III. Approximation Algorithms: Covering Problems

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Outline

Motivation

Many fundamental problems are **NP-complete**, yet they are too important to be abandoned.

Examples: HAMILTON, 3-SAT, VERTEX-COVER, KNAPSACK,...

Strategies to cope with NP-complete problems

1. If inputs (or solutions) are small, an algorithm with exponential running time may be satisfactory.
2. Isolate important special cases which can be solved in polynomial-time.
3. Develop algorithms which find near-optimal solutions in polynomial-time.

We will call these **approximation algorithms**.

Performance Ratios for Approximation Algorithms

Approximation Ratio

An algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size n, the cost C of the returned solution and optimal cost C^* satisfy:

$$\max\left(\frac{C}{C^*}, \frac{C^*}{C}\right) \leq \rho(n).$$

- Maximization problem: $\frac{C^*}{C} \geq 1$
- Minimization problem: $\frac{C}{C^*} \geq 1$

This covers both maximization and minimization problems.

For many problems: tradeoff between runtime and approximation ratio.

Approximation Schemes

An approximation scheme is an approximation algorithm, which given any input and $\epsilon > 0$, is a $(1+\epsilon)$-approximation algorithm.

- It is a polynomial-time approximation scheme (PTAS) if for any fixed $\epsilon > 0$, the runtime is polynomial in n. For example, $O(n^{2/\epsilon})$.
- It is a fully polynomial-time approximation scheme (FPTAS) if the runtime is polynomial in both $1/\epsilon$ and n. For example, $O((1/\epsilon)^2 \cdot n^3)$.

Outline

Introduction

Vertex Cover

The Set-Covering Problem

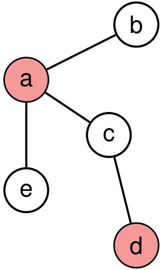
The Vertex-Cover Problem

We are covering edges by picking vertices!

Vertex Cover Problem

- Given: Undirected graph $G = (V, E)$
- Goal: Find a minimum-cardinality subset $V' \subseteq V$ such that if $(u, v) \in E(G)$, then $u \in V'$ or $v \in V'$.

This is an NP-hard problem.



Applications:

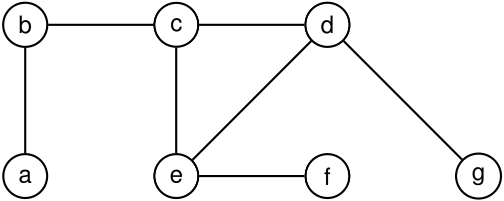
- Every edge forms a task, and every vertex represents a person/machine which can execute that task
- Perform all tasks with the minimal amount of resources
- Extensions: weighted vertices or hypergraphs ($\rightsquigarrow$ Set-Covering Problem)

Exercise: Be creative and design your own algorithm for VERTEX-COVER!

An Approximation Algorithm based on Greedy

APPROX-VERTEX-COVER(G)

$C = \emptyset$
$E' = G.E$
while $E' \neq \emptyset$
  let (u, v) be an arbitrary edge of E'
  $C = C \cup \{u, v\}$
  remove from E' every edge incident on either u or v
return C



An Approximation Algorithm based on Greedy

APPROX-VERTEX-COVER(G)

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C = ∅
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while E' ≠ ∅
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    C = C ∪ {u, v}
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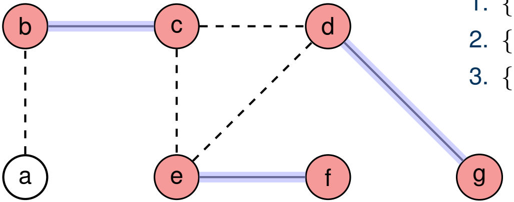
return C
```



Edges removed from E':

1. $\{b, c\}$
2. $\{e, f\}$
3. $\{d, g\}$

APPROX-VERTEX-COVER produces a set of size 6.

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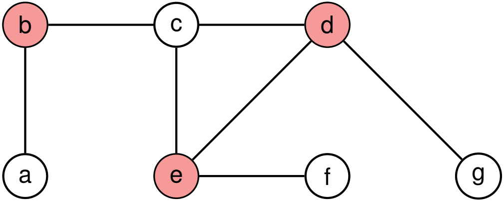


The optimal solution has size 3.

Analysis of Greedy for Vertex Cover

