Advanced Algorithms

I. Course Intro and Sorting Networks

Thomas Sauerwald

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UNIVERSITY OF CAMBRIDGE

Outline

Outline of this Course

Some Highlights

Introduction to Sorting Networks

Batcher's Sorting Network

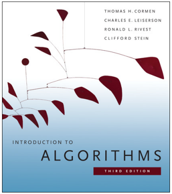
Bonus Material: Construction of an Optimal Sorting Network (non-examinable)

Counting Networks

List of Topics

IA Algorithms → IB Complexity Theory → II Advanced Algorithms

- I. Sorting Networks (Sorting, Counting)
- II. Linear Programming
- III. Approximation Algorithms: Covering Problems
- IV. Approximation Algorithms via Exact Algorithms
- V. Approximation Algorithms: Travelling Salesman Problem
- VI. Approximation Algorithms: Randomisation and Rounding



- closely follow CLRS3 and use the same numberring
- however, slides will be self-contained

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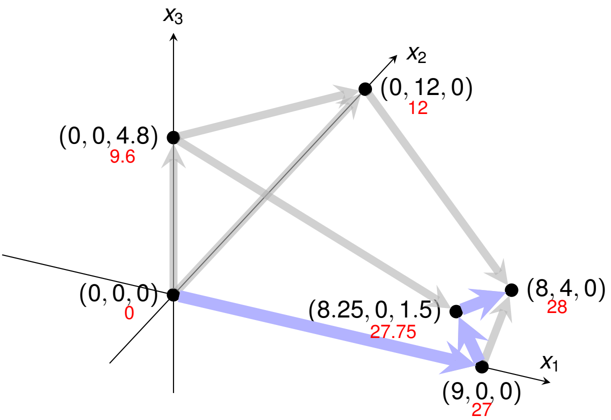
Linear Programming and Simplex



$$
\begin{array}{llllllll}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
& 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
& 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
& & & x_1, x_2, x_3 & & & \geq & 0
\end{array}
$$

The Original Article (1954)

SOLUTION OF A LARGE-SCALE TRAVELING-SALESMAN PROBLEM*

G. DANTZIG, R. FULKERSON, AND S. JOHNSON
The Rand Corporation, Santa Monica, California
(Received August 9, 1954)

It is shown that a certain tour of 49 cities, one in each of the 48 states and Washington, D. C., has the shortest road distance.

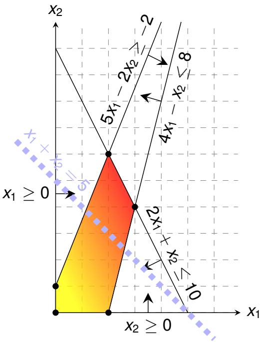
THE TRAVELING-SALESMAN PROBLEM might be described as follows: Find the shortest route (tour) for a salesman starting from a given city, visiting each of a specified group of cities, and then returning to the original point of departure. More generally, given an n by n symmetric matrix $D=(d_{IJ})$, where d_{IJ} represents the 'distance' from I to J, arrange the points in a cyclic order in such a way that the sum of the d_{IJ} between consecutive points is minimal. Since there are only a finite number of possibilities (at most $\frac{1}{2}(n-1)!$) to consider, the problem is to devise a method of picking out the optimal arrangement which is reasonably efficient for fairly large values of n. Although algorithms have been devised for problems of similar nature, e.g., the optimal assignment problem,[3,7,8] little is known about the traveling-salesman problem. We do not claim that this note alters the situation very much; what we shall do is outline a way of approaching the problem that sometimes, at least, enables one to find an optimal path and prove it so. In particular, it will be shown that a certain arrangement of 49 cities, one in each of the 48 states and Washington, D. C., is best, the d_{IJ} used representing road distances as taken from an atlas.

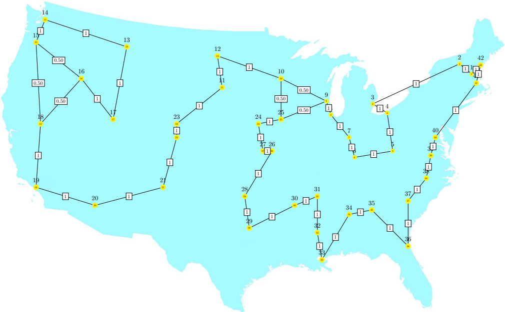
Travelling Salesman Problem: The 42 (49) Cities

