Lecture 8

Recall :

Simply-Typed Lambda Calculus (STLC)

Types: $A, B, C, \ldots ::=$

$G, G', G'' \ldots$	"ground" types
`unit`	unit type
A x B	product type
A -> B	function type

Terms: $s, t, r, \ldots ::=$

c^A	constants (of given type A)
x	variable (countably many)
$()$	unit value
$(s\,,t)$	pair
`fst` t `snd` t	projections
$\lambda x : A.\, t$	function abstraction
$s\, t$	function application

Recall :

Semantics of STLC terms in a ccc

Given a cartesian closed category $\mathbf{C}$,

given any function M mapping

- ground types G to $\mathbf{C}$-objects $M(G)$
- constants c^A to $\mathbf{C}$-morphisms $M(c^A) : 1 \to M[\![A]\!]$

we get a function mapping provable instances of the typing relation $\Gamma \vdash t : A$ to $\mathbf{C}$-morphisms

$$M[\![\Gamma \vdash t : A]\!] : M[\![\Gamma]\!] \to M[\![A]\!]$$

defined by recursing over the proof of $\Gamma \vdash t : A$ from the typing rules (which follows the structure of t):

Semantics of STLC terms in a ccc

Variables:

$$M[\![\Gamma, x : A \vdash x : A]\!] = M[\![\Gamma]\!] \times M[\![A]\!] \xrightarrow{\pi_2} M[\![A]\!]$$

$$M[\![\Gamma, x' : A' \vdash x : A]\!] =$$

$$M[\![\Gamma]\!] \times M[\![A']\!] \xrightarrow{\pi_1} M[\![\Gamma]\!] \xrightarrow{M[\![\Gamma \vdash x : A]\!]} M[\![A]\!]$$

Constants:

$$M[\![\Gamma \vdash c^A : A]\!] = M[\![\Gamma]\!] \xrightarrow{\langle\rangle} 1 \xrightarrow{M(c^A)} M[\![A]\!]$$

Unit value:

$$M[\![\Gamma \vdash () : \mathtt{unit}]\!] = M[\![\Gamma]\!] \xrightarrow{\langle\rangle} 1$$

Semantics of STLC terms in a ccc

Pairing:

$$M[\![\Gamma \vdash (s\,,t) : A \times B]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{\langle M[\![\Gamma \vdash s:A]\!], M[\![\Gamma \vdash t:B]\!]\rangle} M[\![A]\!] \times M[\![B]\!]$$

Projections:

$$M[\![\Gamma \vdash \texttt{fst}\, t : A]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{M[\![\Gamma \vdash t:A \times B]\!]} M[\![A]\!] \times M[\![B]\!] \xrightarrow{\pi_1} M[\![A]\!]$$

Semantics of STLC terms in a ccc

Pairing:

$$M[\![\Gamma \vdash (s\,,t) : A \times B]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{\langle M[\![\Gamma \vdash s:A]\!], M[\![\Gamma \vdash t:B]\!]\rangle} M[\![A]\!] \times M[\![B]\!]$$

Projections:

Given that $\Gamma \vdash \mathtt{fst}\, t : A$ holds, there is a unique type B such that $\Gamma \vdash t : A \times B$ already holds.

$$M[\![\Gamma \vdash \mathtt{fst}\, t : A]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{M[\![\Gamma \vdash t:A\times B]\!]} M[\![A]\!] \times M[\![B]\!] \xrightarrow{\pi_1} M[\![A]\!]$$

Lemma. If $\Gamma \vdash t : A$ and $\Gamma \vdash t : B$ are provable, then $A = B$.

