V. Approx. Algorithms: Travelling Salesman Problem

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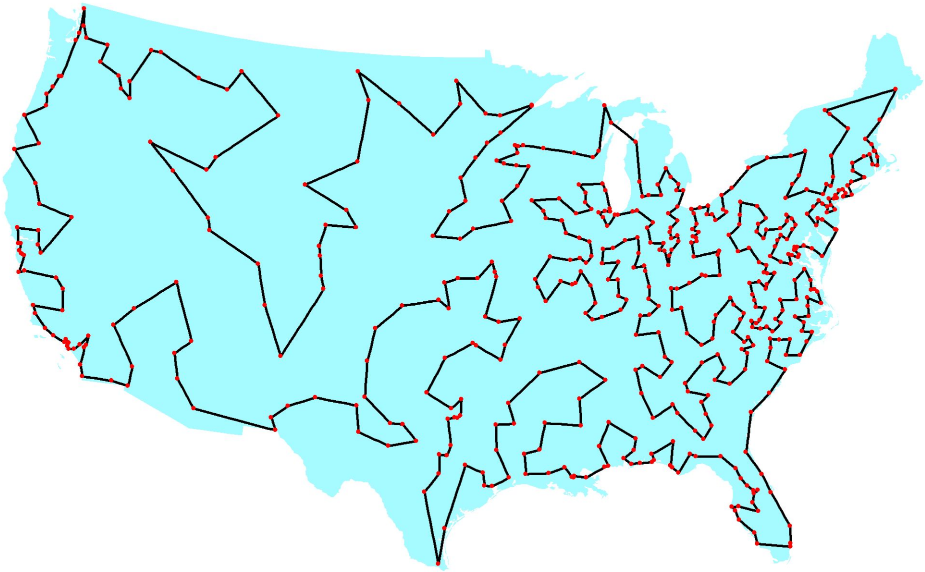
UNIVERSITY OF CAMBRIDGE

Outline

33 city contest (1964)

532 cities (1987 [Padberg, Rinaldi])



13,509 cities (1999 [Applegate, Bixby, Chavatal, Cook])

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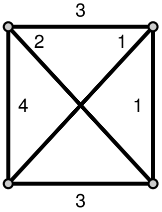

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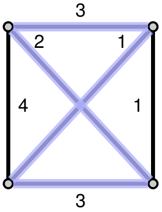

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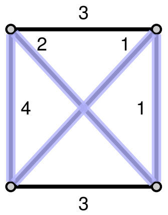

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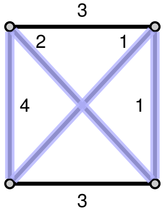

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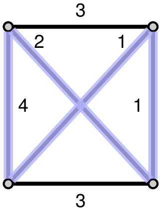

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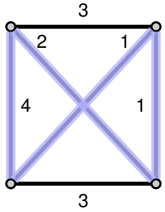

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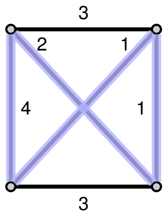

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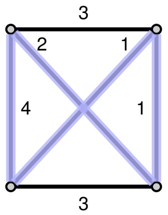

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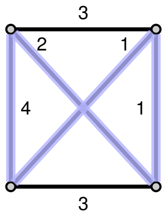

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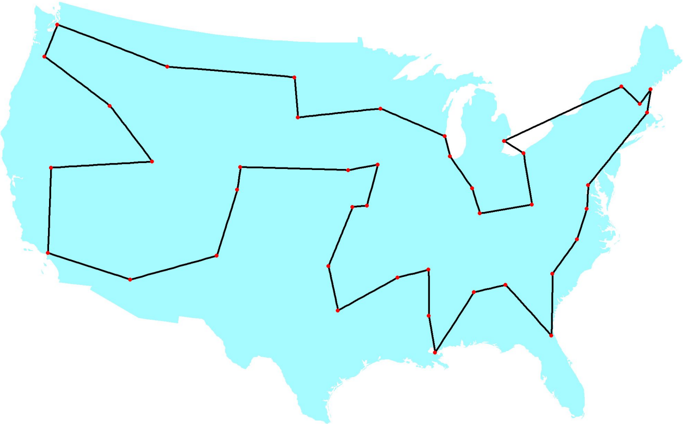
- Euclidean TSP: cities are points in the Euclidean space, costs are equal to their (rounded) Euclidean distance

History of the TSP problem (1954)

Dantzig, Fulkerson and Johnson found an optimal tour through 42 cities.



`http://www.math.uwaterloo.ca/tsp/history/img/dantzig_big.html`

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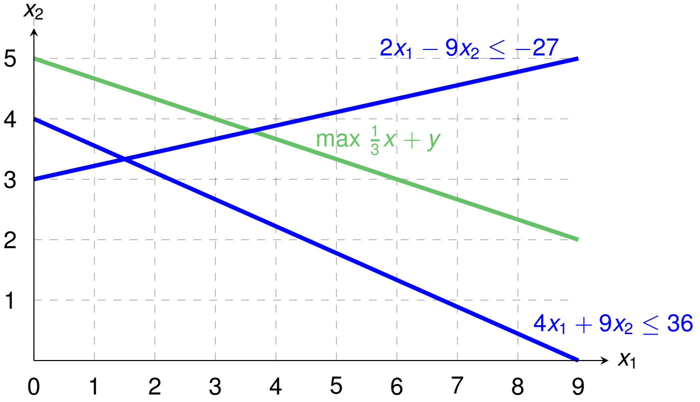

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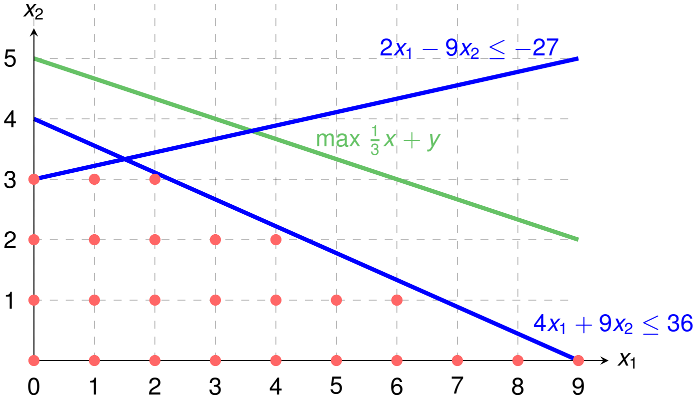

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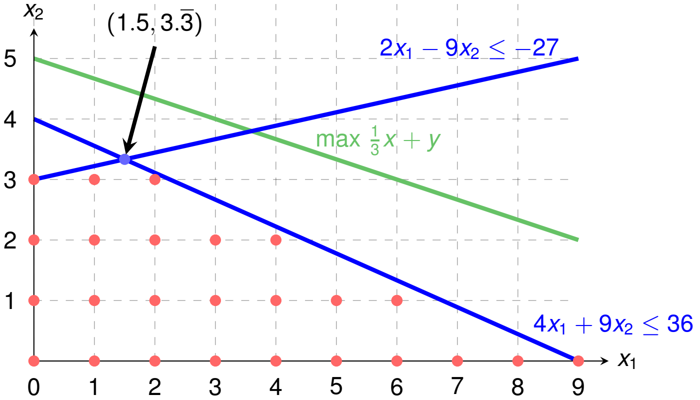

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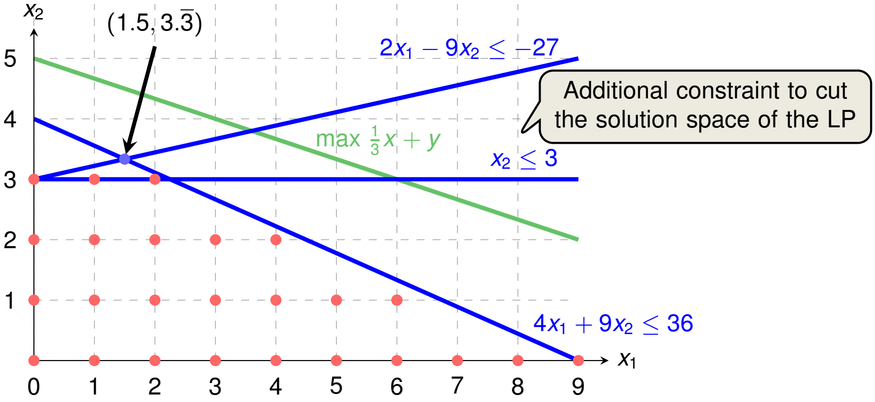

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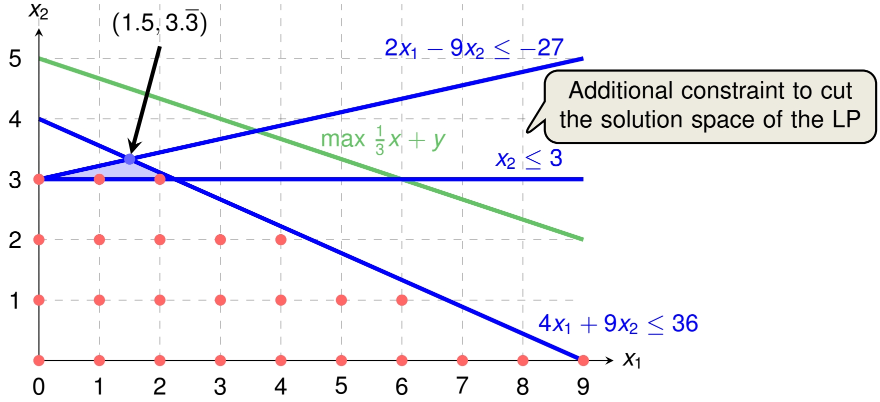