```
APPROX-VERTEX-COVER(G)
1  C = ∅
2  E' = G.E
3  while E' ≠ ∅
4      let (u, v) be an arbitrary edge of E'
5      C = C ∪ {u, v}
6      remove from E' every edge incident on either u or v
7  return C
```

A "vertex-based" Greedy that adds **one** vertex at each iteration fails to achieve an approximation ratio of 2 (Supervision Exercise)!

We can bound the size of the returned solution without knowing the (size of an) optimal solution!

Theorem 35.1

APPROX-VERTEX-COVER is a poly-time 2-approximation algorithm.

Proof:

- Running time is $O(V + E)$ (using adjacency lists to represent E')
- Let $A \subseteq E$ denote the set of edges picked in line 4
- Key Observation: A is a set of vertex-disjoint edges, i.e., A is a matching

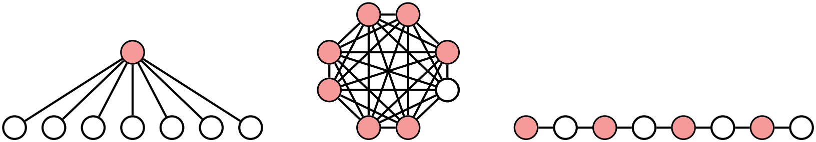
$\Rightarrow$ Every optimal cover C^* must include at least one endpoint: $|C^*| \geq |A|$

- Every edge in A contributes 2 vertices to $|C|$: $|C| = 2|A| \leq 2|C^*|$. □

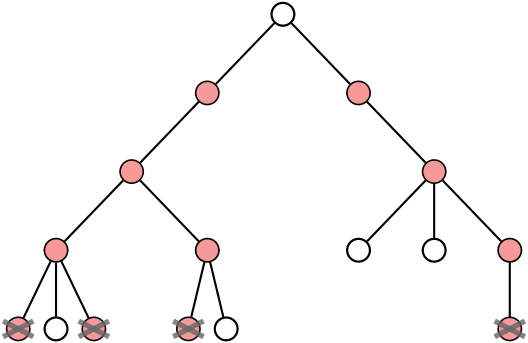
Solving Special Cases

Strategies to cope with NP-complete problems

1. If inputs are small, an algorithm with exponential running time may be satisfactory.
2. Isolate important special cases which can be solved in polynomial-time.
3. Develop algorithms which find near-optimal solutions in polynomial-time.



Vertex Cover on Trees



There exists an optimal vertex cover which does not include any leaves.

Exchange-Argument: Replace any leaf in the cover by its parent.

Solving Vertex Cover on Trees

There exists an optimal vertex cover which does not include any leaves.

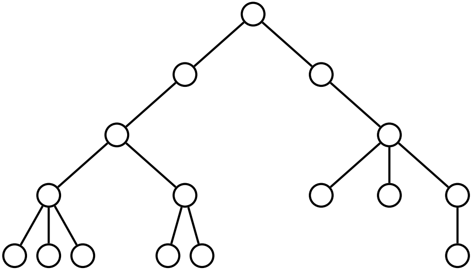
VERTEX-COVER-TREES(G)

```
C = ∅
while ∃ leaves in G
    Add all parents to C
    Remove all leaves and their parents from G
return C
```

Clear: Running time is $O(V)$, and the returned solution is a vertex cover.

Solution is also optimal. (Use inductively the existence of an optimal vertex cover without leaves)

Execution on a Small Example

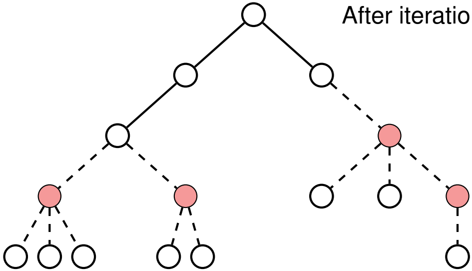


VERTEX-COVER-TREES(G)

1: $C = \emptyset$
2: **while** $\exists$ leaves in G
3: Add all parents to C
4: Remove all leaves and their parents from G
5: **return** C

Execution on a Small Example

After iteration 1



VERTEX-COVER-TREES(G)

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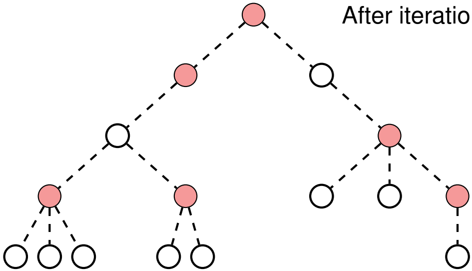
Execution on a Small Example



After iteration 2