Semantics of STLC terms in a ccc

Pairing:

$$M[\![\Gamma \vdash (s\,,t) : A \times B]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{\langle M[\![\Gamma \vdash s:A]\!], M[\![\Gamma \vdash t:B]\!]\rangle} M[\![A]\!] \times M[\![B]\!]$$

Projections:

$$M[\![\Gamma \vdash \mathtt{snd}\, t : B]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{M[\![\Gamma \vdash t:A \times B]\!]} M[\![A]\!] \times M[\![B]\!] \xrightarrow{\pi_2} M[\![B]\!]$$

(As for the case of `fst`, if $\Gamma \vdash \mathtt{snd}\, t : B$, then $\Gamma \vdash t : A \times B$ already holds for a unique type A.)

Semantics of STLC terms in a ccc

Function abstraction:

$$M[\![\Gamma \vdash \lambda x : A.t : A \mathbin{\text{->}} B]\!] = \mathrm{cur}\, f : M[\![\Gamma]\!] \longrightarrow (M[\![A]\!] \rightarrow M[\![B]\!])$$

where

$$f = M[\![\Gamma, x : A \vdash t : B]\!] : M[\![\Gamma]\!] \times M[\![A]\!] \longrightarrow M[\![B]\!]$$

Semantics of STLC terms in a ccc

Function application:

$$M[\![\Gamma \vdash s\,t : B]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{\langle f,g\rangle} (M[\![A]\!] \to M[\![B]\!]) \times M[\![A]\!] \xrightarrow{\mathrm{app}} M[\![B]\!]$$

where

$A =$ unique type such that $\Gamma \vdash s : A \mathbin{\text{->}} B$ and $\Gamma \vdash t : A$ already holds (exists because $\Gamma \vdash s\,t : B$ holds)

$f = M[\![\Gamma \vdash s : A \mathbin{\text{->}} B]\!] : M[\![\Gamma]\!] \longrightarrow (M[\![A]\!] \to M[\![B]\!])$

$g = M[\![\Gamma \vdash t : A]\!] : M[\![\Gamma]\!] \longrightarrow M[\![A]\!]$

Example

Consider $t \triangleq \lambda x : A.\, g\,(f\ x)$ so that $\Gamma \vdash t : A \to C$ when $\Gamma \triangleq \diamond, f : A \to B, g : B \to C$.

Suppose $M[\![A]\!] = X$, $M[\![B]\!] = Y$ and $M[\![C]\!] = Z$ in $\mathbf{C}$. Then

$$M[\![\Gamma]\!] = (1 \times Y^X) \times Z^Y$$
$$M[\![\Gamma, x : A]\!] = ((1 \times Y^X) \times Z^Y) \times X$$

$$M[\![\Gamma, x : A \vdash x : A]\!] = \pi_2$$
$$M[\![\Gamma, x : A \vdash g : B \to C]\!] = \pi_2 \circ \pi_1$$
$$M[\![\Gamma, x : A \vdash f : A \to B]\!] = \pi_2 \circ \pi_1 \circ \pi_1$$
$$M[\![\Gamma, x : A \vdash f\ x : B]\!] = \mathrm{app} \circ \langle \pi_2 \circ \pi_1 \circ \pi_1\,, \pi_2 \rangle$$
$$M[\![\Gamma, x : A \vdash g\,(f\ x) : C]\!] = \mathrm{app} \circ \langle \pi_2 \circ \pi_1\,, \mathrm{app} \circ \langle \pi_2 \circ \pi_1 \circ \pi_1\,, \pi_2 \rangle \rangle$$
$$M[\![\Gamma \vdash t : A \to C]\!] = \mathrm{cur}(\mathrm{app} \circ \langle \pi_2 \circ \pi_1\,, \mathrm{app} \circ \langle \pi_2 \circ \pi_1 \circ \pi_1\,, \pi_2 \rangle \rangle)$$

STLC equations

take the form $\boxed{\Gamma \vdash s = t : A}$ where $\Gamma \vdash s : A$ and $\Gamma \vdash t : A$ are provable.

Such an equation is satisfied by the semantics in a ccc if $M[\![\Gamma \vdash s : A]\!]$ and $M[\![\Gamma \vdash t : A]\!]$ are equal $\mathbf{C}$-morphisms $M[\![\Gamma]\!] \to M[\![A]\!]$.