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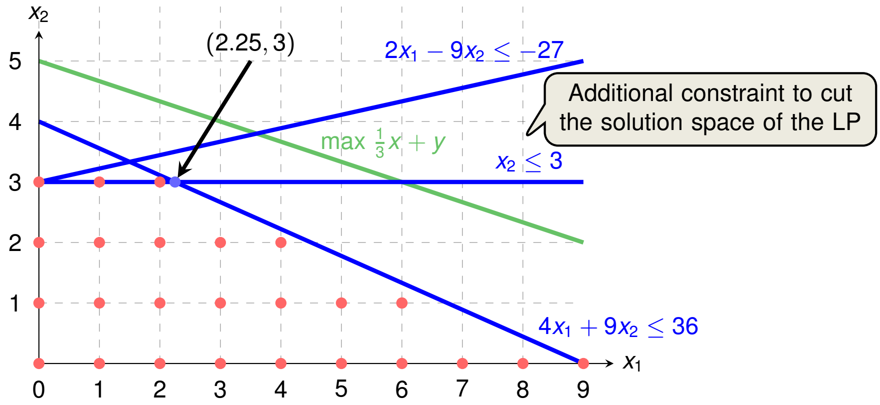

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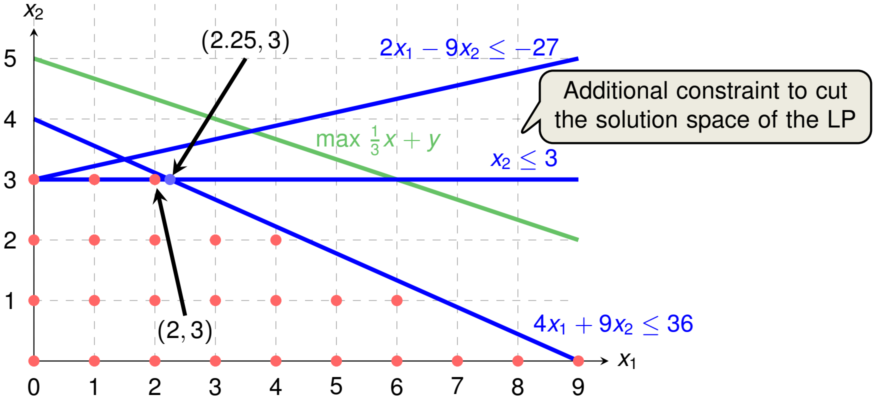

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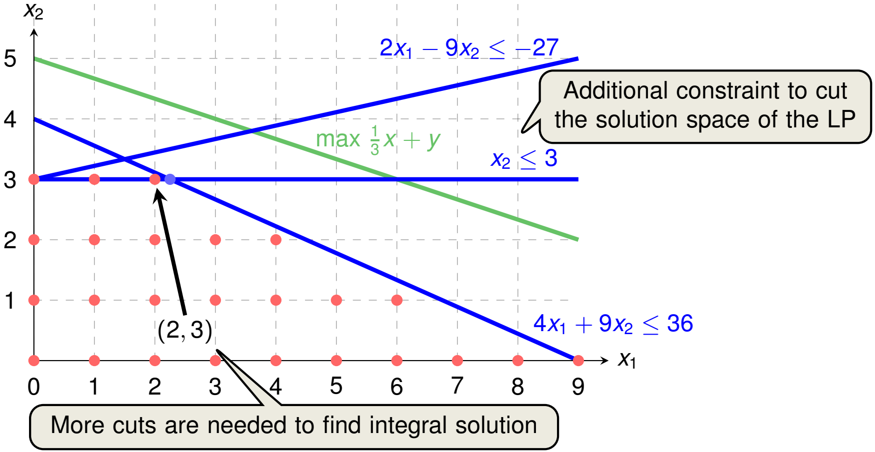

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General TSP

Metric TSP

Hardness of Approximation

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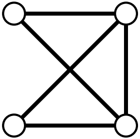
$G = (V, E)$ 

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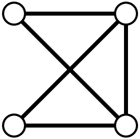

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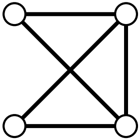

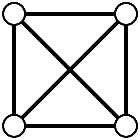

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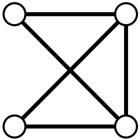

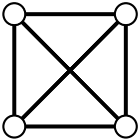

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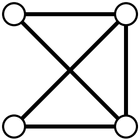

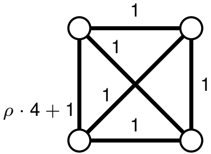

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Large weight will render this edge useless!

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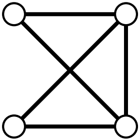

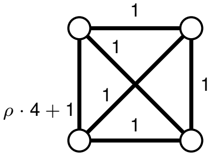

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Can create representations of G' and c in time polynomial in $|V|$ and $|E|$!

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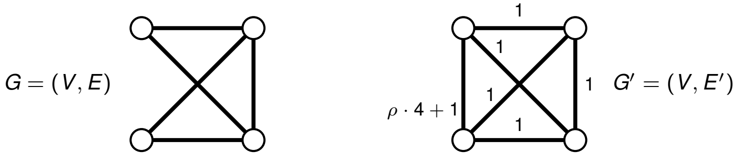

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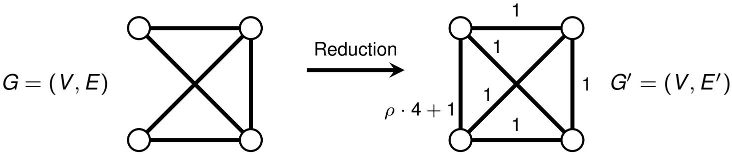

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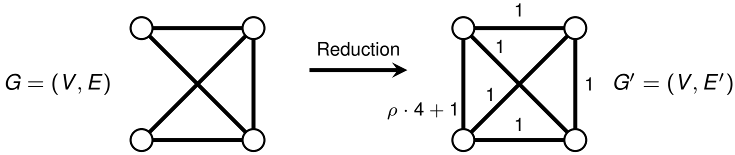

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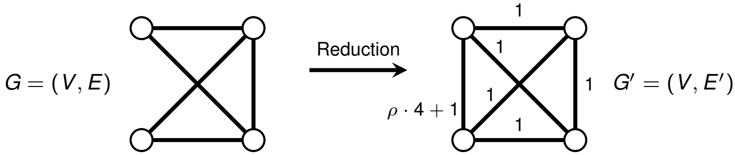

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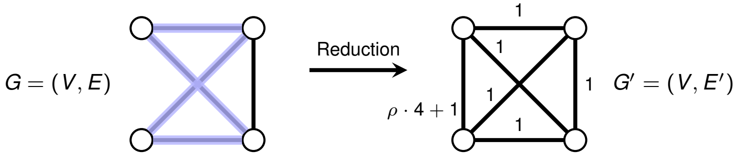

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Theorem 35.3

If P $\neq$ NP, then for any constant $\rho \geq 1$, there is no polynomial-time approximation algorithm with approximation ratio ρ for the general TSP.

Idea: Reduction from the hamiltonian-cycle problem.

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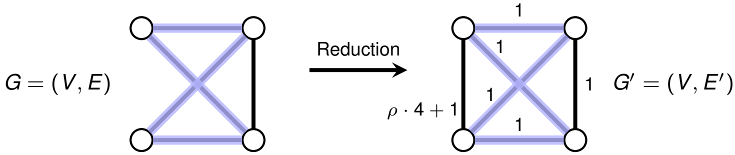

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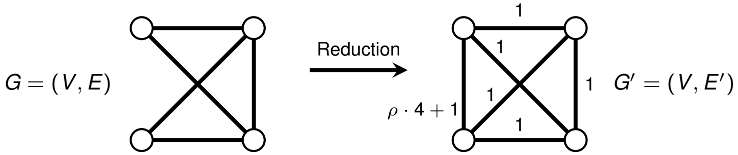

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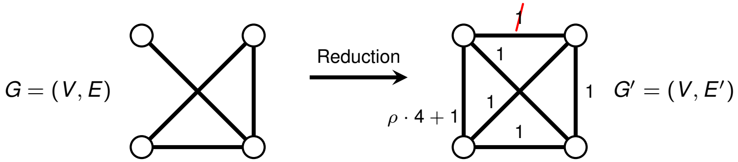

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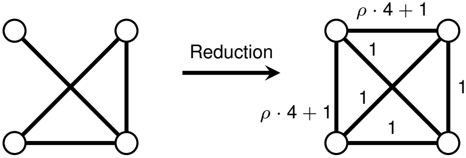

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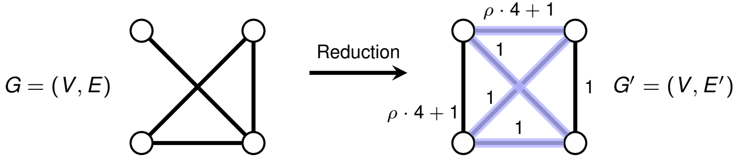

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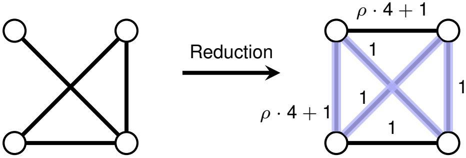

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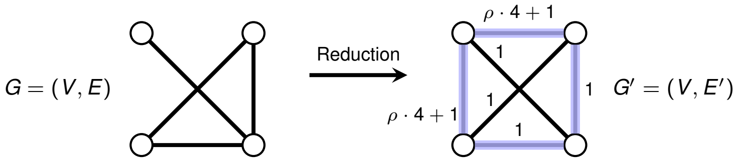