1. Manchester, N. H.
2. Montpelier, Vt.
3. Detroit, Mich.
4. Cleveland, Ohio
5. Charleston, W. Va.
6. Louisville, Ky.
7. Indianapolis, Ind.
8. Chicago, Ill.
9. Milwaukee, Wis.
10. Minneapolis, Minn.
11. Pierre, S. D.
12. Bismarck, N. D.
13. Helena, Mont.
14. Seattle, Wash.
15. Portland, Ore.
16. Boise, Idaho
17. Salt Lake City, Utah
18. Carson City, Nev.
19. Los Angeles, Calif.
20. Phoenix, Ariz.
21. Santa Fe, N. M.
22. Denver, Colo.
23. Cheyenne, Wyo.
24. Omaha, Neb.
25. Des Moines, Iowa
26. Kansas City, Mo.
27. Topeka, Kans.
28. Oklahoma City, Okla.
29. Dallas, Tex.
30. Little Rock, Ark.
31. Memphis, Tenn.
32. Jackson, Miss.
33. New Orleans, La.
34. Birmingham, Ala.
35. Atlanta, Ga.
36. Jacksonville, Fla.
37. Columbia, S. C.
38. Raleigh, N. C.
39. Richmond, Va.
40. Washington, D. C.
41. Boston, Mass.
42. Portland, Me.

A. Baltimore, Md.
B. Wilmington, Del.
C. Philadelphia, Penn.
D. Newark, N. J.
E. New York, N. Y.
F. Hartford, Conn.
G. Providence, R. I.

Computing the Optimal Tour





We are going to use our own implementation of the Simplex-Algorithm along with a visulation to solve a series of linear programs in order to solve the TSP instance optimally!

There are a couple of exercises spread across the recordings to test your understanding!

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Counting Networks

Overview: Sorting Networks

(Serial) Sorting Algorithms

- we already know several (comparison-based) sorting algorithms: Insertion sort, Bubble sort, Merge sort, Quick sort, Heap sort
- execute one operation at a time
- can handle arbitrarily large inputs
- sequence of comparisons is not set in advance

Sorting Networks

- only perform comparisons
- can only handle inputs of a fixed size
- sequence of comparisons is set in advance
- Comparisons can be performed in parallel

Allows to sort n numbers in sublinear time!

Simple concept, but surprisingly deep and complex theory!

Comparison Networks

A sorting network is a comparison network which works correctly (that is, it sorts every input)

Comparison Network

- A comparison network consists solely of wires and comparators:
  - comparator is a device with, on given two inputs, x and y, returns two outputs $x' = \min(x, y)$ and $y' = \max(x, y)$
  - wire connect output of one comparator to the input of another
  - special wires: n input wires $a_1, a_2, \ldots, a_n$ and n output wires $b_1, b_2, \ldots, b_n$

operates in $O(1)$

Convention: use the same name for both a wire and its value.

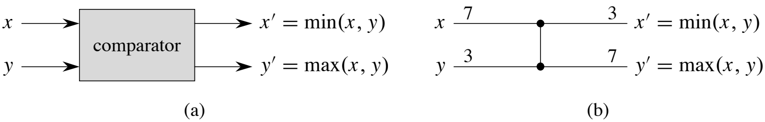
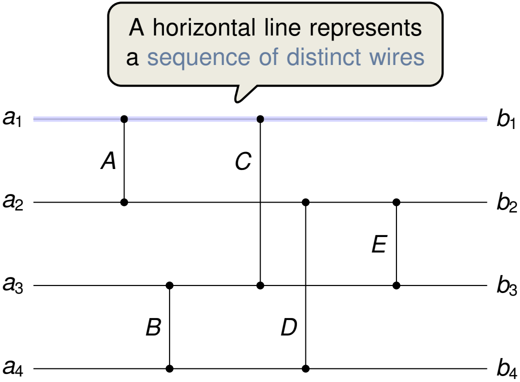


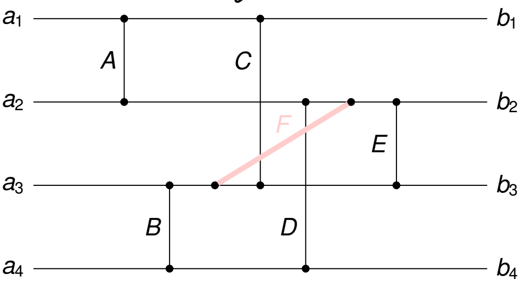
Figure 27.1 **(a)** A comparator with inputs x and y and outputs x' and y'. **(b)** The same comparator, drawn as a single vertical line. Inputs $x = 7$, $y = 3$ and outputs $x' = 3$, $y' = 7$ are shown.

Example of a Comparison Network (Figure 27.2, CLRS2)



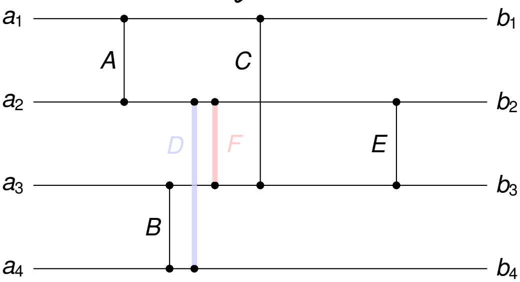
Example of a Comparison Network (Figure 27.2, CLRS2)