Qu: which equations are always satisfied in any ccc?

STLC equations

take the form $\boxed{\Gamma \vdash s = t : A}$ where $\Gamma \vdash s : A$ and $\Gamma \vdash t : A$ are provable.

Such an equation is satisfied by the semantics in a ccc if $M[\![\Gamma \vdash s : A]\!]$ and $M[\![\Gamma \vdash t : A]\!]$ are equal **C**-morphisms $M[\![\Gamma]\!] \to M[\![A]\!]$.

Qu: which equations are always satisfied in any ccc?

Ans: $(\alpha)\beta\eta$-equivalence — to define this, first have to define alpha-equivalence, substitution and its semantics.

Alpha equivalence of STLC terms

The names of λ-bound variables should not affect meaning.

E.g. $\lambda f : A \to B. \lambda x : A. f\, x$ should have the same meaning as $\lambda x : A \to B. \lambda y : A. x\, y$

Alpha equivalence of STLC terms

The names of λ-bound variables should not affect meaning.

E.g. $\lambda f : A \rightarrow B.\, \lambda x : A.\, f\, x$ should have the same meaning as $\lambda x : A \rightarrow B.\, \lambda y : A.\, x\, y$

This issue is best dealt with at the level of syntax rather than semantics: from now on we re-define "STLC term" to mean not an abstract syntax tree (generated as described before), but rather an equivalence class of such trees with respect to alpha-equivalence $\boxed{s =_\alpha t}$, defined as follows. . .

(Alternatively, one can use a "nameless" (de Bruijn) representation of terms.)

Alpha equivalence of STLC terms

$$\frac{}{c^A =_\alpha c^A}$$

$$\frac{}{x =_\alpha x}$$

$$\frac{}{() =_\alpha ()}$$

$$\frac{s =_\alpha s' \qquad t =_\alpha t'}{(s\,,t) =_\alpha (s'\,,t')}$$

$$\frac{t =_\alpha t'}{\texttt{fst}\,t =_\alpha \texttt{fst}\,t'}$$

$$\frac{t =_\alpha t'}{\texttt{snd}\,t =_\alpha \texttt{snd}\,t'}$$

$$\frac{s =_\alpha s' \qquad t =_\alpha t'}{s\,t =_\alpha s'\,t'}$$

$$\frac{(y\ x)\cdot t =_\alpha (y\ x')\cdot t' \qquad y \text{ does not occur in } \{x, x', t, t'\}}{\lambda x : A.\,t =_\alpha \lambda x' : A.\,t'}$$

Alpha equivalence of STLC terms

$$\frac{}{c^A =_\alpha c^A} \qquad \frac{}{x =_\alpha x} \qquad \frac{}{() =_\alpha ()} \qquad \frac{s =_\alpha s' \qquad t =_\alpha t'}{(s\,,t) =_\alpha (s'\,,t')}$$

$$\frac{t =_\alpha t'}{\mathtt{fst}\; t =_\alpha \mathtt{fst}\; t'} \qquad \frac{t =_\alpha t'}{\mathtt{snd}\; t =_\alpha \mathtt{snd}\; t'} \qquad \frac{s =_\alpha s' \qquad t =_\alpha t'}{s\,t =_\alpha s'\,t'}$$

$$\frac{(y\ x) \cdot t =_\alpha (y\ x') \cdot t' \qquad y \text{ does not occur in } \{x, x', t, t'\}}{\lambda x : A.\, t =_\alpha \lambda x' : A.\, t'}$$

result of replacing all occurrences of x with y in t

Alpha equivalence of STLC terms

$$\frac{}{c^A =_\alpha c^A} \qquad \frac{}{x =_\alpha x} \qquad \frac{}{() =_\alpha ()} \qquad \frac{s =_\alpha s' \qquad t =_\alpha t'}{(s\,,t) =_\alpha (s'\,,t')}$$