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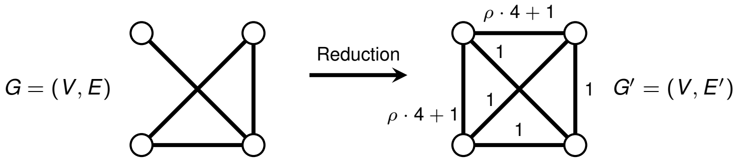

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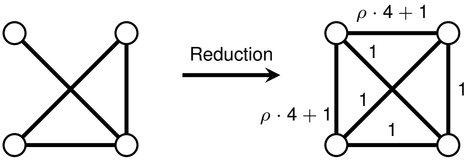

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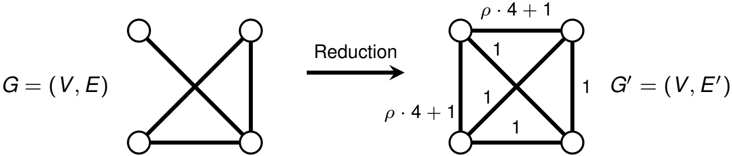

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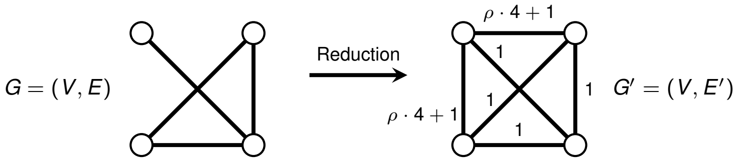

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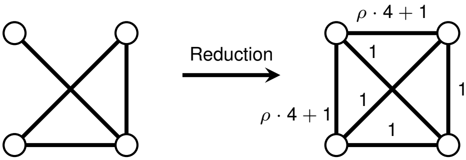

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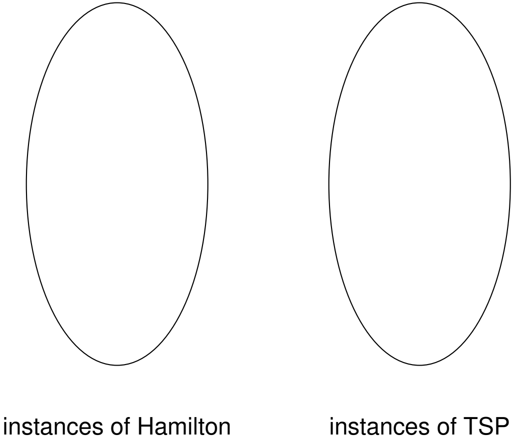
Proof of Theorem 35.3 from a higher perspective



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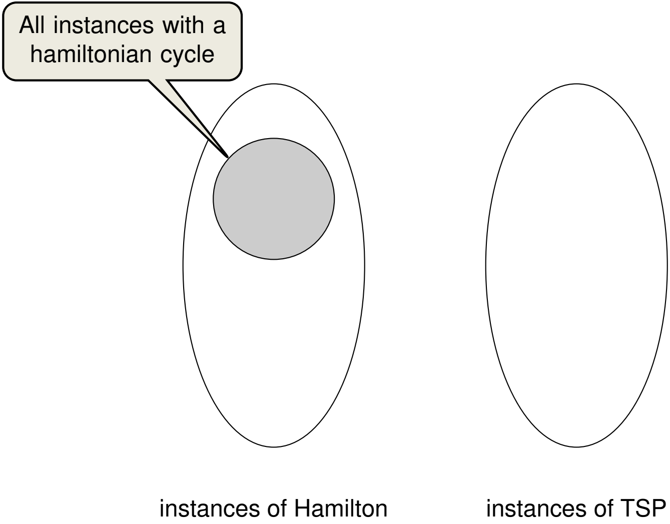

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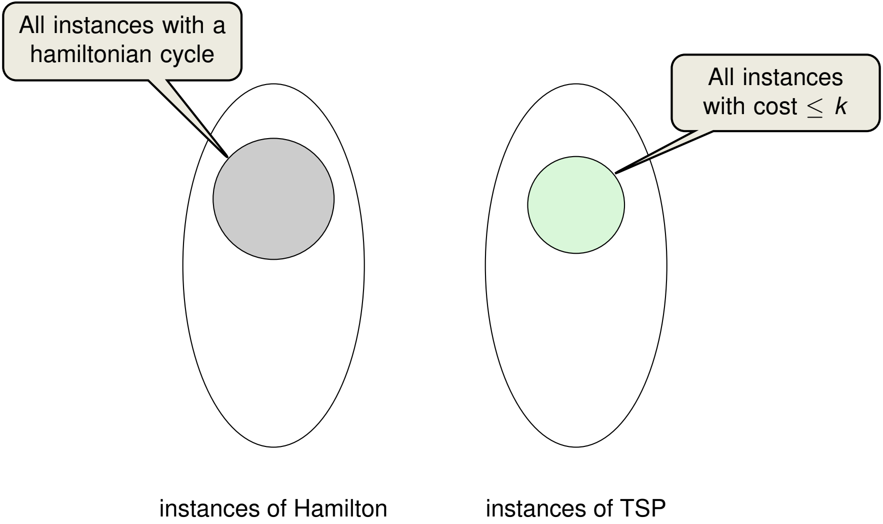

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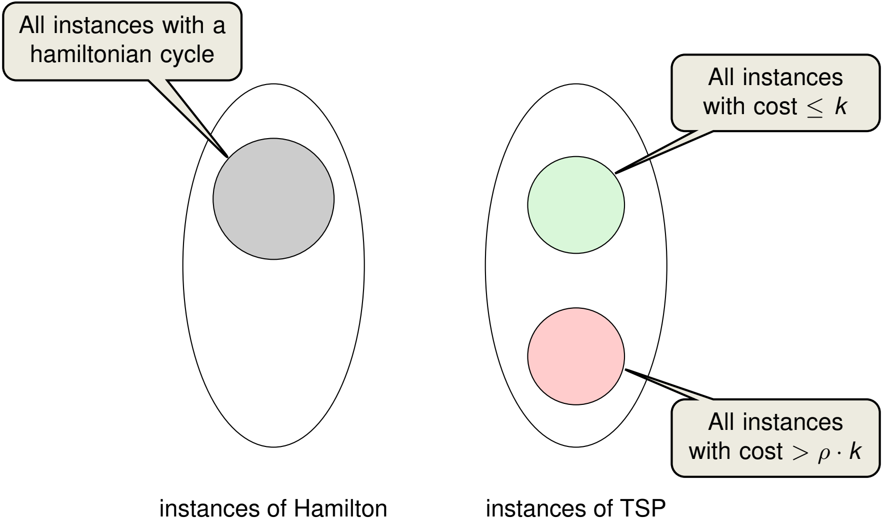

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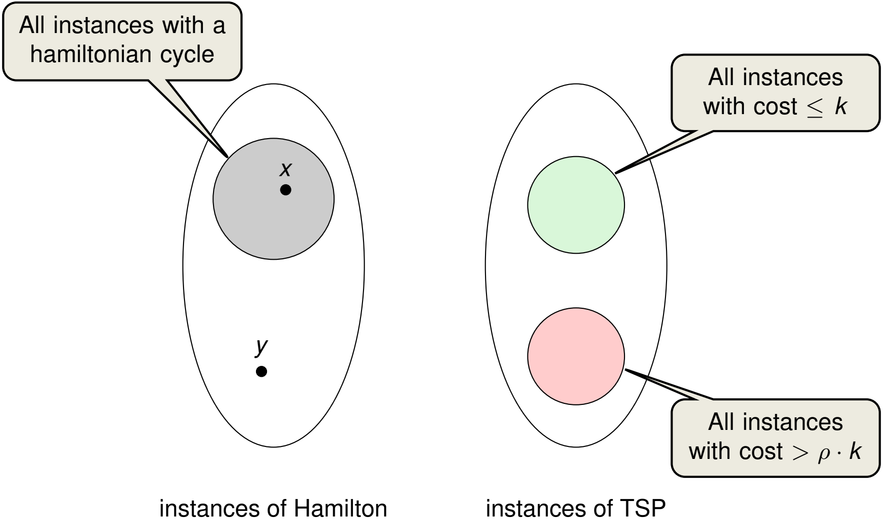

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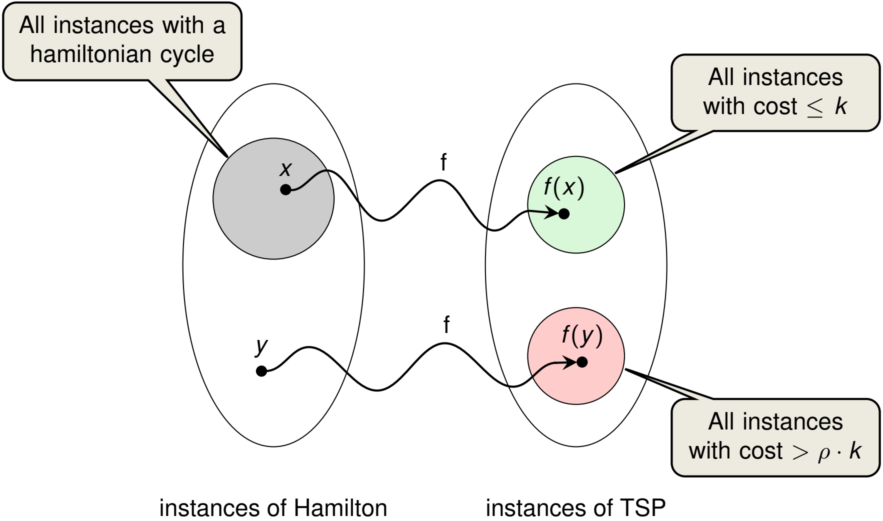