```
VERTEX-COVER-TREES(G)
C = ∅
while ∃ leaves in G
    Add all parents to C
    Remove all leaves and their parents from G
return C
```

Problem can be also solved on bipartite graphs, using Max-Flows and Min-Cuts.

Exact Algorithms

Such algorithms are called exact algorithms.

Strategies to cope with NP-complete problems

1. If inputs (or solutions) are small, an algorithm with exponential running time may be satisfactory
2. Isolate important special cases which can be solved in polynomial-time.
3. Develop algorithms which find near-optimal solutions in polynomial-time.

Focus on instances where the minimum vertex cover is small, that is, **less or equal** than some given integer k.

Simple Brute-Force Search would take $\approx \binom{n}{k} = \Theta(n^k)$ time.

Towards a more efficient Search

Substructure Lemma

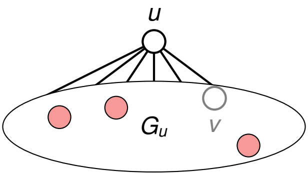
Consider a graph $G = (V, E)$, edge $\{u, v\} \in E(G)$ and integer $k \geq 1$. Let G_u be the graph obtained by deleting u and its incident edges (G_v is defined similarly). Then G has a vertex cover of size k if and only if G_u or G_v (or both) have a vertex cover of size $k - 1$.

Reminiscent of Dynamic Programming.

Proof:

$\Leftarrow$ Assume G_u has a vertex cover C_u of size $k - 1$.
Adding u yields a vertex cover of G which is of size k

$\Rightarrow$ Assume G has a vertex cover C of size k, which contains, say u.
Removing u from C yields a vertex cover of G_u which is of size $k - 1$. □



A More Efficient Search Algorithm

```
VERTEX-COVER-SEARCH(G, k)
if E = ∅ return ∅
if k = 0 and E ≠ ∅ return ⊥
Pick an arbitrary edge (u, v) ∈ E
S_1 = VERTEX-COVER-SEARCH(G_u, k − 1)
S_2 = VERTEX-COVER-SEARCH(G_v, k − 1)
if S_1 ≠ ⊥ return S_1 ∪ {u}
if S_2 ≠ ⊥ return S_2 ∪ {v}
return ⊥
```

Correctness follows by the Substructure Lemma and induction.

Running time:

- Depth k, branching factor 2 $\Rightarrow$ total number of calls is $O(2^k)$
- $O(E)$ worst-case time for one call (computing G_u or G_v could take $\Theta(E)$!)
- Total runtime: $O(2^k \cdot E)$.

exponential in k, but much better than $\Theta(n^k)$ (i.e., still polynomial for $k = O(\log n)$)

Outline

Introduction

Vertex Cover

The Set-Covering Problem

The Set-Covering Problem

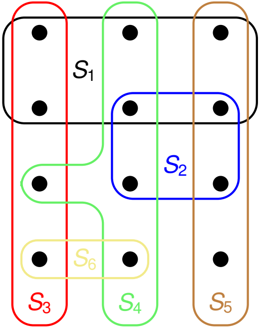
Set Cover Problem

- Given: set X of size n and family of subsets $\mathcal{F}$
- Goal: Find a minimum-size subset $\mathcal{C} \subseteq \mathcal{F}$

s.t. $$X = \bigcup_{S \in \mathcal{C}} S.$$

Number of sets (and not elements)

Only solvable if $\bigcup_{S \in \mathcal{F}} S = X$!



Remarks:

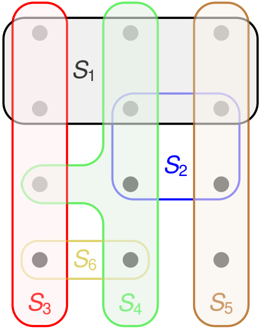
- generalisation of the vertex-cover problem and hence also NP-hard.
- models resource allocation problems, e.g., wireless coverage

Greedy

> Strategy: Pick the set S that covers the largest number of uncovered elements.

GREEDY-SET-COVER$(X, \mathcal{F})$