Interconnections between comparators must be acyclic



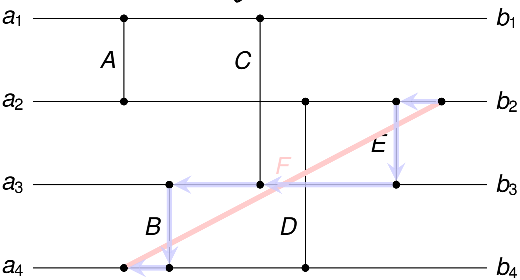
Example of a Comparison Network (Figure 27.2, CLRS2)

Interconnections between comparators must be acyclic ✓



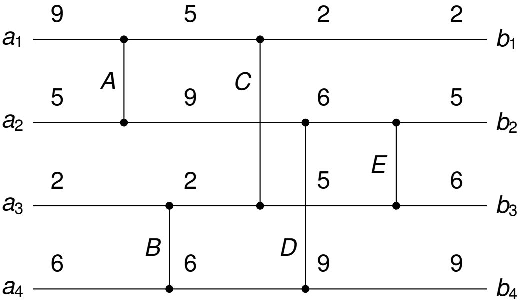
Example of a Comparison Network (Figure 27.2, CLRS2)

Interconnections between comparators must be acyclic



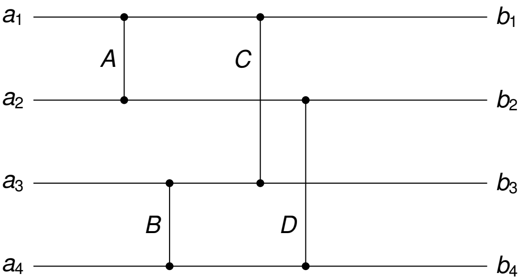
Tracing back a path must never cycle back on itself and go through the same comparator twice.

Example of a Comparison Network (Figure 27.2, CLRS2)



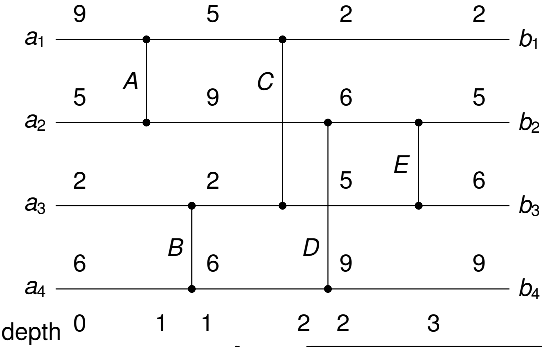
This network is in fact a sorting network (**Exercise 1**)

Example of a Comparison Network (Figure 27.2, CLRS2)



This network would not be a sorting network (**Exercise 2**)

Example of a Comparison Network (Figure 27.2, CLRS2)



Maximum depth of an output wire equals total running time

Depth of a wire:

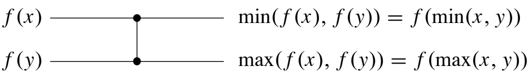
- Input wire has depth 0
- If a comparator has two inputs of depths d_x and d_y, then outputs have depth $\max\{d_x, d_y\} + 1$

Zero-One Principle

Zero-One Principle: A sorting networks works correctly on arbitrary inputs if it works correctly on binary inputs.

Lemma 27.1

If a comparison network transforms the input $a = \langle a_1, a_2, \ldots, a_n \rangle$ into the output $b = \langle b_1, b_2, \ldots, b_n \rangle$, then for any monotonically increasing function f, the network transforms $f(a) = \langle f(a_1), f(a_2), \ldots, f(a_n) \rangle$ into $f(b) = \langle f(b_1), f(b_2), \ldots, f(b_n) \rangle$.



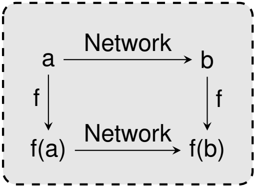


Figure 27.4 The operation of the comparator in the proof of Lemma 27.1. The function f is monotonically increasing.

Zero-One Principle

Zero-One Principle: A sorting networks works correctly on arbitrary inputs if it works correctly on binary inputs.

Lemma 27.1

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Theorem 27.2 (Zero-One Principle)

If a comparison network with n inputs sorts all 2^n possible sequences of 0's and 1's correctly, then it sorts all sequences of arbitrary numbers correctly.

Proof of the Zero-One Principle

Theorem 27.2 (Zero-One Principle)

If a comparison network with n inputs sorts all 2^n possible sequences of 0's and 1's correctly, then it sorts all sequences of arbitrary numbers correctly.