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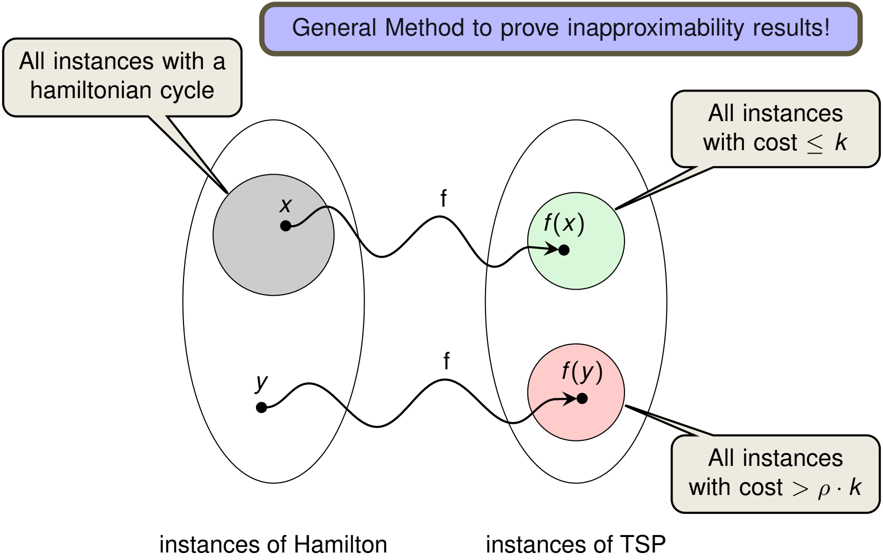

Outline

Introduction

General TSP

Metric TSP

Metric TSP (TSP Problem with the Triangle Inequality)

Idea: First compute an MST, and then create a tour based on the tree.

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APPROX-TSP-TOUR(G, c)
select a vertex r ∈ G.V to be a "root" vertex
compute a minimum spanning tree T_min for G from root r
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let H be a list of vertices, ordered according to when they are first visited
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return the hamiltonian cycle H
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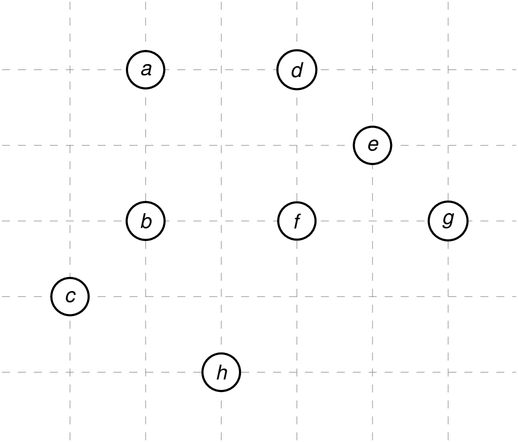
Remember: In the Metric-TSP problem, G is a complete graph.

Run of APPROX-TSP-TOUR



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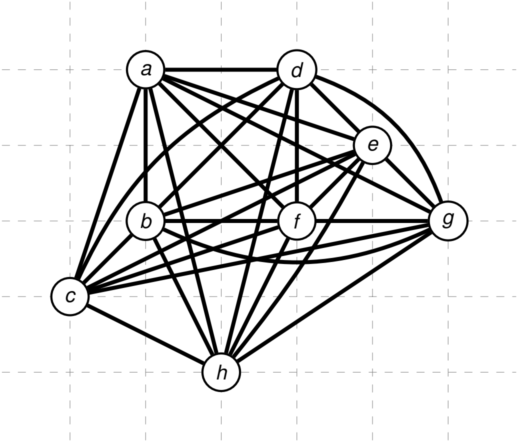

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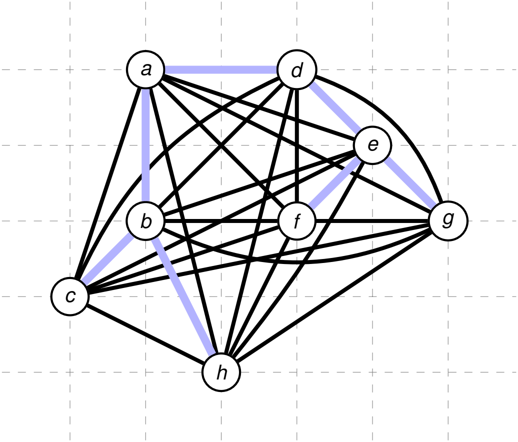

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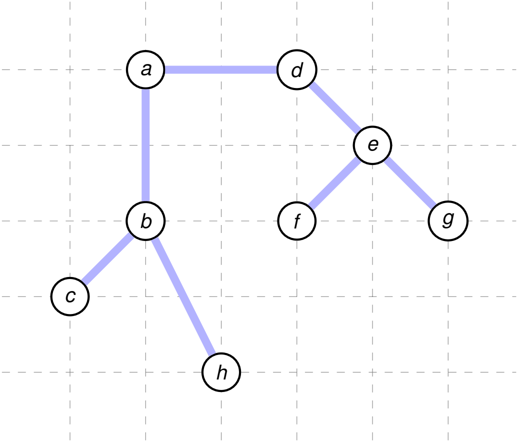

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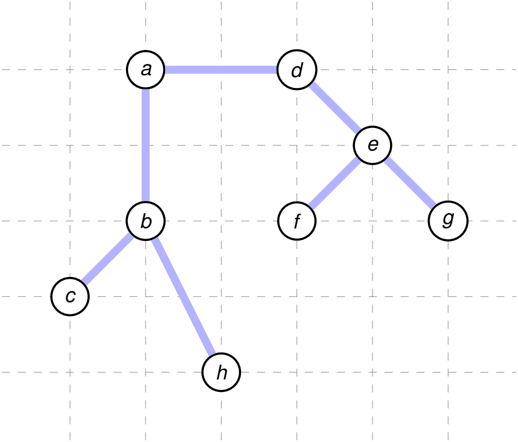

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2. Perform preorder walk on MST $T_{\min}$

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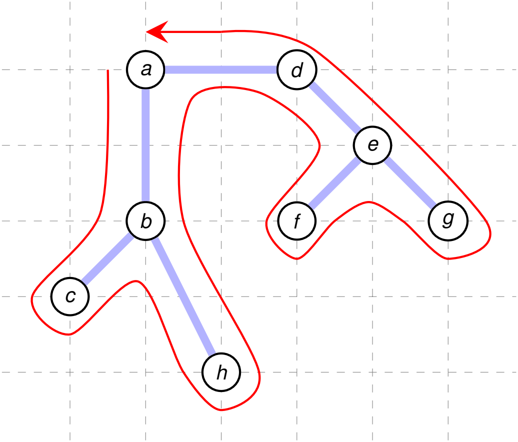

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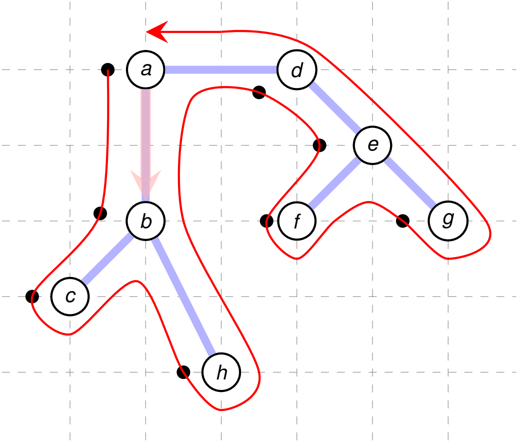

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3. Return list of vertices according to the preorder tree walk

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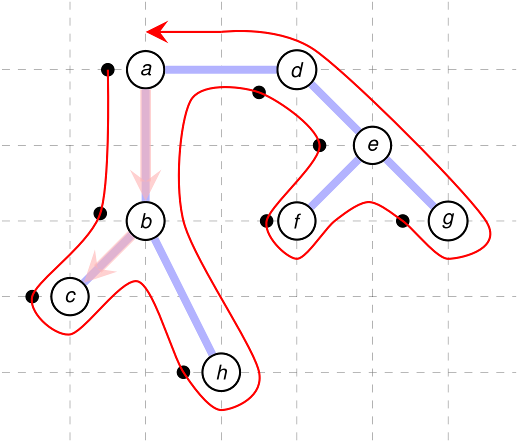

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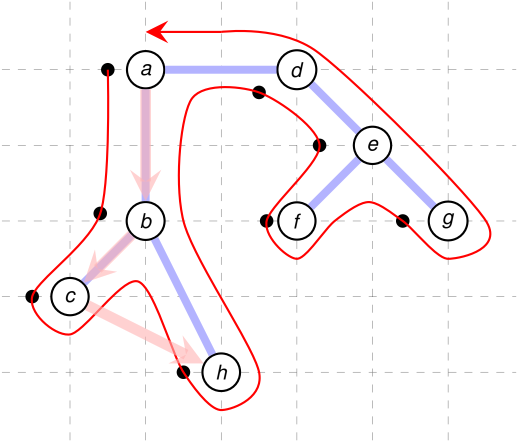

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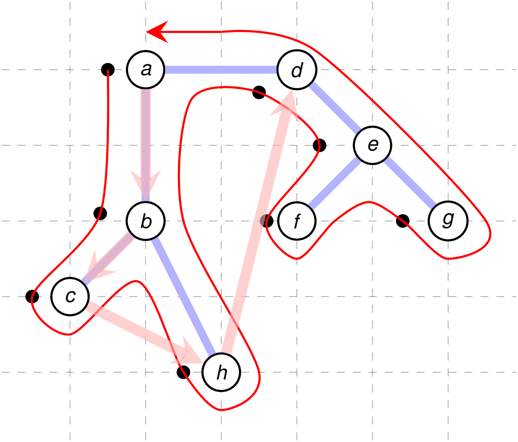

