V. Approx. Algorithms: Travelling Salesman Problem

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Outline

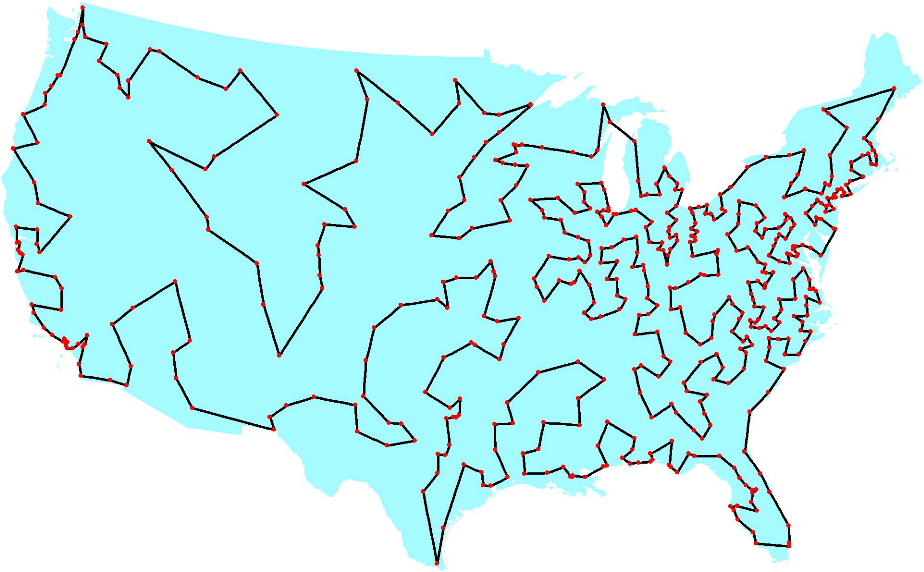
Introduction

General TSP

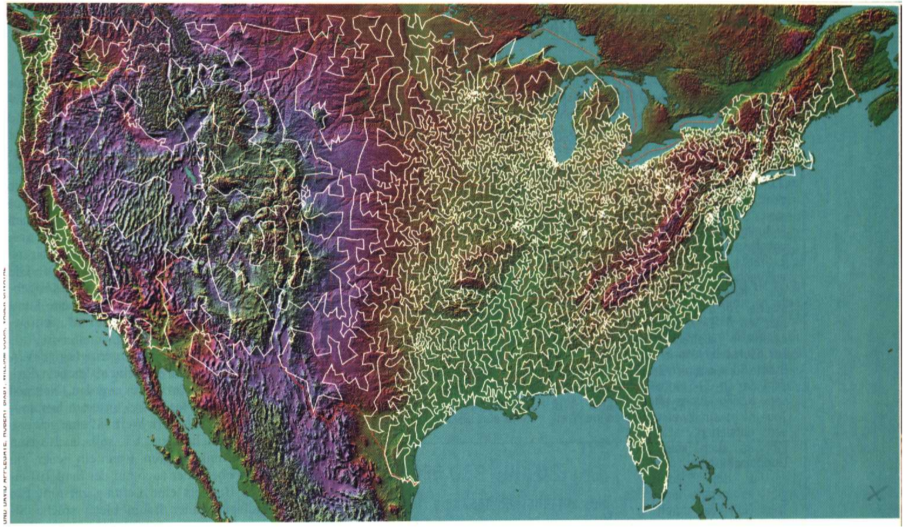
Metric TSP

33 city contest (1964)

532 cities (1987 [Padberg, Rinaldi])



13,509 cities (1999 [Applegate, Bixby, Chavatal, Cook])



The Traveling Salesman Problem (TSP)

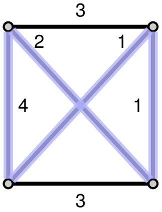
Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to your starting point.

Formal Definition

- Given: A complete undirected graph $G = (V, E)$ with nonnegative integer cost $c(u, v)$ for each edge $(u, v) \in E$
- Goal: Find a hamiltonian cycle of G with minimum cost.