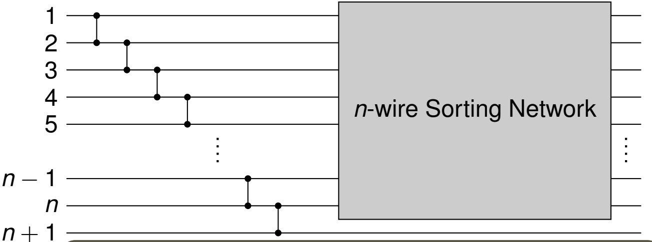
Proof:

- For the sake of contradiction, suppose the network does not correctly sort.
- Let $a = \langle a_1, a_2, \ldots, a_n \rangle$ be the input with $a_i < a_j$, but the network places a_j before a_i in the output
- Define a monotonically increasing function f as:

$$f(x) = \begin{cases} 0 & \text{if } x \leq a_i, \\ 1 & \text{if } x > a_i. \end{cases}$$

- Since the network places a_j before a_i, by the previous lemma $\Rightarrow f(a_j)$ is placed before $f(a_i)$
- But $f(a_j) = 1$ and $f(a_i) = 0$, which contradicts the assumption that the network sorts all sequences of 0's and 1's correctly $\square$

Some Basic (Recursive) Sorting Networks



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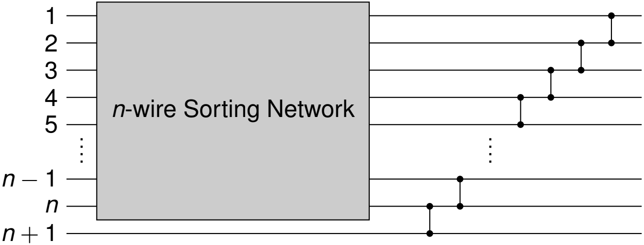
These are Sorting Networks, but with depth $\Theta(n)$.



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Counting Networks

Bitonic Sequences

Bitonic Sequence

A sequence is bitonic if it monotonically increases and then monotonically decreases, or can be circularly shifted to become monotonically increasing and then monotonically decreasing.

Sequences of one or two numbers are defined to be bitonic.

Examples:

- $\langle 1, 4, 6, 8, 3, 2 \rangle$ ✓
- $\langle 6, 9, 4, 2, 3, 5 \rangle$ ✓
- $\langle 9, 8, 3, 2, 4, 6 \rangle$ ✓
- ~~$\langle 4, 5, 7, 1, 2, 6 \rangle$~~
- binary sequences: $0^i 1^j 0^k$, or, $1^i 0^j 1^k$, for $i, j, k \geq 0$.

Towards Bitonic Sorting Networks

Half-Cleaner

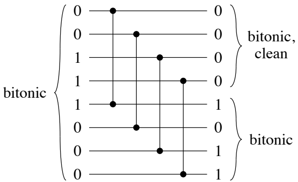
A half-cleaner is a comparison network of depth 1 in which input wire i is compared with wire $i + n/2$ for $i = 1, 2, \ldots, n/2$.

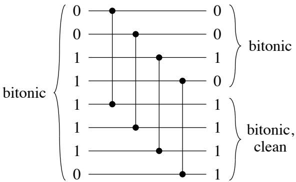
We always assume that n is even.

Lemma 27.3

If the input to a half-cleaner is a bitonic sequence of 0's and 1's, then the output satisfies the following properties:

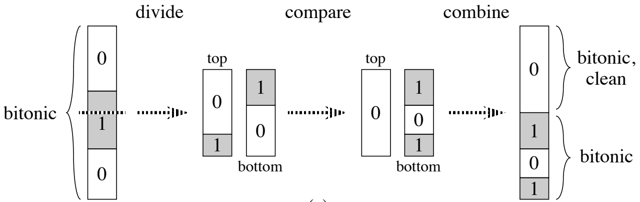
- both the top half and the bottom half are bitonic,
- every element in the top is not larger than any element in the bottom,
- at least one half is clean.



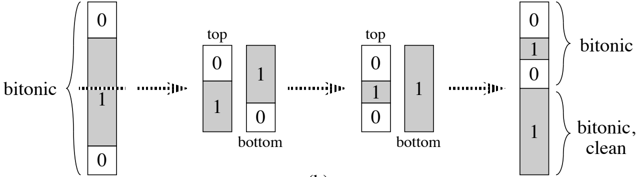


Proof of Lemma 27.3

W.l.o.g. assume that the input is of the form $0^i 1^j 0^k$, for some $i, j, k \geq 0$.



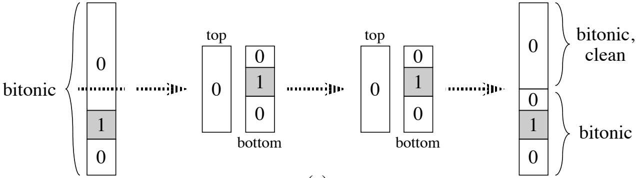
(a)



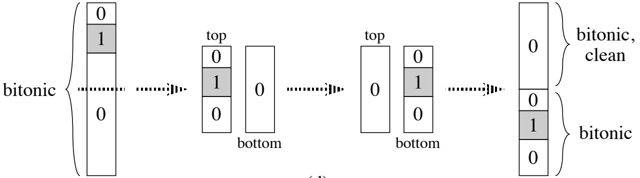
(b)

Proof of Lemma 27.3

W.l.o.g. assume that the input is of the form $0^i 1^j 0^k$, for some $i, j, k \geq 0$.



