Lecture 9

STLC equations

take the form $\boxed{\Gamma \vdash s = t : A}$ where $\Gamma \vdash s : A$ and $\Gamma \vdash t : A$ are provable.

Such an equation is satisfied by the semantics in a ccc if $M[\![\Gamma \vdash s : A]\!]$ and $M[\![\Gamma \vdash t : A]\!]$ are equal **C**-morphisms $M[\![\Gamma]\!] \to M[\![A]\!]$.

Qu: which equations are always satisfied in any ccc?

Ans: $\beta\eta$-equivalence. . .

STLC $\beta\eta$-Equality

The relation $\boxed{\Gamma \vdash s =_{\beta\eta} t : A}$ (where Γ ranges over typing environments, s and t over terms and A over types) is inductively defined by the following rules:

STLC $\beta\eta$-Equality

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- β-conversions

$$\frac{\Gamma, x : A \vdash t : B \qquad \Gamma \vdash s : A}{\Gamma \vdash (\lambda x : A.\, t)s =_{\beta\eta} t[s/x] : B}$$

$$\frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : B}{\Gamma \vdash \mathtt{fst}(s\,,t) =_{\beta\eta} s : A} \qquad \frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : B}{\Gamma \vdash \mathtt{snd}(s\,,t) =_{\beta\eta} t : B}$$

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- η-conversions

$$\frac{\Gamma \vdash t : A \to B \qquad x \text{ does not occur in } t}{\Gamma \vdash t =_{\beta\eta} (\lambda x : A.\, t\, x) : A \to B}$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash t =_{\beta\eta} (\mathtt{fst}\, t, \mathtt{snd}\, t) : A \times B} \qquad \frac{\Gamma \vdash t : \mathtt{unit}}{\Gamma \vdash t =_{\beta\eta} () : \mathtt{unit}}$$

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- η-conversions
- congruence rules

$$\frac{\Gamma, x : A \vdash t =_{\beta\eta} t' : B}{\Gamma \vdash \lambda x : A.\, t =_{\beta\eta} \lambda x : A.\, t' : A \to B}$$

$$\frac{\Gamma \vdash s =_{\beta\eta} s' : A \to B \qquad \Gamma \vdash t =_{\beta\eta} t' : A}{\Gamma \vdash s\, t =_{\beta\eta} s'\, t' : B}$$

etc

STLC $\beta\eta$-Equality

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- β-conversions
- η-conversions
- congruence rules
- $=_{\beta\eta}$ is reflexive, symmetric and transitive

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t =_{\beta\eta} t : A} \qquad \frac{\Gamma \vdash s =_{\beta\eta} t : A}{\Gamma \vdash t =_{\beta\eta} s : A}$$

$$\frac{\Gamma \vdash r =_{\beta\eta} s : A \qquad \Gamma \vdash s =_{\beta\eta} t : A}{\Gamma \vdash r =_{\beta\eta} t : A}$$

STLC $\beta\eta$-Equality

Soundness Theorem for semantics of STLC in a ccc.
If $\Gamma \vdash s =_{\beta\eta} t : A$ is provable, then in any ccc

$$M[\![\Gamma \vdash s : A]\!] = M[\![\Gamma \vdash t : A]\!]$$

are equal **C**-morphisms $M[\![\Gamma]\!] \to M[\![A]\!]$.

Proof is by induction on the structure of the proof of $\Gamma \vdash s =_{\beta\eta} t : A$.

Here we just check the case of β-conversion for functions.

So suppose we have $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$. We have to see that

$$M[\![\Gamma \vdash (\lambda x : A.\, t)s : B]\!] = M[\![\Gamma \vdash t[s/x] : B]\!]$$

Suppose
$$M[\![\Gamma]\!] = X$$
$$M[\![A]\!] = Y$$
$$M[\![B]\!] = Z$$
$$M[\![\Gamma, x : A \vdash t : B]\!] = f : X \times Y \to Z$$
$$M[\![\Gamma \vdash s : A]\!] = g : X \to Z$$

Then
$$M[\![\Gamma \vdash \lambda x : A.\, t : A \mathbin{-\!\!>} B]\!] = \mathtt{cur}\, f : X \to Z^Y$$

and hence

$$\begin{aligned}
&M[\![\Gamma \vdash (\lambda x : A.\, t) s : B]\!] \\
&= \mathtt{app} \circ \langle \mathtt{cur}\, f\,, g \rangle \\
&= \mathtt{app} \circ (\mathtt{cur}\, f \times \mathtt{id}_Y) \circ \langle \mathtt{id}_X\,, g \rangle && \text{since } (a \times b) \circ \langle c\,, d \rangle = \langle a \circ c\,, b \circ d \rangle \\
&= f \circ \langle \mathtt{id}_X\,, g \rangle && \text{by definition of } \mathtt{cur}\, f \\
&= M[\![\Gamma \vdash t[s/x] : B]\!] && \text{by the Substitution Theorem}
\end{aligned}$$

as required.

The internal language of a ccc, **C**

- one ground type for each **C**-object X
- for each $X \in \mathbf{C}$, one constant f^X for each **C**-morphism $f : 1 \to X$ ("global element" of the object X)

The types and terms of STLC over this language usefully describe constructions on the objects and morphisms of **C** using its cartesian closed structure, but in an "element-theoretic" way.

For example...