Solution space consists of at most $n!$ possible tours!

Actually the right number is $(n-1)!/2$



$2 + 4 + 1 + 1 = 8$

Special Instances

- Metric TSP: costs satisfy triangle inequality:

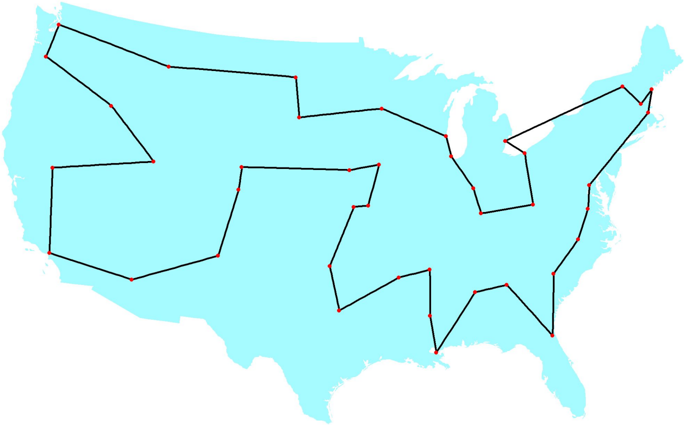
Even this version is NP hard (Ex. 35.2-2)

$$\forall u, v, w \in V: \qquad c(u, w) \leq c(u, v) + c(v, w).$$

- Euclidean TSP: cities are points in the Euclidean space, costs are equal to their (rounded) Euclidean distance

History of the TSP problem (1954)

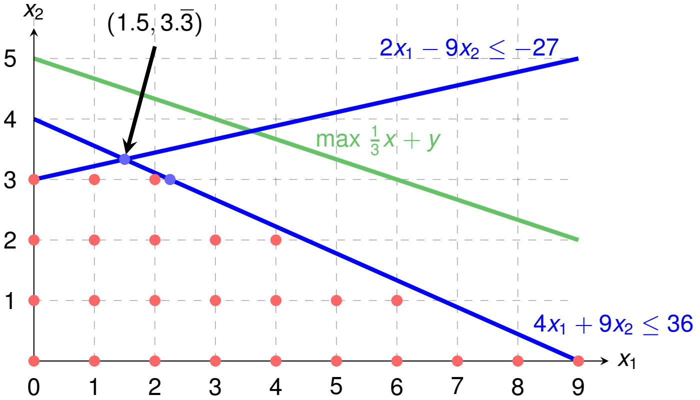
Dantzig, Fulkerson and Johnson found an optimal tour through 42 cities.



`http://www.math.uwaterloo.ca/tsp/history/img/dantzig_big.html`

The Dantzig-Fulkerson-Johnson Method

1. Create a linear program (variable $x(u,v) = 1$ iff tour goes between u and v)
2. Solve the linear program. If the solution is integral and forms a tour, stop. Otherwise find a new constraint to add (cutting plane)



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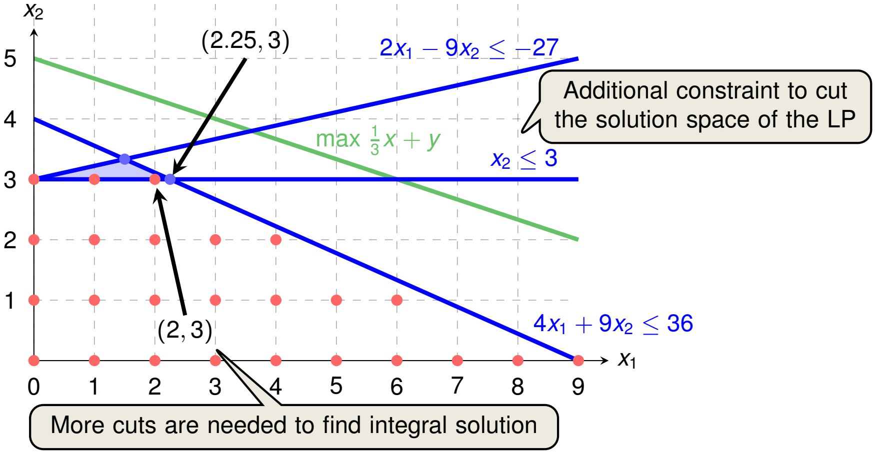