```
U = X
𝒞 = ∅
while U ≠ ∅
    select an S ∈ 𝓕 that maximizes |S ∩ U|
    U = U − S
    𝒞 = 𝒞 ∪ {S}
return 𝒞
```



Greedy chooses S_1, S_4, S_5 and S_3 (or S_6), which is a cover of size 4.

Greedy

Strategy: Pick the set S that covers the largest number of uncovered elements.

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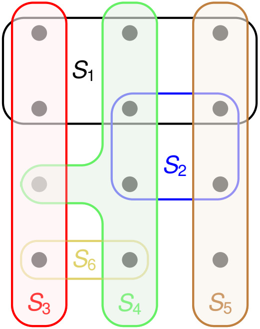


Can be easily implemented to run in time polynomial in $|X|$ and $|\mathcal{F}|$

Optimal cover is $\mathcal{C} = \{S_3, S_4, S_5\}$

How good is the approximation ratio?

Approximation Ratio of Greedy

Theorem 35.4

GREEDY-SET-COVER is a polynomial-time $\rho(n)$-algorithm, where

$$\rho(n) = H(\max\{|S| \colon S \in \mathcal{F}\}) \leq \ln(n) + 1.$$

$H(k) := \sum_{i=1}^{k} \frac{1}{i} \leq \ln(k) + 1$

Idea: Distribute cost of 1 for each added set over newly covered elements.

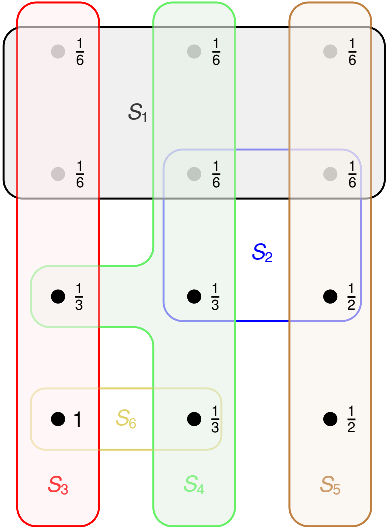
Definition of cost

If an element x is covered for the first time by set S_i in iteration i, then

$$c_x := \frac{1}{|S_i \setminus (S_1 \cup S_2 \cup \cdots \cup S_{i-1})|}.$$

Notice that in the mathematical analysis, S_i is the set chosen in iteration i - not to be confused with the sets $S_1, S_2, \ldots, S_6$ in the example.

Illustration of Costs for Greedy picking S_1, S_4, S_5 and S_3



$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{2} + \frac{1}{2} + 1 = 4$$

Proof of Theorem 35.4 (1/2)

Definition of cost

If x is covered for the first time by a set S_i, then $c_x := \frac{1}{\left|S_i \setminus (S_1 \cup S_2 \cup \cdots \cup S_{i-1})\right|}$.

Proof.

- Each step of the algorithm assigns one unit of cost, so

$$|\mathcal{C}| = \sum_{x \in X} c_x \tag{1}$$

- Each element $x \in X$ is in at least one set in the optimal cover $\mathcal{C}^*$, so

$$\sum_{S \in \mathcal{C}^*} \sum_{x \in S} c_x \geq \sum_{x \in X} c_x \tag{2}$$

- Combining 1 and 2 gives

$$|\mathcal{C}| \leq \sum_{S \in \mathcal{C}^*} \sum_{x \in S} c_x \leq \sum_{S \in \mathcal{C}^*} H(|S|) \leq |\mathcal{C}^*| \cdot H(\max\{|S| : S \in \mathcal{F}\})$$ □

Key Inequality: $\sum_{x \in S} c_x \leq H(|S|)$.

Proof of Theorem 35.4 (2/2)

Proof of the Key Inequality $\sum_{x \in S} c_x \leq H(|S|)$

- For any $S \in \mathcal{F}$ and $i = 1, 2, \ldots, |\mathcal{C}| = k$ let $u_i := |S \setminus (S_1 \cup S_2 \cup \cdots \cup S_i)|$