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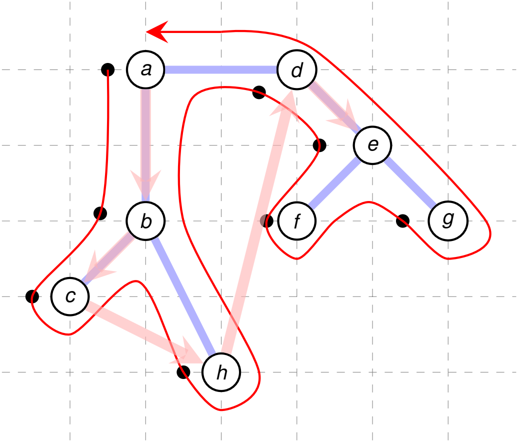

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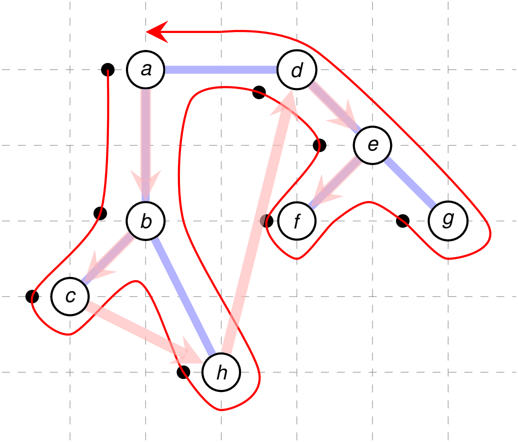

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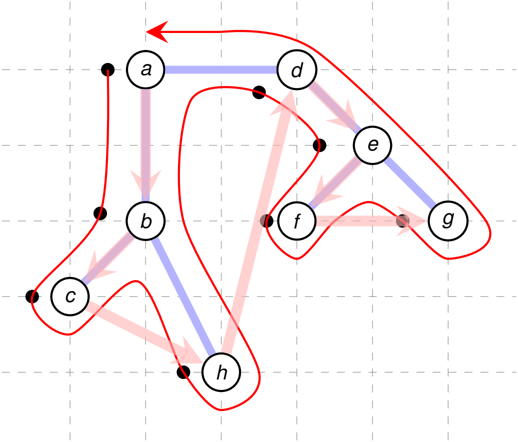

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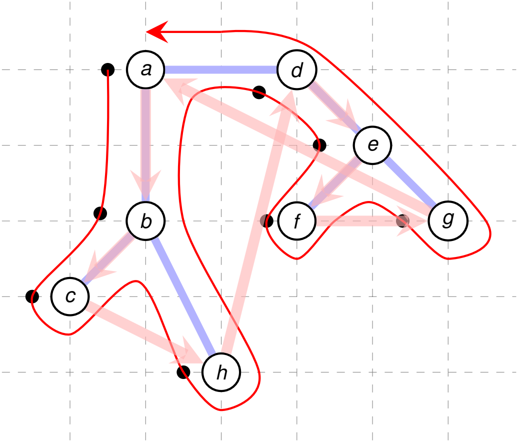

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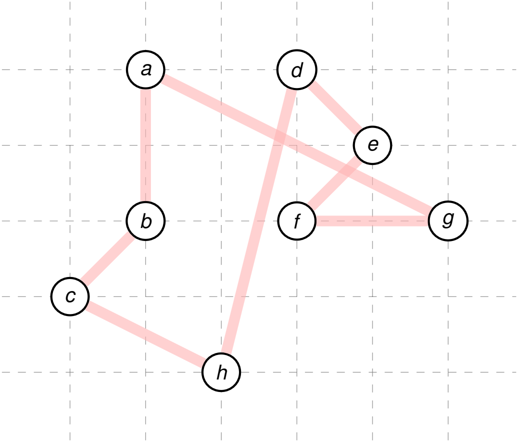

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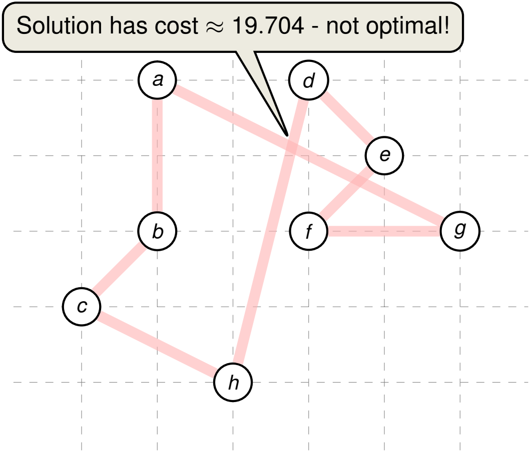
Run of APPROX-TSP-TOUR



1. Compute MST $T_{\min}$ ✓
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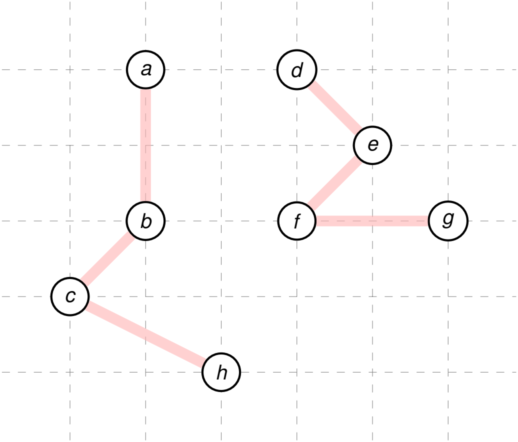

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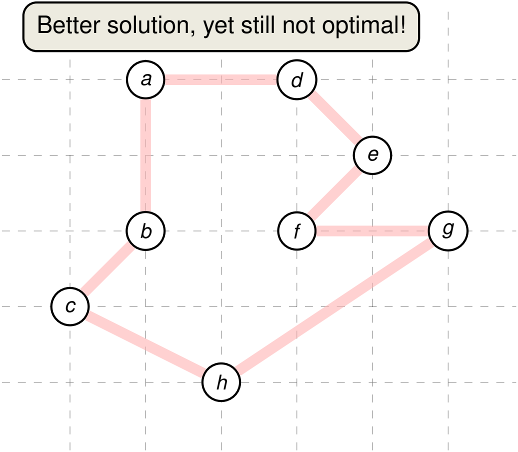
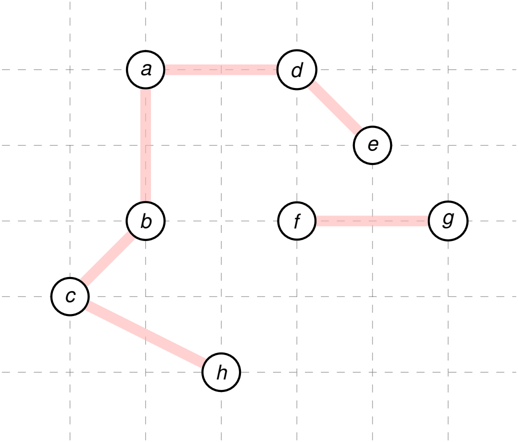


1. Compute MST T_{min} ✓
2. Perform preorder walk on MST T_{min} ✓
3. Return list of vertices according to the preorder tree walk ✓

Run of APPROX-TSP-TOUR



1. Compute MST $T_{\min}$ ✓
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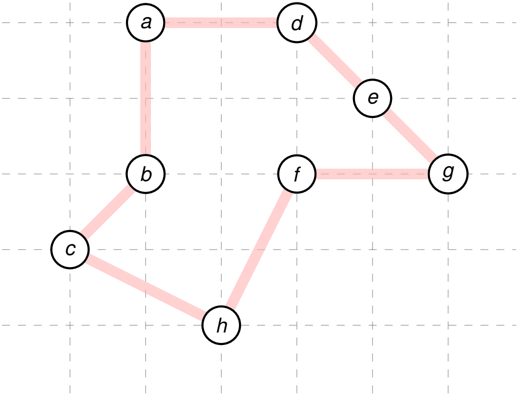
Run of APPROX-TSP-TOUR

This is the optimal solution (cost ≈ 14.715).