Outline

Hardness of Approximation

Theorem 35.3

If P $\neq$ NP, then for any constant $\rho \geq 1$, there is no polynomial-time approximation algorithm with approximation ratio ρ for the general TSP.

Idea: Reduction from the hamiltonian-cycle problem.

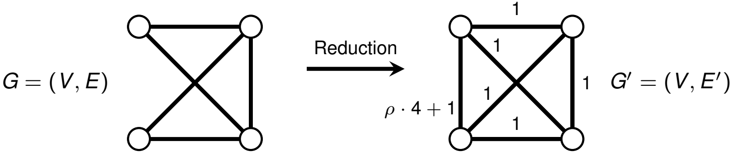
Proof:

- Let $G = (V, E)$ be an instance of the hamiltonian-cycle problem
- Let $G' = (V, E')$ be a complete graph with costs for each $(u, v) \in E'$:

Can create representations of G' and c in time polynomial in $|V|$ and $|E|$!

$$c(u, v) = \begin{cases} 1 & \text{if } (u, v) \in E, \\ \rho|V| + 1 & \text{otherwise.} \end{cases}$$

Large weight will render this edge useless!



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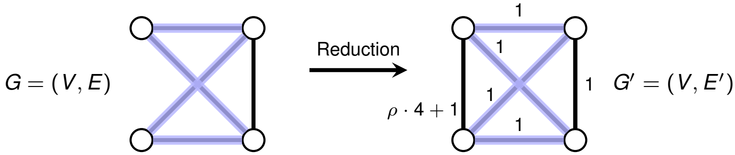
- If G has a hamiltonian cycle H, then (G', c) contains a tour of cost $|V|$



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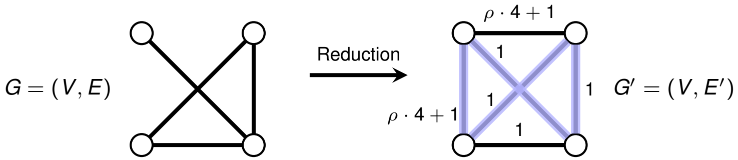
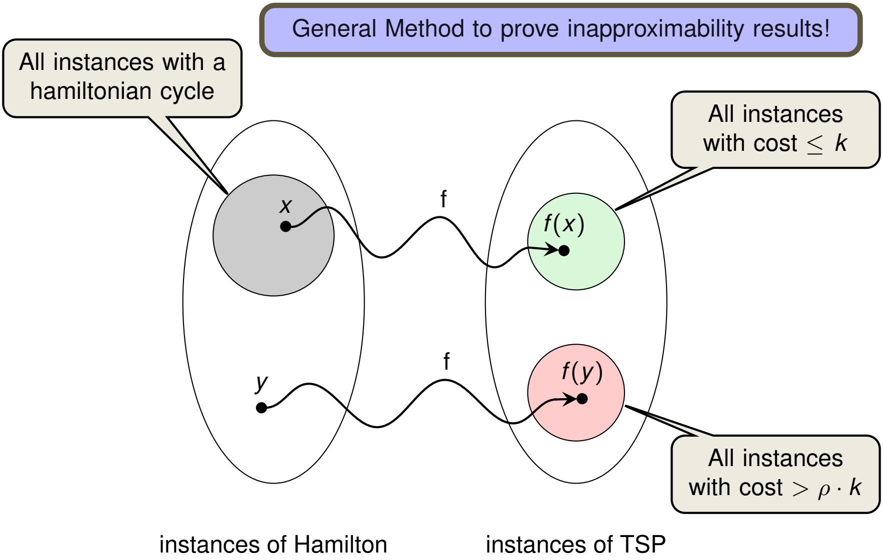
- If G has a hamiltonian cycle H, then (G', c) contains a tour of cost $|V|$
- If G does not have a hamiltonian cycle, then any tour T must use some edge $\notin E$,