(c)



(d)

This suggests a recursive approach, since it now suffices to sort the top and bottom half separately.

The Bitonic Sorter

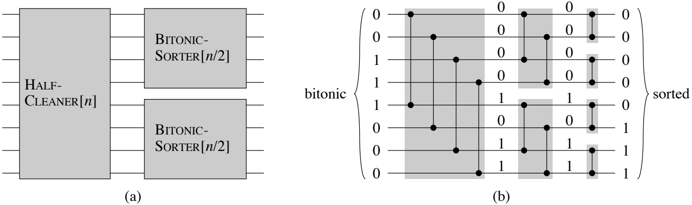


Figure 27.9 The comparison network BITONIC-SORTER[n], shown here for $n = 8$. **(a)** The recursive construction: HALF-CLEANER[n] followed by two copies of BITONIC-SORTER[$n/2$] that operate in parallel. **(b)** The network after unrolling the recursion. Each half-cleaner is shaded. Sample zero-one values are shown on the wires.

Henceforth we will always assume that n is a power of 2.

Recursive Formula for depth $D(n)$:

$$D(n) = \begin{cases} 0 & \text{if } n = 1, \\ D(n/2) + 1 & \text{if } n = 2^k. \end{cases}$$

BITONIC-SORTER[n] has depth $\log n$ and sorts any zero-one bitonic sequence.

Merging Networks

Merging Networks

- can merge two sorted input sequences into one sorted output sequence
- will be based on a modification of BITONIC-SORTER[n]

Basic Idea:

- consider two given sequences $X = 00000111$, $Y = 00001111$
- concatenating X with Y^R (the reversal of Y) $\Rightarrow$ 0000011111110000

This sequence is bitonic!

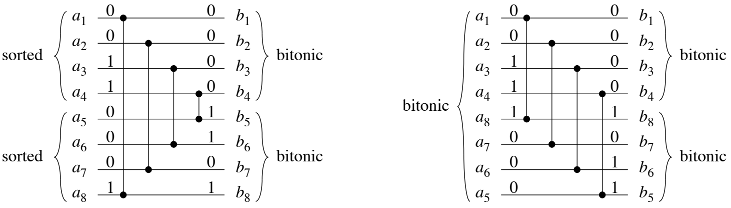
Hence in order to merge the sequences X and Y, it suffices to perform a bitonic sort on X concatenated with Y^R.

Construction of a Merging Network (1/2)

- Given two sorted sequences $\langle a_1, a_2, \ldots, a_{n/2} \rangle$ and $\langle a_{n/2+1}, a_{n/2+2}, \ldots, a_n \rangle$
- We know it suffices to bitonically sort $\langle a_1, a_2, \ldots, a_{n/2}, a_n, a_{n-1}, \ldots, a_{n/2+1} \rangle$
- Recall: first half-cleaner of BITONIC-SORTER[n] compares i and $n/2 + i$

$\Rightarrow$ First part of MERGER[n] compares inputs i and $n - i + 1$ for $i = 1, 2, \ldots, n/2$

- Remaining part is identical to BITONIC-SORTER[n]



(a) (b)

Figure 27.10 Comparing the first stage of MERGER[n] with HALF-CLEANER[n], for $n = 8$. **(a)** The first stage of MERGER[n] transforms the two monotonic input sequences $\langle a_1, a_2, \ldots, a_{n/2} \rangle$ and $\langle a_{n/2+1}, a_{n/2+2}, \ldots, a_n \rangle$ into two bitonic sequences $\langle b_1, b_2, \ldots, b_{n/2} \rangle$ and $\langle b_{n/2+1}, b_{n/2+2}, \ldots, b_n \rangle$. **(b)** The equivalent operation for HALF-CLEANER[n]. The bitonic input sequence $\langle a_1, a_2, \ldots, a_{n/2-1}, a_{n/2}, a_n, a_{n-1}, \ldots, a_{n/2+2}, a_{n/2+1} \rangle$ is transformed into the two bitonic sequences $\langle b_1, b_2, \ldots, b_{n/2} \rangle$ and $\langle b_n, b_{n-1}, \ldots, b_{n/2+1} \rangle$.