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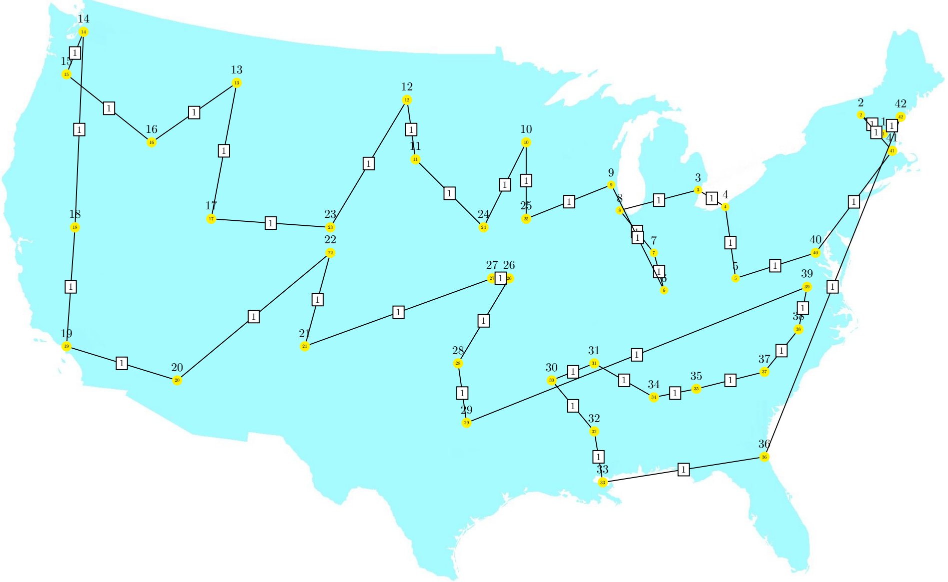
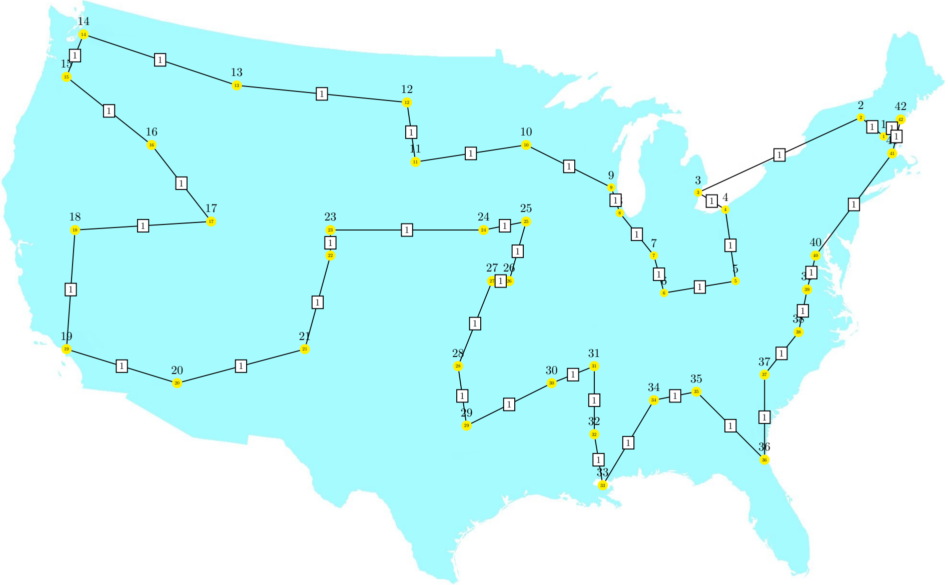
Approximate Solution: Objective 921



Optimal Solution: Objective 699



Proof of the Approximation Ratio

Theorem 35.2

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

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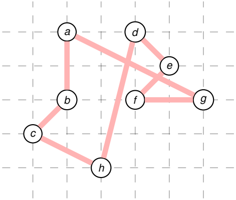

solution H of APPROX-TSP

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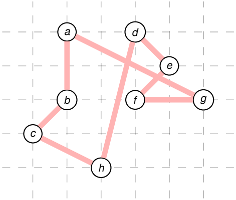

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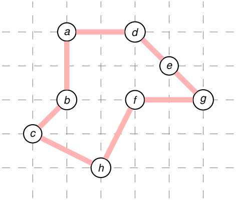

optimal solution H^*

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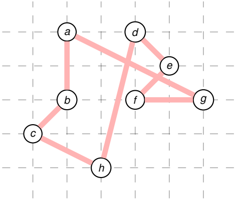

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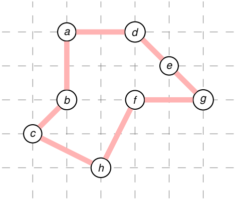

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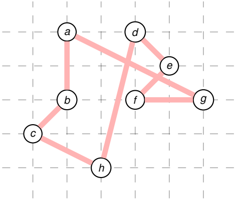

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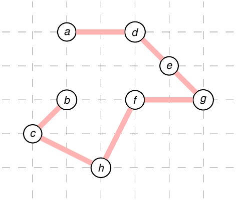

spanning tree T as a subset of H^*

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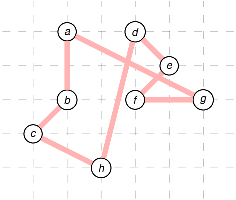

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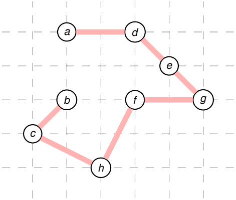

spanning tree T as a subset of H^*

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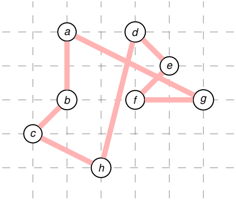

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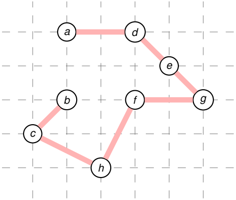

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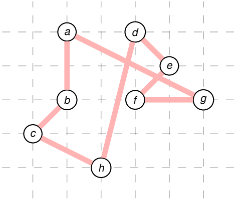
exploiting that all edge costs are non-negative!



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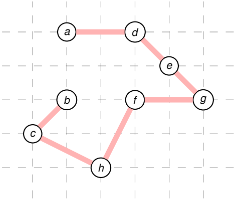

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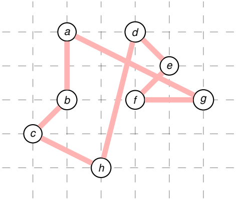

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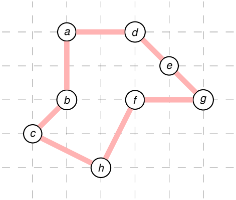

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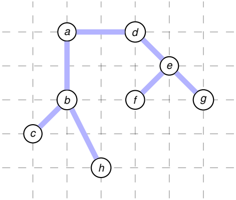
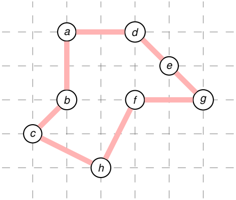


minimum spanning tree $T_{\min}$



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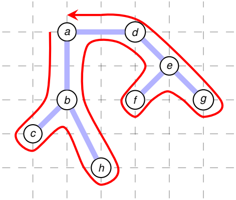
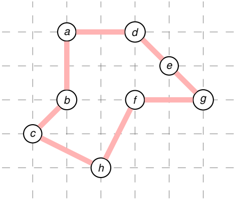


Walk $W = (a, b, c, b, h, b, a, d, e, f, e, g, e, d, a)$



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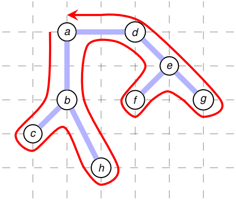

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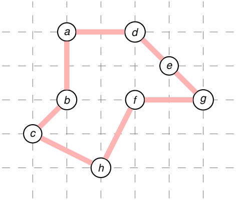

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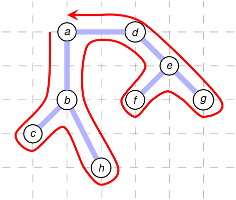

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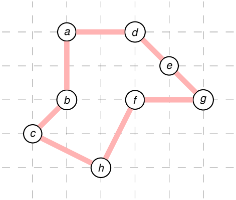

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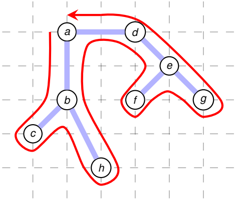

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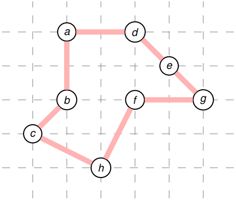

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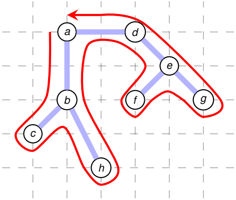

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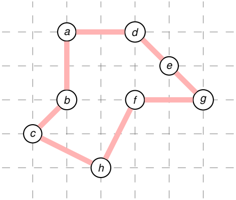

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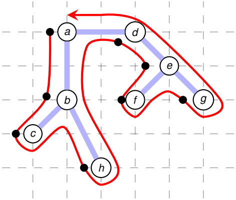

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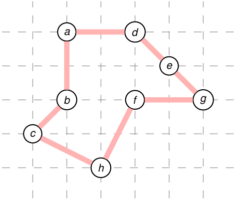

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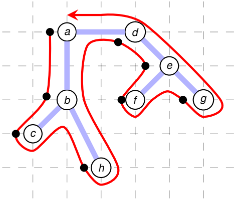

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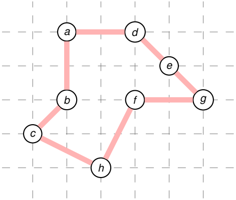

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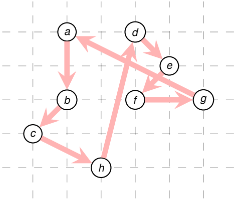
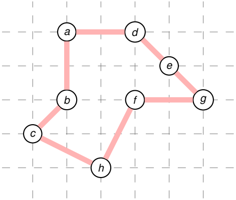


Tour $H = (a, b, c, h, d, e, f, g, a)$



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exploiting triangle inequality!