$$\Rightarrow \qquad c(T) \geq (\rho|V| + 1) + (|V| - 1) = (\rho + 1)|V|.$$

- Gap of $\rho + 1$ between tours which are using only edges in G and those which don't
- ρ-Approximation of TSP in G' computes hamiltonian cycle in G (if one exists) $\square$



Proof of Theorem 35.3 from a higher perspective



Outline

Introduction

General TSP

Metric TSP

Metric TSP (TSP Problem with the Triangle Inequality)

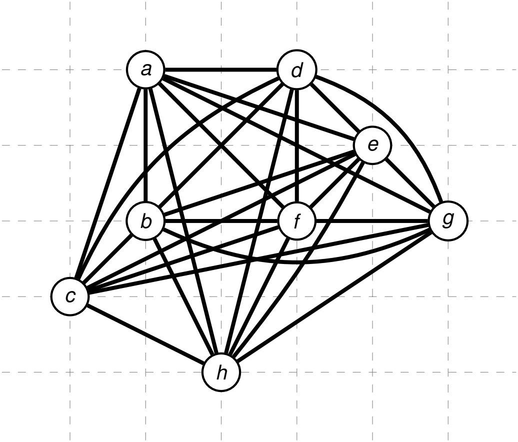
Idea: First compute an MST, and then create a tour based on the tree.

APPROX-TSP-TOUR(G, c)
select a vertex $r \in G.V$ to be a "root" vertex
compute a minimum spanning tree $T_{\min}$ for G from root r
    using MST-PRIM(G, c, r)
let H be a list of vertices, ordered according to when they are first visited
    in a preorder walk of $T_{\min}$
return the hamiltonian cycle H

Runtime is dominated by MST-PRIM, which is $\Theta(V^2)$.

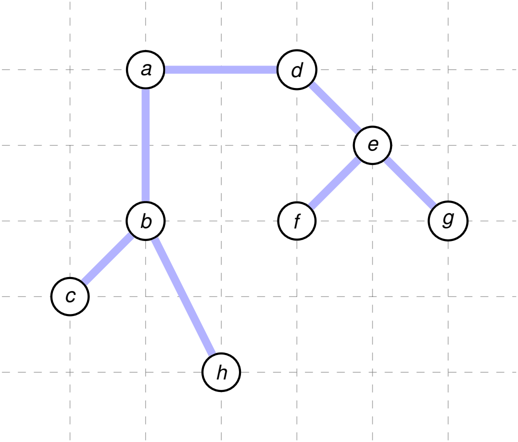
Remember: In the Metric-TSP problem, G is a complete graph.