$\Rightarrow$ $|S| = u_0 \geq u_1 \geq \cdots \geq u_{|\mathcal{C}|} = 0$ and $u_{i-1} - u_i$ counts the items in S covered first time by S_i.

$\Rightarrow$

$$\sum_{x \in S} c_x = \sum_{i=1}^{k} (u_{i-1} - u_i) \cdot \frac{1}{|S_i \setminus (S_1 \cup S_2 \cup \cdots \cup S_{i-1})|}$$

- Further, by definition of the GREEDY-SET-COVER:

$$|S_i \setminus (S_1 \cup S_2 \cup \cdots \cup S_{i-1})| \geq |S \setminus (S_1 \cup S_2 \cup \cdots \cup S_{i-1})| = u_{i-1}.$$

- Combining the last inequalities gives:

$$\begin{aligned}
\sum_{x \in S} c_x \leq \sum_{i=1}^{k} (u_{i-1} - u_i) \cdot \frac{1}{u_{i-1}} &= \sum_{i=1}^{k} \sum_{j=u_i+1}^{u_{i-1}} \frac{1}{u_{i-1}} \\
&\leq \sum_{i=1}^{k} \sum_{j=u_i+1}^{u_{i-1}} \frac{1}{j} \\
&= \sum_{i=1}^{k} (H(u_{i-1}) - H(u_i)) = H(u_0) - H(u_k) = H(|S|). \qquad \square
\end{aligned}$$

Set-Covering Problem (Summary)

The same approach also gives an approximation ratio of $O(\ln(n))$ if there exists a cost function $c : \mathcal{F} \to \mathbb{R}^+$

Theorem 35.4

GREEDY-SET-COVER is a polynomial-time $\rho(n)$-algorithm, where

$$\rho(n) = H(\max\{|S| : S \in \mathcal{F}\}) \leq \ln(n) + 1.$$

- Is the bound on the approximation ratio in Theorem 35.4 tight?
- Is there a better algorithm?

Lower Bound

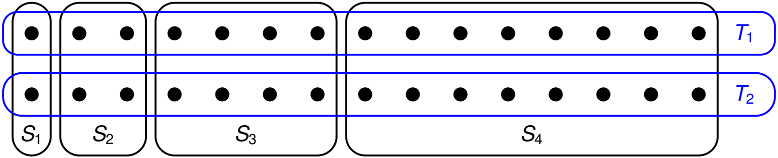
Unless P=NP, there is no $c \cdot \ln(n)$ polynomial-time approximation algorithm for some constant $0 < c < 1$.

Example where the solution of Greedy is bad

Instance

- Given any integer $k \geq 3$
- There are $n = 2^{k+1} - 2$ elements overall (so $k \approx \log_2 n$)
- Sets $S_1, S_2, \ldots, S_k$ are pairwise disjoint and each set contains $2, 4, \ldots, 2^k$ elements
- Sets T_1, T_2 are disjoint and each set contains half of the elements of each set $S_1, S_2, \ldots, S_k$

$k = 4, n = 30$:



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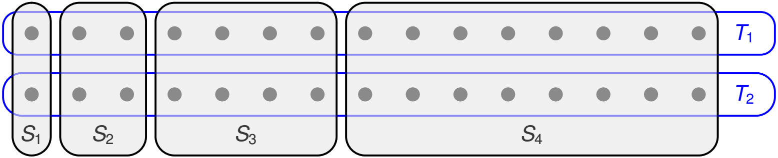

Solution of Greedy consists of k sets.

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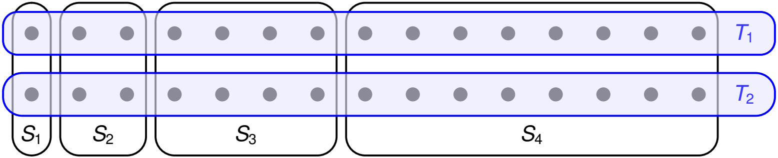

Optimum consists of 2 sets.

Exercise: Consider the vertex cover problem, restricted to a graph where every vertex has exactly 3 neighbours. Which approximation ratio can we obtain?

1. 1 (i.e., I can solve it exactly!!!)
2. 2
3. $11/6 = 2 - 1/6$
4. $H(n) \leq log(n)$