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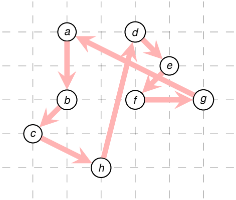

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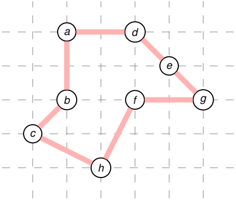

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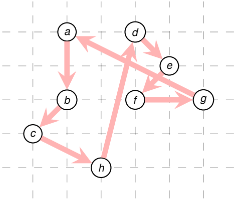

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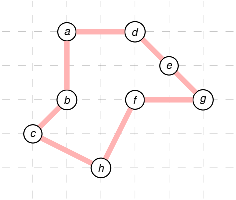

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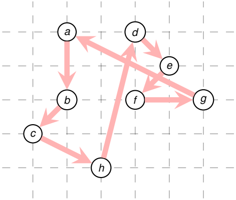
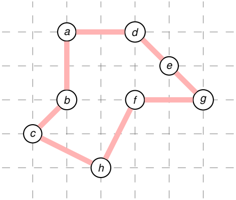


Tour $H = (a, b, c, h, d, e, f, g, a)$



optimal solution H^*

Proof of the Approximation Ratio

Theorem 35.2

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

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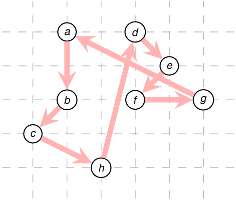

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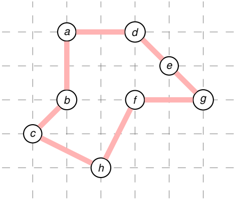

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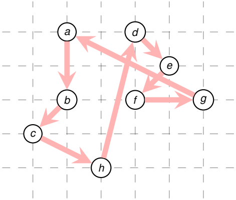

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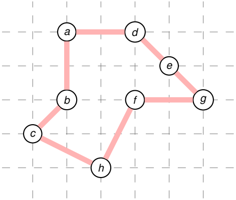

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Christofides Algorithm

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Can we get a better approximation ratio?