Construction of a Merging Network (2/2)

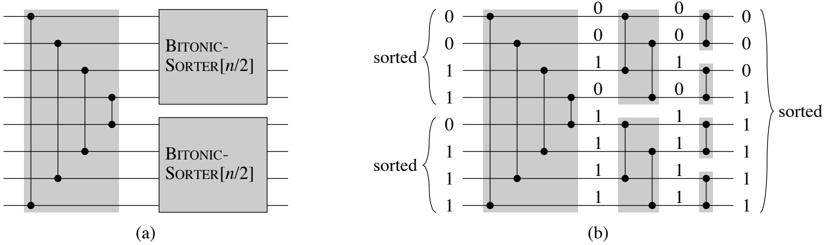


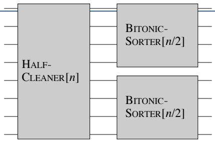
Figure 27.11 A network that merges two sorted input sequences into one sorted output sequence. The network MERGER[n] can be viewed as BITONIC-SORTER[n] with the first half-cleaner altered to compare inputs i and $n-i+1$ for $i = 1, 2, \ldots, n/2$. Here, $n = 8$. **(a)** The network decomposed into the first stage followed by two parallel copies of BITONIC-SORTER[$n/2$]. **(b)** The same network with the recursion unrolled. Sample zero-one values are shown on the wires, and the stages are shaded.

Construction of a Sorting Network

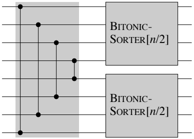
Main Components

1. BITONIC-SORTER[n]
   - sorts any bitonic sequence
   - depth $\log n$
2. MERGER[n]
   - merges two sorted input sequences
   - depth $\log n$

1.



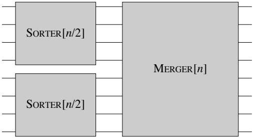
2.



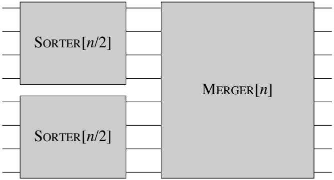
Batcher's Sorting Network

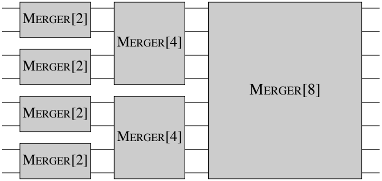
- SORTER[n] is defined recursively:
  - If $n = 2^k$, use two copies of SORTER[$n/2$] to sort two subsequences of length $n/2$ each. Then merge them using MERGER[n].
  - If $n = 1$, network consists of a single wire.

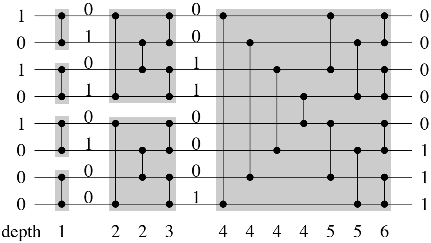
can be seen as a parallel version of merge sort



Unrolling the Recursion (Figure 27.12)







Recursion for $D(n)$:

$$D(n) = \begin{cases} 0 & \text{if } n = 1, \\ D(n/2) + \log n & \text{if } n = 2^k. \end{cases}$$

Solution: $D(n) = \Theta(\log^2 n)$.

SORTER[n] has depth $\Theta(\log^2 n)$ and sorts any input.

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Bonus Material: Construction of an Optimal Sorting Network (non-examinable)

Counting Networks

A Glimpse at the AKS Network

Ajtai, Komlós, Szemerédi (1983)

There exists a sorting network with depth $O(\log n)$.

Quite elaborate construction, and involves huges constants.

Perfect Halver

A perfect halver is a comparison network that, given any input, places the $n/2$ smaller keys in $b_1, \ldots, b_{n/2}$ and the $n/2$ larger keys in $b_{n/2+1}, \ldots, b_n$.

Perfect halver of depth $\log n$ exist $\rightsquigarrow$ yields sorting networks of depth $\Theta((\log n)^2)$.

Approximate Halver

An (n, ϵ)-approximate halver, $\epsilon < 1$, is a comparison network that for every $k = 1, 2, \ldots, n/2$ places at most ϵk of its k smallest keys in $b_{n/2+1}, \ldots, b_n$ and at most ϵk of its k largest keys in $b_1, \ldots, b_{n/2}$.

We will prove that such networks can be constructed in constant depth!

Expander Graphs

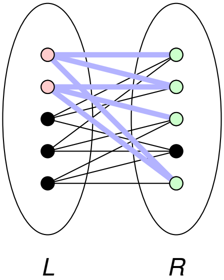
Expander Graphs

A bipartite (n, d, μ)-expander is a graph with:

- G has n vertices ($n/2$ on each side)
- the edge-set is union of d perfect matchings
- For every subset $S \subseteq V$ being in one part,