Run of APPROX-TSP-TOUR



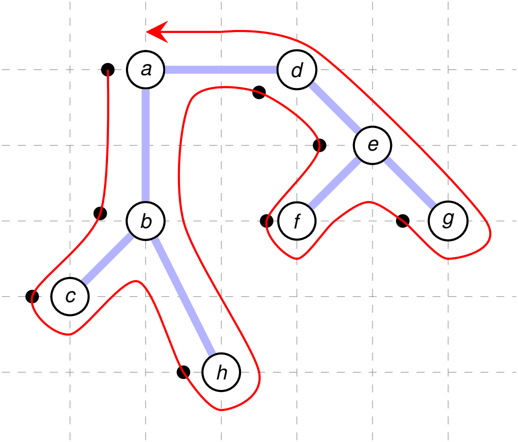
1. Compute MST $T_{\min}$

Run of APPROX-TSP-TOUR



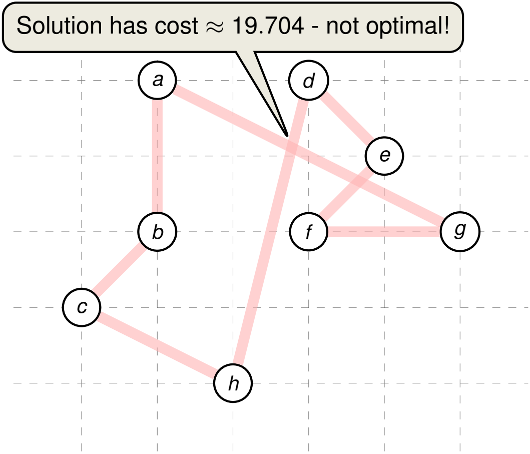
1. Compute MST $T_{\min}$ ✓
2. Perform preorder walk on MST $T_{\min}$

Run of Approx-Tsp-Tour



1. Compute MST $T_{\min}$ ✓
2. Perform preorder walk on MST $T_{\min}$ ✓
3. Return list of vertices according to the preorder tree walk

Run of Approx-Tsp-Tour



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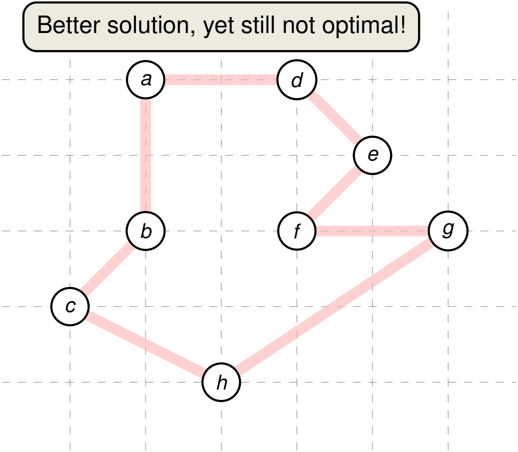
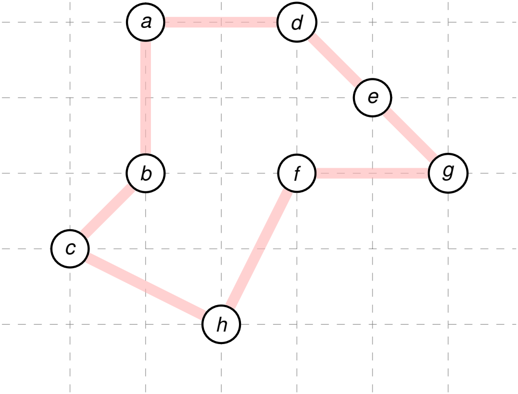
Run of APPROX-TSP-TOUR



1. Compute MST T_{min} ✓
2. Perform preorder walk on MST T_{min} ✓
3. Return list of vertices according to the preorder tree walk ✓