Example

In any ccc **C**, for any $X, Y, Z \in \mathbf{C}$ there is an isomorphism

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Example

In any ccc **C**, for any $X, Y, Z \in \mathbf{C}$ there is an isomorphism
$$Z^{(X \times Y)} \cong (Z^Y)^X$$

which in the internal language of **C** is described by the terms

$$\diamond \vdash s : ((X \times Y) \to Z) \to (X \to (Y \to Z))$$
$$\diamond \vdash t : (X \to (Y \to Z)) \to ((X \times Y) \to Z)$$

where $\begin{cases} s & \triangleq \lambda f : (X \times Y) \to Z.\, \lambda x : X.\, \lambda y : Y.\, f\,(x\,,y) \\ t & \triangleq \lambda g : X \to (Y \to Z).\, \lambda z : X \times Y.\, g\,(\mathtt{fst}\,z)\,(\mathtt{snd}\,z) \end{cases}$

and which satisfy $\begin{cases} \diamond, f : (X \times Y) \to Z \vdash t(s\,f) =_{\beta\eta} f \\ \diamond, g : X \to (Y \to Z) \vdash s(t\,g) =_{\beta\eta} g \end{cases}$

Free cartesian closed categories

The Soundness Theorem has a converse—completeness.

In fact for a given set of ground types and typed constants there is a single ccc **F** (the free ccc for that language) with an interpretation function M so that $\Gamma \vdash s =_{\beta\eta} t : A$ is provable iff $M[\![\Gamma \vdash s : A]\!] = M[\![\Gamma \vdash t : A]\!]$ in **F**.

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- **F**-objects are the STLC types over the given set of ground types
- **F**-morphisms $A \to B$ are equivalence classes of STLC terms t satisfying $\diamond \vdash t : A \mathbin{->} B$ (so t is a *closed* term—it has no free variables) with respect to the equivalence relation equating s and t if $\diamond \vdash s =_{\beta\eta} t : A \mathbin{->} B$ is provable.
- identity morphism on A is the equivalence class of $\diamond \vdash \lambda x : A.\, x : A \mathbin{->} A$.
- composition of a morphism $A \to B$ represented by $\diamond \vdash s : A \mathbin{->} B$ and a morphism $B \to C$ represented by $\diamond \vdash t : B \mathbin{->} C$ is represented by $\diamond \vdash \lambda x : A.\, t(s\, x) : A \mathbin{->} C$.

Curry-Howard correspondence

Logic		**Type Theory**
propositions	$\leftrightarrow$	types
proofs	$\leftrightarrow$	terms

E.g. IPL *versus* STLC.

Curry-Howard for IPL *vs* STLC

Proof of $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$ in IPL

$$
\dfrac{
  \dfrac{\dfrac{}{\;\cdots\;}\,(\text{AX})}{\Phi \vdash \psi \Rightarrow \theta}\,(\text{WK})
  \qquad
  \dfrac{
    \dfrac{\dfrac{\dfrac{}{\;\cdots\;}\,(\text{AX})}{\;\cdots\;}\,(\text{WK})}{\Phi \vdash \varphi \Rightarrow \psi}\,(\text{WK})
    \qquad
    \dfrac{}{\Phi \vdash \varphi}\,(\text{AX})
  }{\Phi \vdash \psi}\,({\Rightarrow}\text{E})
}{
  \dfrac{\Phi \vdash \theta}{\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\,({\Rightarrow}\text{I})
}\,({\Rightarrow}\text{E})
$$

where $\Phi = \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$

Curry-Howard for IPL *vs* STLC

and a corresponding STLC term

$$
\dfrac{
  \dfrac{\dfrac{}{\cdots}\,(\text{AX})}{\Phi \vdash z : \psi \Rightarrow \theta}\,(\text{WK})
  \qquad
  \dfrac{
    \dfrac{\dfrac{\dfrac{}{\cdots}\,(\text{AX})}{\cdots}\,(\text{WK})}{\Phi \vdash y : \varphi \Rightarrow \psi}\,(\text{WK})
    \qquad
    \dfrac{}{\Phi \vdash x : \varphi}\,(\text{AX})
  }{\Phi \vdash y\, x : \psi}\,({\Rightarrow}\text{E})
}{
  \dfrac{\Phi \vdash z(y\, x) : \theta}{\diamond, y : \varphi \Rightarrow \psi, z : \psi \Rightarrow \theta \vdash \lambda x : \varphi .\, z(y\, x) : \varphi \Rightarrow \theta}\,({\Rightarrow}\text{I})
}\,({\Rightarrow}\text{E})
$$

where $\Phi = \diamond, y : \varphi \Rightarrow \psi, z : \psi \Rightarrow \theta, x : \varphi$

Curry-Howard-Lawvere/Lambek correspondence

Logic		**Type Theory**		**Category Theory**
propositions	$\leftrightarrow$	types	$\leftrightarrow$	objects
proofs	$\leftrightarrow$	terms	$\leftrightarrow$	morphisms

E.g. IPL *versus* STLC *versus* CCCs

Curry-Howard-Lawvere/Lambek correspondence

Logic		**Type Theory**		**Category Theory**
propositions	$\leftrightarrow$	types	$\leftrightarrow$	objects
proofs	$\leftrightarrow$	terms	$\leftrightarrow$	morphisms

E.g. IPL *versus* STLC *versus* CCCs

These correspondences can be made into category-theoretic equivalences—we first need to define the notions of functor and natural transformation in order to define the notion of equivalence of categories.