$$|N(S)| > \min\{\mu \cdot |S|, n/2 - |S|\}$$

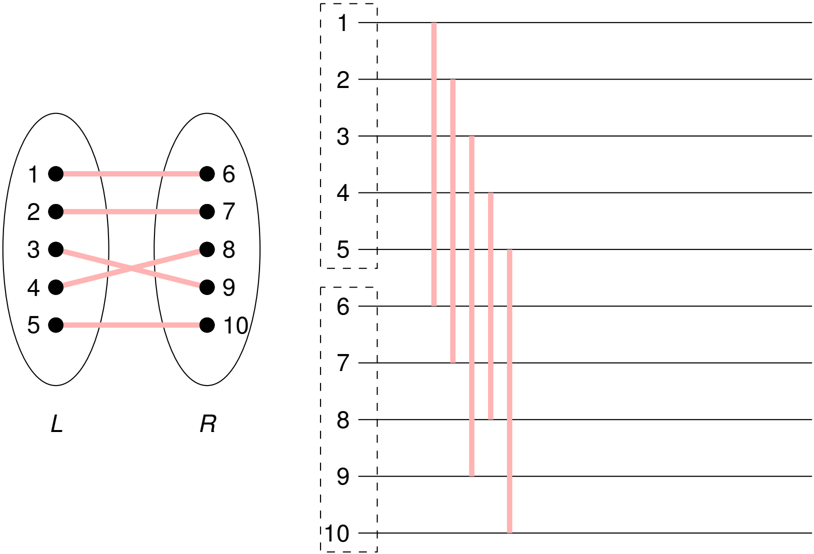
Specific definition tailored for sorting network - many other variants exist!



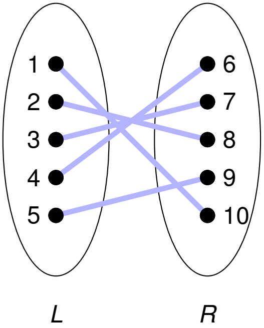
Expander Graphs:

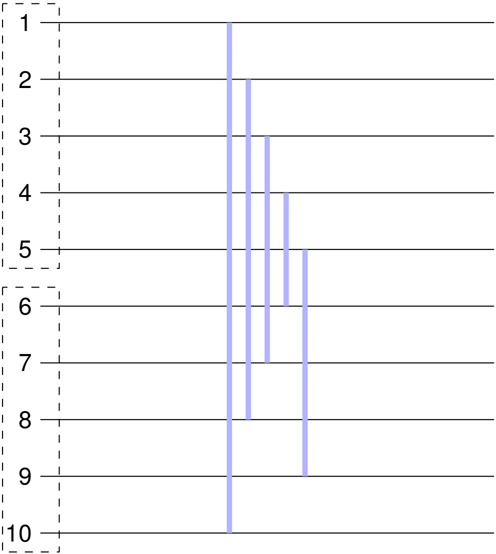
- **probabilistic construction** "easy": take d (disjoint) random matchings
- **explicit construction** is a deep mathematical problem with ties to number theory, group theory, combinatorics etc.
- **many applications** in networking, complexity theory and coding theory

From Expanders to Approximate Halvers

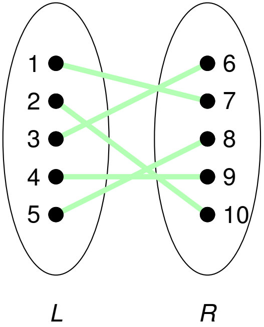


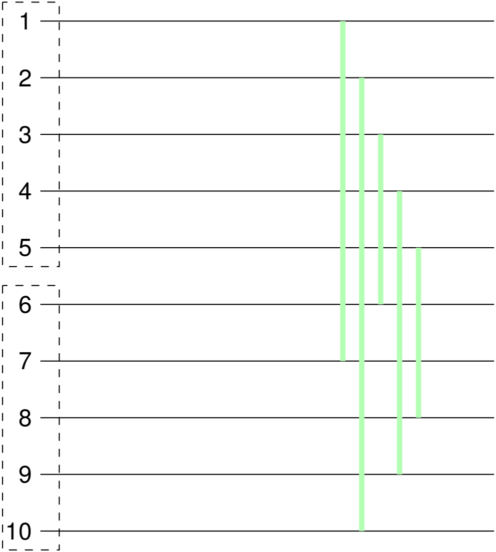
From Expanders to Approximate Halvers



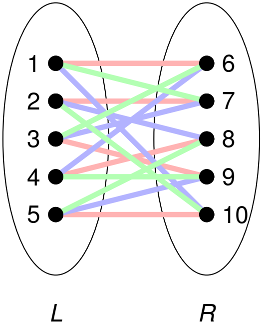


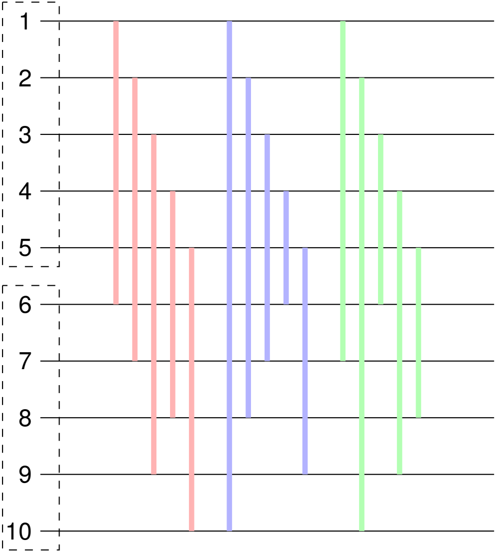
From Expanders to Approximate Halvers





From Expanders to Approximate Halvers





Existence of Approximate Halvers (non-examinable)

