I\}$ is compatible.
($\textit{Sufficient to check the `drop condition' for } y \longrightarrow\!\!\!\!\subset x_1, \cdots, x_n$)

Theorem. Continuous valuations restrict to configuration-valuations.
A configuration-valuation extends to a unique continuous valuation on open sets, and that to a unique probabilistic measure on Borel subsets of configurations.
(*The result holds in greater generality, for Scott domains*)

Example



Probabilistic event structure with polarities

Let E be an event structure in which (not necessarily all) events carry $+/-$.
Write $x \subseteq^p y$ if $x \subseteq y$ and no event in $y \setminus x$ has polarity $-$.

Now, a **configuration-valuation** is a function $v : \mathcal{C}(E) \to [0,1]$ for which

$v(\emptyset) = 1$, $\quad x \subseteq^- y \Rightarrow v(x) = v(y)$, for all $x, y \in \mathcal{C}(E)$,

and the "drop condition"

$$d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$$

for all $n \in \omega$ and $y \subseteq^p x_1, \cdots, x_n$ in $\mathcal{C}(E)$.
(*Sufficient to check the 'drop condition' for* $y \longrightarrow\!\subset^p x_1, \cdots, x_n$)

A **probabilistic event structure with polarity** comprises E an event structure with polarity together with a configuration-valuation $v_E : \mathcal{C}(E) \to [0,1]$.

Probabilistic strategies

Assume games are *race-free*, $i.e.$ there is no immediate conflict between events of opposite polarity.

A **probabilistic strategy** in A comprises a strategy $\sigma : A$ with a valuation $v : \sigma \cong \mathcal{C}(\mathrm{Pr}(\sigma)) \to [0,1]$.

A race-free game A has a **probabilistic copy-cat** by taking $v_{\text{æ}_A}$ constantly 1 —this is a valuation as æ_A is deterministic for race-free A.

The **interaction** and **composition** valuations:

$$v_{\tau \circledast \sigma}(r) = \sum \{v_\tau(q).v_\sigma(p) \mid q \circledast p = r\} \text{ for } r \in \tau \circledast \sigma \text{ and}$$

$$v_{\tau \odot \sigma}(r) = \sum \{v_\tau(q).v_\sigma(p) \mid p, q \text{ minimum s.t. } q \odot p = r\} \text{ for } r \in \tau \odot \sigma\,.$$

$\rightsquigarrow$ *a category of probabilistic strategies on race-free games*

A special case of composition without hiding: play-off

Given a probabilistic strategy $v_\sigma, \sigma : A$ and counter-strategy $v_\tau, \tau : A^\perp$ we obtain a probabilistic bare strategy $\tau \circledast \sigma : A^0$ with valuation

$$v_{\tau \circledast \sigma} : \tau \circledast \sigma \to [0,1]\,.$$

Via $\mathcal{C}(\mathrm{Pr}(\tau \circledast \sigma)) \cong \tau \circledast \sigma$ this makes $\mathrm{Pr}(\tau \circledast \sigma)$ a probabilistic event structure so generates a measure μ on $\mathcal{C}^\infty(A)$.

Adding **pay-off** as a random variable from $\mathcal{C}^\infty(A)$ get **expected pay-off** as the Lebesgue integral

$$\int_{x \in \mathcal{C}^\infty(A)} X(x)\ d\mu(x)\,.$$

Extensions

Winning conditions - determinacy

Coping with deadlock, a process language view

Imperfect information

Probabilistic and quantum strategies

Parallel causes - needed for a full account of probabilistic strategies

Continuous distributions - for probabilistic programming

Symmetry - for (co)monads and nonlinearity. Semantics of prog langs, applications

Determinacy and value theorems - programming optimal strategies

Learning strategies? ...