$$\frac{t =_\alpha t'}{\mathtt{fst}\, t =_\alpha \mathtt{fst}\, t'} \qquad \frac{t =_\alpha t'}{\mathtt{snd}\, t =_\alpha \mathtt{snd}\, t'} \qquad \frac{s =_\alpha s' \qquad t =_\alpha t'}{s\,t =_\alpha s'\,t'}$$

$$\frac{(y\ x) \cdot t =_\alpha (y\ x') \cdot t' \qquad y \text{ does not occur in } \{x, x', t, t'\}}{\lambda x : A.\, t =_\alpha \lambda x' : A.\, t'}$$

E.g.

$\lambda x : A.\, x\, x =_\alpha \lambda y : A.\, y\, y \neq_\alpha \lambda x : A.\, x\, y$

$(\lambda y : A.\, y)\, x =_\alpha (\lambda x : A.\, x)\, x \neq_\alpha (\lambda x : A.\, x)\, y$

Substitution

$t[s/x]$ = result of replacing all free occurrences of variable x in term t (i.e. those not occurring within the scope of a $\lambda x : A.\_$ binder) by the term s, alpha-converting λ-bound variables in t to avoid them "capturing" any free variables of t.

E.g. $(\lambda y : A.\,(y\,,x))[y/x]$ is $\lambda z : A.\,(z\,,y)$ and is not $\lambda y : A.\,(y\,,y)$

Substitution

$t[s/x]$ = result of replacing all free occurrences of variable x in term t (i.e. those not occurring within the scope of a $\lambda x : A.\_$ binder) by the term s, alpha-converting λ-bound variables in t to avoid them "capturing" any free variables of t.

E.g. $(\lambda y : A.\,(y\,,x))[y/x]$ is $\lambda z : A.\,(z\,,y)$ and is not $\lambda y : A.\,(y\,,y)$

The relation $t[s/x] = t'$ can be inductively defined by the following rules. . .

Substitution

$$\frac{}{c^A[s/x] = c^A}$$

$$\frac{}{x[s/x] = s}$$

$$\frac{y \neq x}{y[s/x] = y}$$

$$\frac{}{()[s/x] = ()}$$

$$\frac{t_1[s/x] = t_1' \qquad t_2[s/x] = t_2'}{(t_1, t_2)[s/x] = (t_1', t_2')}$$

$$\frac{t[s/x] = t'}{(\mathtt{fst}\, t)[s/x] = \mathtt{fst}\, t'}$$

$$\frac{t[s/x] = t'}{(\mathtt{snd}\, t)[s/x] = \mathtt{snd}\, t'}$$

$$\frac{t_1[s/x] = t_1' \qquad t_2[s/x] = t_2'}{(t_1\, t_2)[s/x] = t_1' t_2'}$$

$$\frac{t[s/x] = t' \qquad y \neq x \text{ and } y \text{ does not occur in } s}{(\lambda y : A.\, t)[s/x] = \lambda y : A.\, t'}$$

Semantics of substitution in a ccc

Substitution Lemma If $\Gamma \vdash s : A$ and $\Gamma, x : A \vdash t : B$ are provable, then so is $\Gamma \vdash t[s/x] : B$.

Substitution Theorem If $\Gamma \vdash s : A$ and $\Gamma, x : A \vdash t : B$ are provable, then in any ccc the following diagram commutes:

$$
\begin{array}{ccc}
M[\![\Gamma]\!] & \xrightarrow{\langle \mathtt{id}, M[\![\Gamma \vdash s : A]\!]\rangle} & M[\![\Gamma]\!] \times M[\![A]\!] \\
 & \searrow^{M[\![\Gamma \vdash t[s/x] : B]\!]} & \downarrow{\scriptstyle M[\![\Gamma, x : A \vdash t : B]\!]} \\
 & & M[\![B]\!]
\end{array}
$$