#### Lecture 9

#### STLC equations

take the form  $\Gamma \vdash s = t : A$  where  $\Gamma \vdash s : A$  and  $\Gamma \vdash t : A$  are provable.

Such an equation is satisfied by the semantics in a ccc if  $M[\Gamma \vdash s : A]$  and  $M[\Gamma \vdash t : A]$  are equal C-morphisms  $M[\Gamma] \rightarrow M[A]$ .

Qu: which equations are always satisfied in any ccc? Ans:  $\beta\eta$ -equivalence...

## STLC βη-Equality

$$\begin{array}{l}
\beta-\text{conversions} \\
\hline \Gamma, x : A \vdash t : B & \Gamma \vdash s : A \\
\hline \Gamma \vdash (\lambda x : A.t)s =_{\beta\eta} t[s/x] : B \\
\hline \Gamma \vdash s : A & \Gamma \vdash t : B \\
\hline \Gamma \vdash fst(s,t) =_{\beta\eta} s : A & \Gamma \vdash s : A & \Gamma \vdash t : B \\
\hline \Gamma \vdash snd(s,t) =_{\beta\eta} t : B \\
\end{array}$$

The relation  $\Gamma \vdash s =_{\beta\eta} t : A$  (where  $\Gamma$  ranges over typing environments, s and t over terms and A over types) is inductively defined by the following rules:



 $\blacktriangleright \eta$ -conversions

 $\Gamma \vdash t : A \rightarrow B \qquad x \text{ does not occur in } t$ 

 $\Gamma \vdash t =_{\beta\eta} (\lambda x : A \cdot t x) : A \to B$ 

 $\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash t =_{\beta\eta} (\text{fst}\,t\,,\,\text{snd}\,t) : A \times B} \quad \frac{\Gamma \vdash t : \text{unit}}{\Gamma \vdash t =_{\beta\eta} () : \text{unit}}$ 

- $\triangleright \beta$ -conversions
- $\blacktriangleright \eta$ -conversions
- congruence rules

$$\frac{\Gamma, x : A \vdash t =_{\beta\eta} t' : B}{\Gamma \vdash \lambda x : A . t =_{\beta\eta} \lambda x : A . t' : A \Rightarrow B}$$
$$\frac{\Gamma \vdash s =_{\beta\eta} s' : A \Rightarrow B \qquad \Gamma \vdash t =_{\beta\eta} t' : A}{\Gamma \vdash s t =_{\beta\eta} s' t' : B}$$
etc

- $\blacktriangleright \beta$ -conversions
- $\blacktriangleright \eta$ -conversions
- congruence rules

$$=_{\beta\eta} \text{ is reflexive, symmetric and transitive}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t =_{\beta\eta} t : A} \begin{bmatrix} \Gamma \vdash s =_{\beta\eta} t : A \\ \Gamma \vdash t =_{\beta\eta} s : A \end{bmatrix}$$

$$\frac{\Gamma \vdash r =_{\beta\eta} s : A}{\Gamma \vdash r =_{\beta\eta} t : A}$$

#### STLC *βη*-Equality

**Soundness Theorem** for semantics of STLC in a ccc. If  $\Gamma \vdash s =_{\beta\eta} t : A$  is provable, then in any ccc

$$M[\![\Gamma \vdash s : A]\!] = M[\![\Gamma \vdash t : A]\!]$$

are equal C-morphisms  $M[[\Gamma]] \rightarrow M[[A]]$ .

**Proof** is by induction on the structure of the proof of  $\Gamma \vdash s =_{\beta\eta} t : A$ . Here we just check the case of  $\beta$ -conversion for functions.

So suppose we have  $\Gamma, x : A \vdash t : B$  and  $\Gamma \vdash s : A$ . We have to see that

 $M\llbracket\Gamma \vdash (\lambda x : A. t)s : B\rrbracket = M\llbracket\Gamma \vdash t[s/x] : B\rrbracket$ 

#### Suppose $M[[\Gamma]] = X$ M[[A]] = Y M[[B]] = Z $M[[\Gamma, x : A \vdash t : B]] = f : X \times Y \rightarrow Z$ $M[[\Gamma \vdash s : A]] = g : X \rightarrow Z$

Then

$$M\llbracket\Gamma \vdash \lambda x : A.t : A \Rightarrow B\rrbracket = \operatorname{cur} f : X \to Z^{Y}$$

and hence

$$\begin{split} M[\![\Gamma \vdash (\lambda x : A. t)s : B]\!] &= \operatorname{app} \circ \langle \operatorname{cur} f, g \rangle \\ &= \operatorname{app} \circ (\operatorname{cur} f \times \operatorname{id}_Y) \circ \langle \operatorname{id}_X, g \rangle \quad \operatorname{since} (a \times b) \circ \langle c, d \rangle = \langle a \circ c, b \circ d \rangle \\ &= f \circ \langle \operatorname{id}_X, g \rangle \quad \text{by definition of cur } f \\ &= M[\![\Gamma \vdash t[s/x]] : B]\!] \quad \text{by the Substitution Theorem} \end{split}$$

as required.

## The internal language of a ccc, C

• one ground type for each C-object X

▶ for each  $X \in C$ , one constant  $f^X$  for each C-morphism  $f: 1 \to X$  ("global element" of the object X)

The types and terms of STLC over this language usefully describe constructions on the objects and morphisms of **C** using its cartesian closed structure, but in an "element-theoretic" way.

For example...