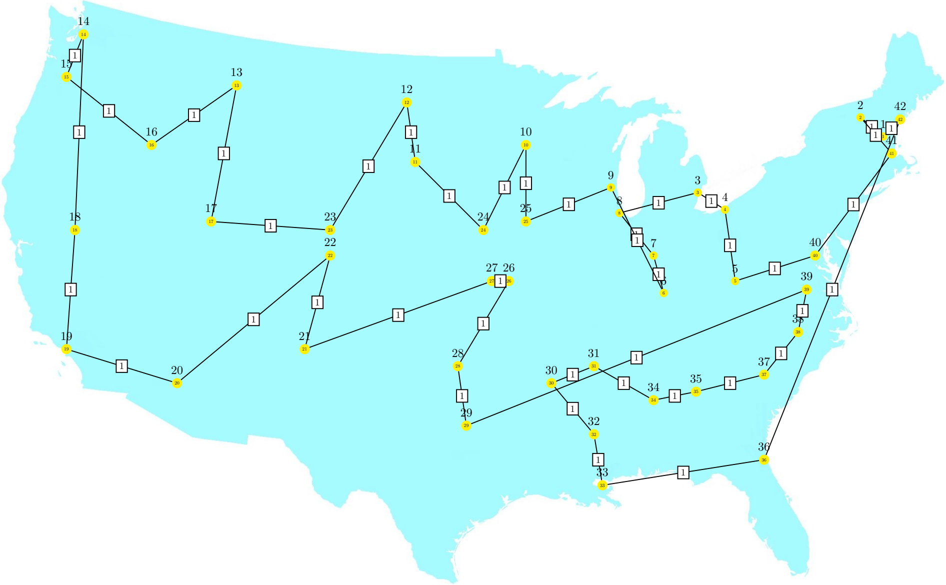
Run of APPROX-TSP-TOUR

This is the optimal solution (cost ≈ 14.715).

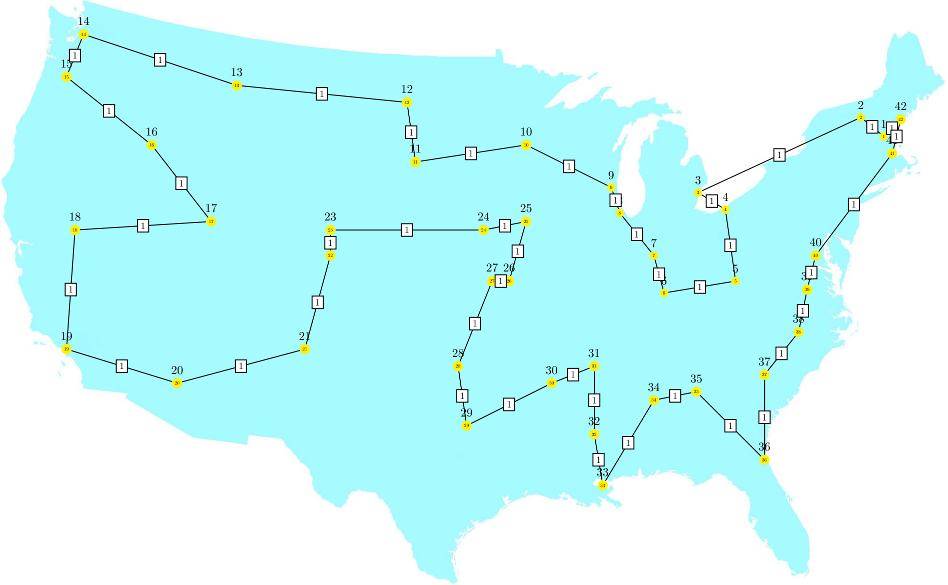


1. Compute MST T_{min} ✓
2. Perform preorder walk on MST T_{min} ✓
3. Return list of vertices according to the preorder tree walk ✓

Approximate Solution: Objective 921



Optimal Solution: Objective 699



Proof of the Approximation Ratio

Theorem 35.2

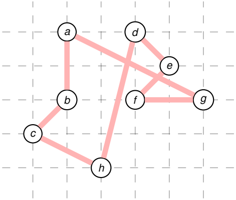
APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

Proof:

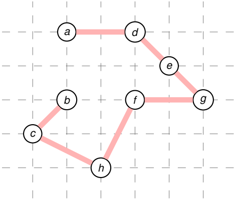
- Consider the optimal tour H^* and remove an arbitrary edge

$\Rightarrow$ yields a spanning tree T and $c(T_{\min}) \leq c(T) \leq c(H^*)$

exploiting that all edge costs are non-negative!



solution H of APPROX-TSP



spanning tree T as a subset of H^*

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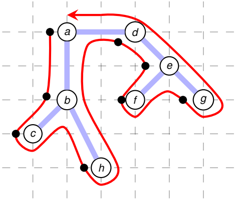
$\Rightarrow$ yields a spanning tree T and $c(T_{\min}) \leq c(T) \leq c(H^*)$