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Can we get a better approximation ratio?

```
CHRISTOFIDES(G, c)
select a vertex r ∈ G.V to be a "root" vertex
compute a minimum spanning tree T_min for G from root r
        using MST-PRIM(G, c, r)
compute a perfect matching M_min with minimum weight in the complete graph
        over the odd-degree vertices in T_min
let H be a list of vertices, ordered according to when they are first visited
        in a Eulearian circuit of T_min ∪ M_min
return the hamiltonian cycle H
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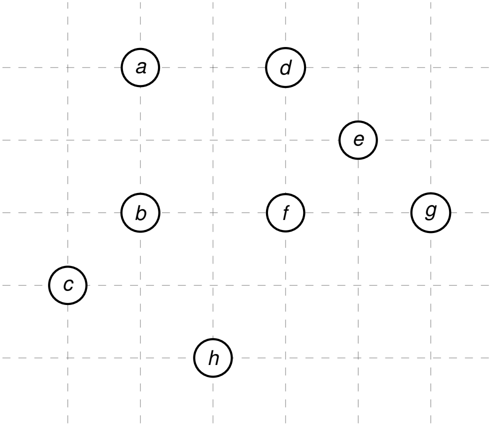
Theorem (Christofides'76)

There is a polynomial-time $\frac{3}{2}$-approximation algorithm for the travelling salesman problem with the triangle inequality.

Run of Christofides



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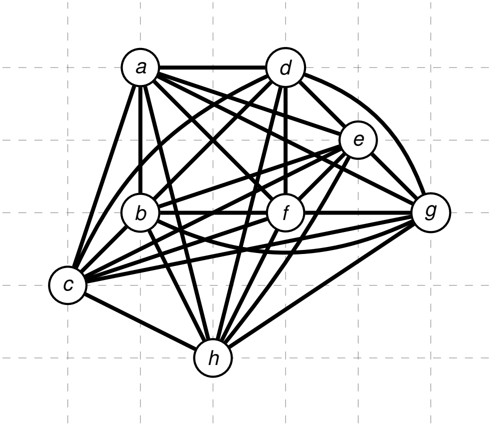
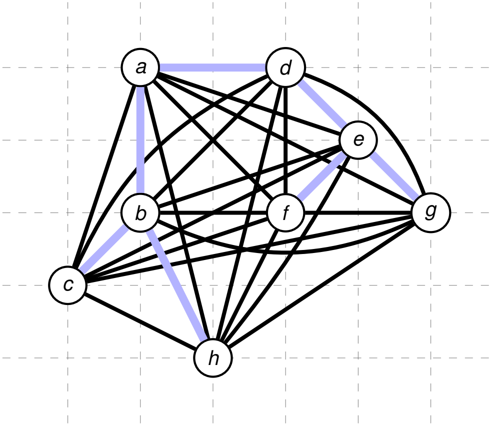


1. Compute MST $T_{\min}$

Run of CHRISTOFIDES



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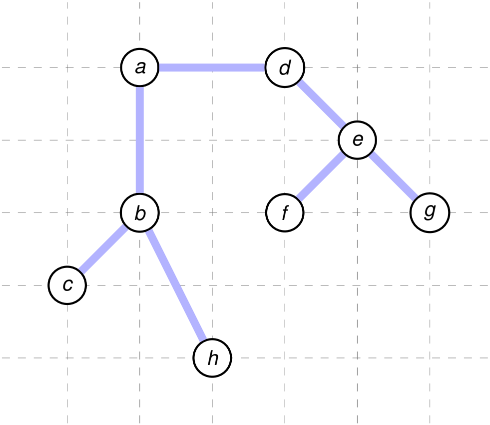

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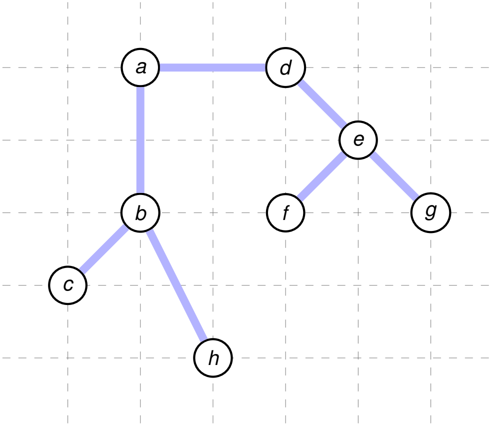
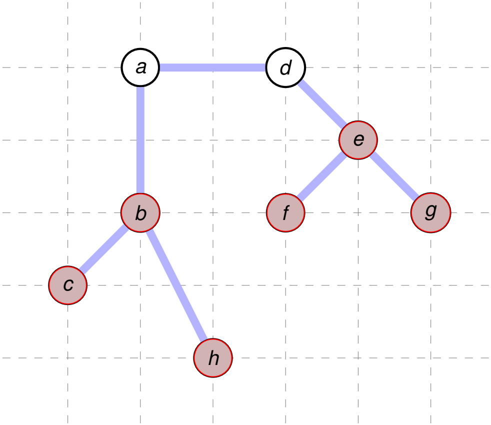


1. Compute MST T_{min} ✓
2. Add a minimum-weight perfect matching M_{min} of the odd vertices in T_{min}

Run of CHRISTOFIDES



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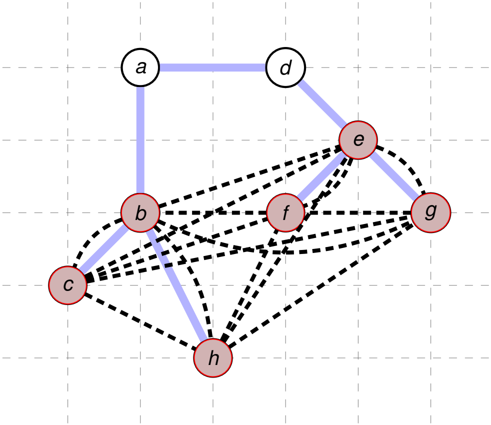

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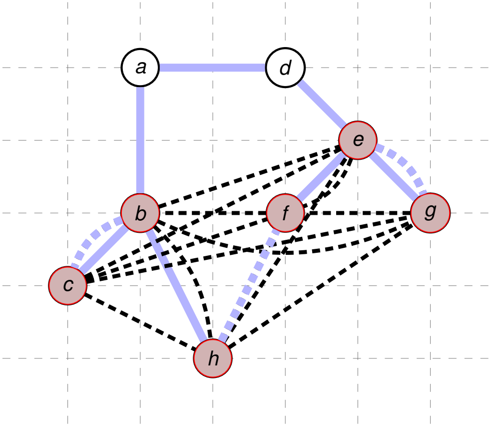

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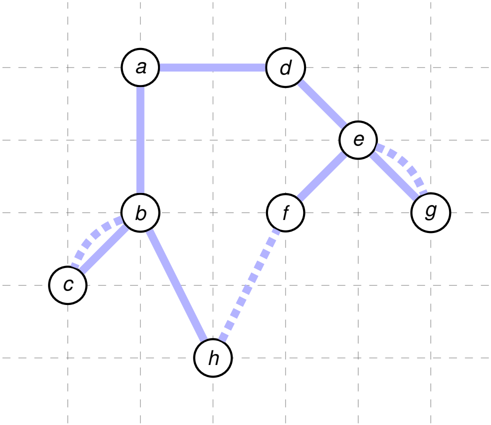

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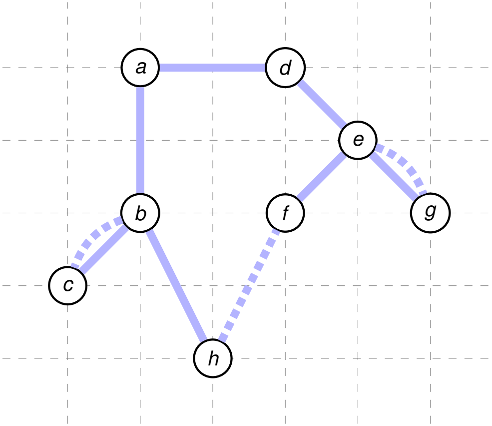

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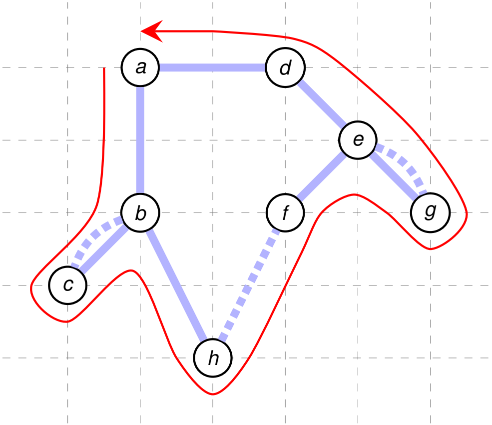
All vertices in $T_{\min} \cup M_{\min}$ have even degree!

Run of CHRISTOFIDES



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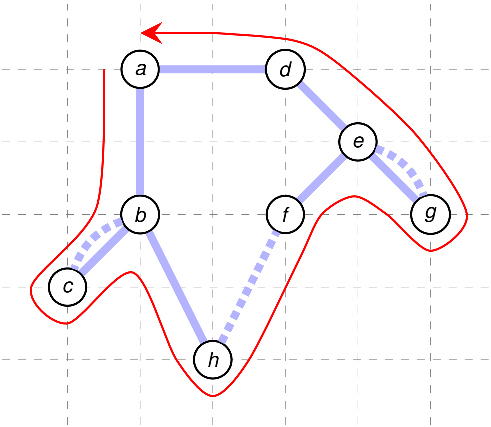

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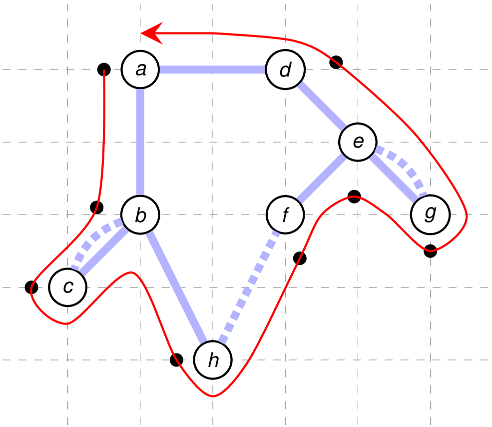

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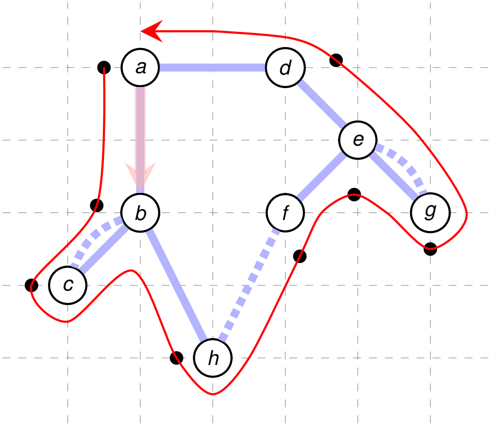

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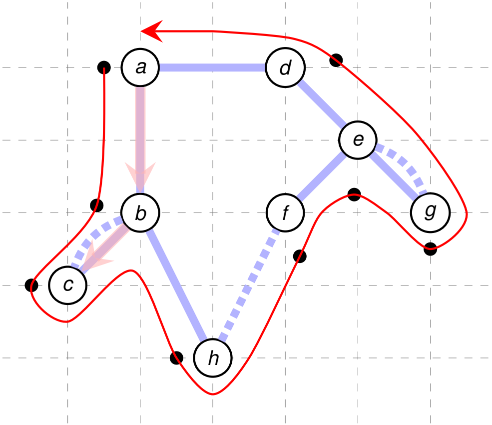

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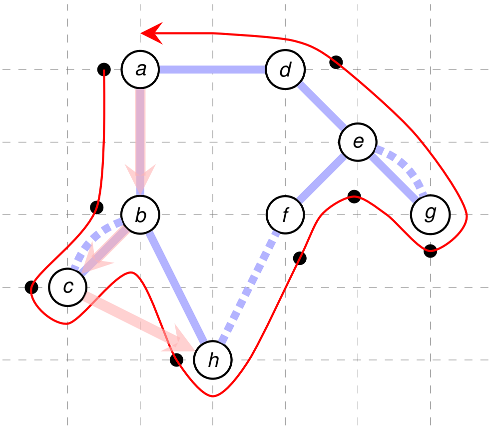

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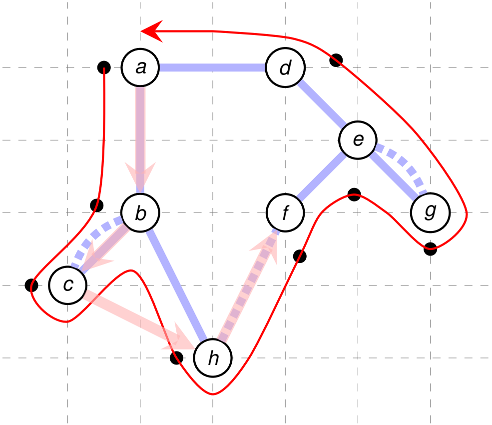

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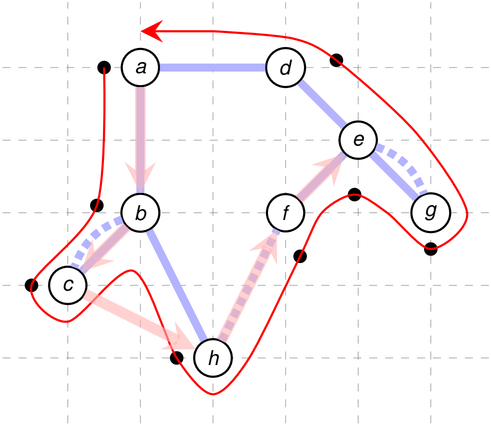

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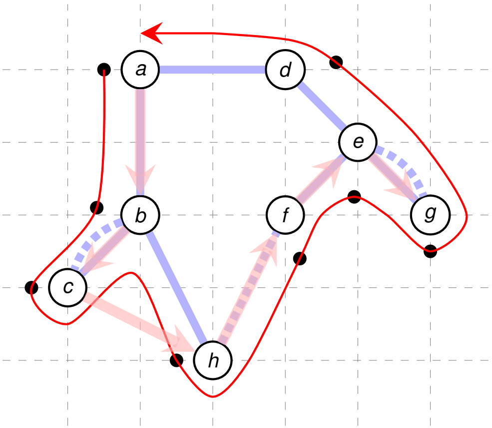

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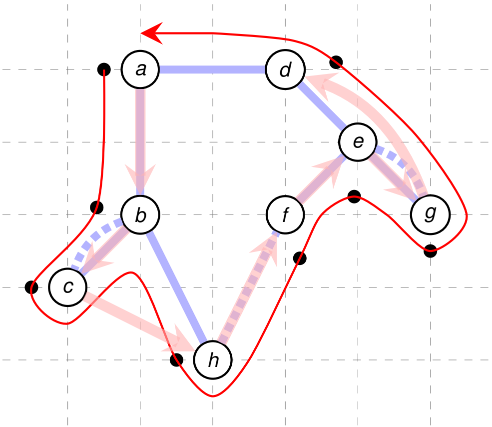

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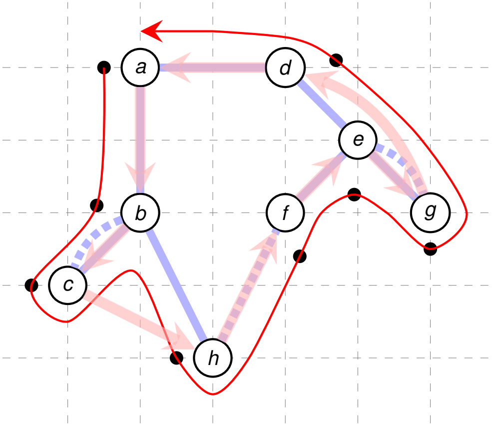

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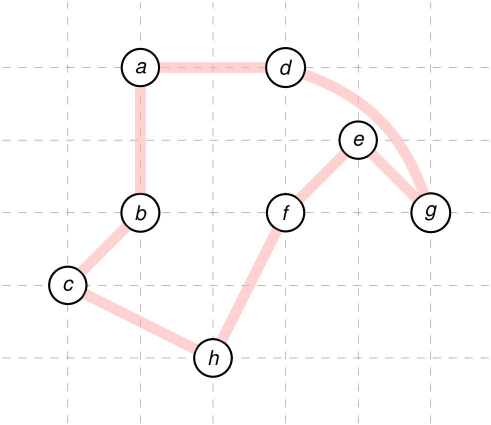

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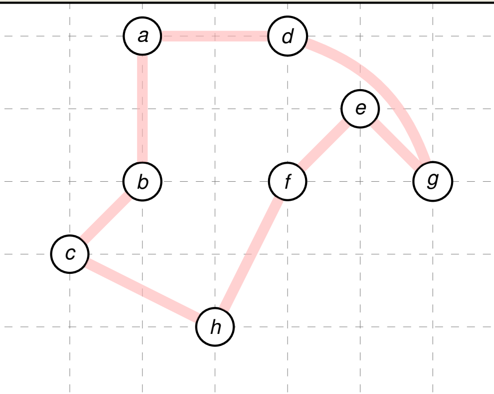
Solution has cost ≈ 15.54 - within 10% of the optimum!



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Theorem (Christofides'76)

There is a polynomial-time $\frac{3}{2}$-approximation algorithm for the travelling salesman problem with the triangle inequality.

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Concluding Remarks

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still the best algorithm for the metric TSP problem(!)

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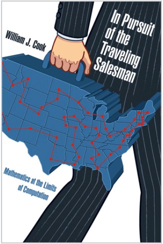

Exercise: Prove that the approximation ratio of APPROX-TSP-TOUR satisfies $\rho(n) < 2$.

Hint: Consider the effect of the shortcutting, but note that edge costs might be zero!