#### Example

In any ccc **C**, for any  $X, Y, Z \in \mathbf{C}$  there is an isomorphism  $Z^{(X \times Y)} \cong (Z^Y)^X$ 

#### Example

In any ccc **C**, for any  $X, Y, Z \in \mathbf{C}$  there is an isomorphism  $Z^{(X \times Y)} \cong (Z^Y)^X$ 

which in the internal language of C is described by the terms

$$\diamond \vdash s : ((X \times Y) \Rightarrow Z) \Rightarrow (X \Rightarrow (Y \Rightarrow Z)) \\ \diamond \vdash t : (X \Rightarrow (Y \Rightarrow Z)) \Rightarrow ((X \times Y) \Rightarrow Z)$$
where
$$\begin{cases} s &\triangleq \lambda f : (X \times Y) \Rightarrow Z. \lambda x : X. \lambda y : Y. f(x, y) \\ t &\triangleq \lambda g : X \Rightarrow (Y \Rightarrow Z). \lambda z : X \times Y. g \text{ (fst } z) \text{ (snd } z) \end{cases}$$
and which satisfy
$$\begin{cases} \diamond, f : (X \times Y) \Rightarrow Z \vdash t(s f) =_{\beta \eta} f \\ \diamond, g : X \Rightarrow (Y \Rightarrow Z) \vdash s(t g) =_{\beta \eta} g \end{cases}$$

#### Free cartesian closed categories

The <u>Soundness Theorem</u> has a converse—completeness.

In fact for a given set of ground types and typed constants there is a single ccc  $\mathbf{F}$  (the free ccc for that language) with an interpretation function M so that  $\Gamma \vdash s =_{\beta\eta} t : A$  is provable iff  $M[\Gamma \vdash s : A] = M[\Gamma \vdash t : A]$  in  $\mathbf{F}$ .

#### Free cartesian closed categories

The <u>Soundness Theorem</u> has a converse—completeness.

In fact for a given set of ground types and typed constants there is a single ccc **F** (the free ccc for that language) with an interpretation function M so that  $\Gamma \vdash s =_{\beta\eta} t : A$  is provable iff  $M[\Gamma \vdash s : A] = M[\Gamma \vdash t : A]$  in **F**.

- **F**-objects are the STLC types over the given set of ground types
- F-morphisms  $A \to B$  are equivalence classes of STLC terms t satisfying  $\diamond \vdash t : A \Rightarrow B$  (so t is a *closed* term—it has no free variables) with respect to the equivalence relation equating s and t if  $\diamond \vdash s =_{\beta\eta} t : A \Rightarrow B$  is provable.
- identity morphism on A is the equivalence class of  $\diamond \vdash \lambda x : A \cdot x : A \rightarrow A$ .
- composition of a morphism  $A \to B$  represented by  $\diamond \vdash s : A \to B$  and a morphism  $B \to C$  represented by  $\diamond \vdash t : B \to C$  is represented by  $\diamond \vdash \lambda x : A \cdot t(s x) : A \to C$ .

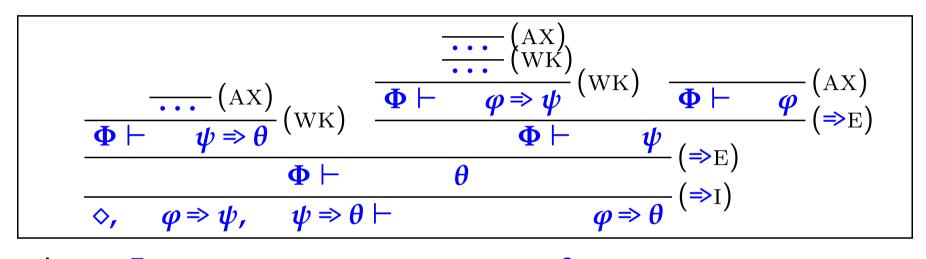
# Curry-Howard correspondence

		Туре
Logic		Theory
propositions	$\leftrightarrow$	types
proofs	$\leftrightarrow$	terms

E.g. IPL versus STLC.

#### Curry-Howard for IPL vs STLC

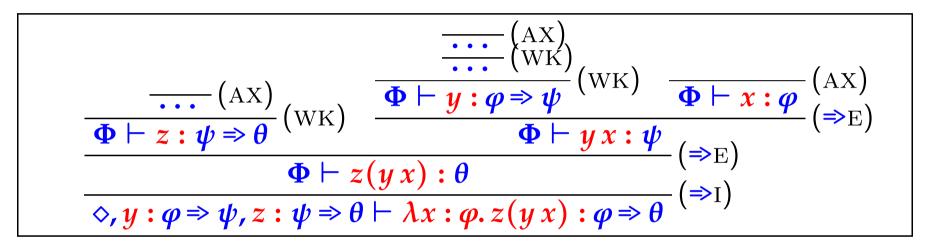
Proof of  $\diamond, \phi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \phi \Rightarrow \theta$  in IPL



where  $\Phi = \diamond$ ,  $\varphi \Rightarrow \psi$ ,  $\psi \Rightarrow \theta$ ,  $\varphi$ 

#### Curry-Howard for IPL vs STLC

and a corresponding STLC term



where  $\Phi = \diamond, y : \phi \Rightarrow \psi, z : \psi \Rightarrow \theta, x : \phi$ 

# Curry-Howard-Lawvere/Lambek correspondence

Logic	Type Theory		Category Theory	
propositions	$\leftrightarrow$	types	$\leftrightarrow$	objects
proofs	$\leftrightarrow$	terms	$\leftrightarrow$	morphisms

E.g. IPL versus STLC versus CCCs

# Curry-Howard-Lawvere/Lambek correspondence

Logic	Type Theory		Category Theory	
propositions	$\leftrightarrow$	types	$\leftrightarrow$	objects
proofs	$\leftrightarrow$	terms	$\leftrightarrow$	morphisms

#### E.g. IPL versus STLC versus CCCs

These correspondences can be made into category-theoretic equivalences—we first need to define the notions of functor and natural transformation in order to define the notion of equivalence of categories.