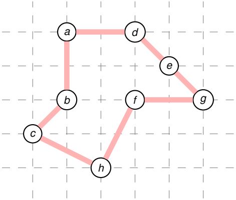
- Let W be the full walk of the minimum spanning tree $T_{\min}$ (including repeated visits)

$\Rightarrow$ Full walk traverses every edge exactly twice, so

$$c(W) = 2c(T_{\min}) \leq 2c(T) \leq 2c(H^*)$$



Walk $W = (a, b, c, b, h, b, a, d, e, f, e, g, e, d, a)$



optimal solution H^*

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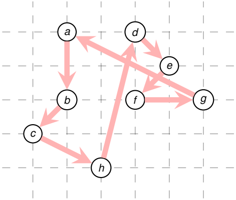
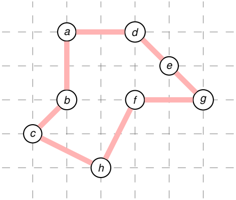
exploiting triangle inequality!

- Deleting duplicate vertices from W yields a tour H with smaller cost:

$$c(H) \leq c(W) \leq 2c(H^*) \qquad \square$$



Walk W = (a, b, c, ~~b~~, h, ~~b~~, ~~a~~, d, e, f, ~~e~~, g, ~~e~~, ~~d~~, a)



optimal solution H^*

Christofides Algorithm

Theorem 35.2

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

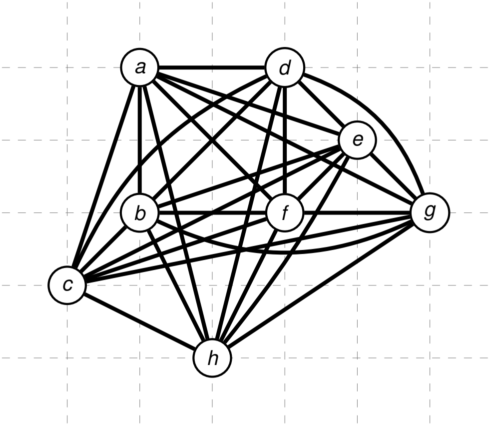
Can we get a better approximation ratio?

```
CHRISTOFIDES(G, c)
select a vertex r ∈ G.V to be a "root" vertex
compute a minimum spanning tree T_min for G from root r
        using MST-PRIM(G, c, r)
compute a perfect matching M_min with minimum weight in the complete graph
        over the odd-degree vertices in T_min
let H be a list of vertices, ordered according to when they are first visited
        in a Eulearian circuit of T_min ∪ M_min
return the hamiltonian cycle H
```

Theorem (Christofides'76)

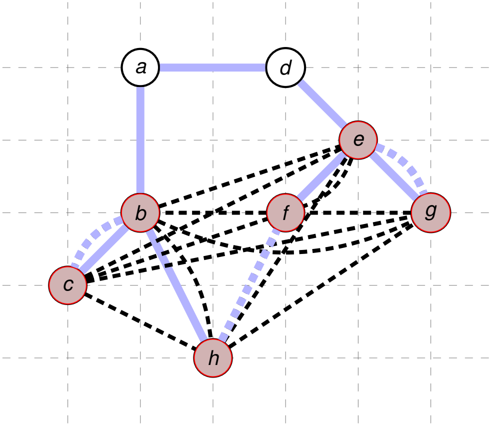
There is a polynomial-time $\frac{3}{2}$-approximation algorithm for the travelling salesman problem with the triangle inequality.

Run of CHRISTOFIDES

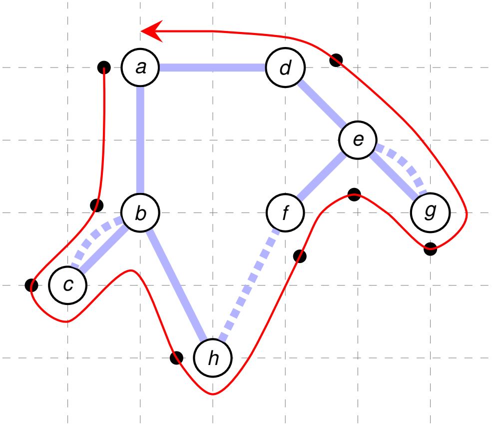


1. Compute MST T_{min}

Run of CHRISTOFIDES



1. Compute MST $T_{\min}$ ✓
2. Add a minimum-weight perfect matching $M_{\min}$ of the odd vertices in $T_{\min}$ ✓

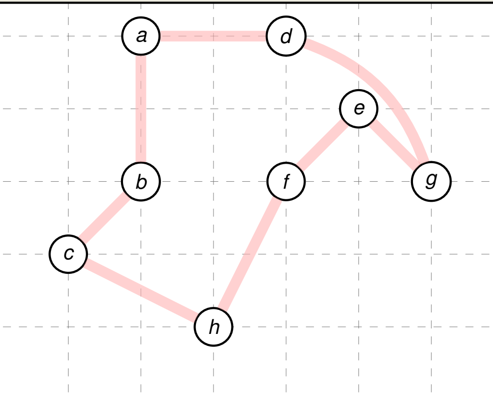


1. Compute MST $T_{\min}$ ✓
2. Add a minimum-weight perfect matching $M_{\min}$ of the odd vertices in $T_{\min}$ ✓
3. Find an Eulerian Circuit in $T_{\min} \cup M_{\min}$ ✓

All vertices in $T_{\min} \cup M_{\min}$ have even degree!

Run of CHRISTOFIDES

Solution has cost ≈ 15.54 - within 10% of the optimum!



1. Compute MST $T_{\min}$ ✓
2. Add a minimum-weight perfect matching $M_{\min}$ of the odd vertices in $T_{\min}$ ✓
3. Find an Eulerian Circuit in $T_{\min} \cup M_{\min}$ ✓
4. Transform the Circuit into a Hamiltonian Cycle ✓

Proof of the Approximation Ratio

Theorem (Christofides'76)

There is a polynomial-time $\frac{3}{2}$-approximation algorithm for the travelling salesman problem with the triangle inequality.

Proof is quite similar to the previous analysis

Proof (Approximation Ratio):

- As before, let H^* denote the optimal tour
- The Eulerian Circuit W uses each edge of the minimum spanning tree $T_{\min}$ and the minimum-weight matching $M_{\min}$ exactly once:

$$c(W) = c(T_{\min}) + c(M_{\min}) \leq c(H^*) + c(M_{\min}) \qquad (1)$$

- Let H^*_{odd} be an optimal tour on the odd-degree vertices in $T_{\min}$
- Taking edges alternately, we obtain two matchings M_1 and M_2 such that $c(M_1) + c(M_2) = c(H^*_{odd})$
- By shortcutting and the triangle inequality,

Number of odd-degree vertices is even!

$$c(M_{\min}) \leq \frac{1}{2} c({H_{odd}}^*) \leq \frac{1}{2} c(H^*). \qquad (2)$$

- Combining 1 with 2 yields

$$c(W) \leq c(H^*) + c(M_{\min}) \leq c(H^*) + \frac{1}{2} c(H^*) = \frac{3}{2} c(H^*).$$

Concluding Remarks

Theorem (Christofides'76)

There is a polynomial-time $\frac{3}{2}$-approximation algorithm for the travelling salesman problem with the triangle inequality.

still the best algorithm for the metric TSP problem(!)

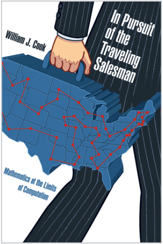
Theorem (Arora'96, Mitchell'96)

There is a PTAS for the Euclidean TSP Problem.

Both received the Gödel Award 2010

"Christos Papadimitriou told me that the traveling salesman problem is not a problem. It's an addiction."

Jon Bentley 1991



Exercise: Prove that the approximation ratio of APPROX-TSP-TOUR satisfies $\rho(n) < 2$.

Hint: Consider the effect of the shortcutting, but note that edge costs might be zero!