Proof:

- $X :=$ keys with the k smallest inputs
- $Y :=$ wires in lower half with k smallest outputs
- For every $u \in N(Y)$: $\exists$ comparat. (u, v), $v \in Y$
- Let u_t, v_t be their keys after the comparator
  Let u_d, v_d be their keys at the output (note $v_d \in X$)
- Further: $u_d \leq u_t \leq v_t \leq v_d \Rightarrow u_d \in X$
- Since u was arbitrary:

$$|Y| + |N(Y)| \leq k.$$

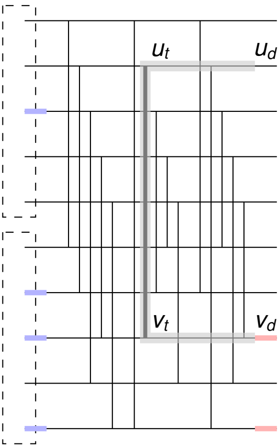
- Since G is a bipartite (n, d, μ)-expander:

$$|Y| + |N(Y)| > |Y| + \min\{\mu|Y|, n/2 - |Y|\}$$
$$= \min\{(1 + \mu)|Y|, n/2\}.$$

- Combining the two bounds above yields:

$$(1 + \mu)|Y| \leq k.$$

- Same argument $\Rightarrow$ at most $\epsilon \cdot k$, $\epsilon := 1/(\mu + 1)$, of the k largest input keys are placed in $b_1, \ldots, b_{n/2}$. $\square$



- typical application of expander graphs in parallel algorithms
- Much more work needed to construct the AKS sorting network

AKS network vs. Batcher's network

Donald E. Knuth (Stanford)

"Batcher's method is much better, unless n exceeds the total memory capacity of all computers on earth!"

Richard J. Lipton (Georgia Tech)

*"The AKS sorting network is **galactic**: it needs that n be larger than 2^{78} or so to finally be smaller than Batcher's network for n items."*

Outline

Outline of this Course

Some Highlights

Introduction to Sorting Networks

Batcher's Sorting Network

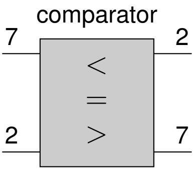
Bonus Material: Construction of an Optimal Sorting Network (non-examinable)

Counting Networks

Siblings of Sorting Network

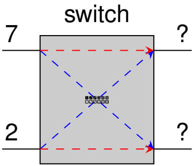
Sorting Networks

- sorts any input of size n
- special case of Comparison Networks



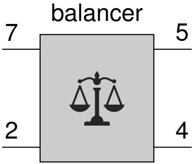
Switching (Shuffling) Networks

- creates a random permutation of n items
- special case of Permutation Networks



Counting Networks

- balances any stream of tokens over n wires
- special case of Balancing Networks



Counting Network

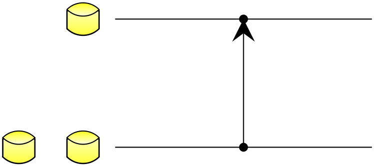
Distributed Counting

Processors collectively assign successive values from a given range.

Values could represent addresses in memories or destinations on an interconnection network

Balancing Networks

- constructed in a similar manner like sorting networks
- instead of comparators, consists of balancers
- balancers are asynchronous flip-flops that forward tokens from its inputs to one of its two outputs alternately (top, bottom, top,...)



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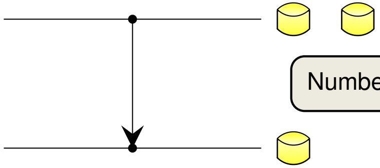

Number of tokens differs by at most one

Bitonic Counting Network

Counting Network (Formal Definition)

1. Let $x_1, x_2, \ldots, x_n$ be the number of tokens (ever received) on the designated input wires
2. Let $y_1, y_2, \ldots, y_n$ be the number of tokens (ever received) on the designated output wires
3. In a quiescent state: $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$
4. A counting network is a balancing network with the **step-property:**

$$0 \leq y_i - y_j \leq 1 \text{ for any } i < j.$$

Bitonic Counting Network: Take Batcher's Sorting Network and replace each comparator by a balancer.

Correctness of the Bitonic Counting Network (non-examinable)

Facts

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have the step property. Then:

1. We have $\sum_{i=1}^{n/2} x_{2i-1} = \left\lceil \frac{1}{2} \sum_{i=1}^{n} x_i \right\rceil$, and $\sum_{i=1}^{n/2} x_{2i} = \left\lfloor \frac{1}{2} \sum_{i=1}^{n} x_i \right\rfloor$
2. If $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, then $x_i = y_i$ for $i = 1, \ldots, n$.
3. If $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i + 1$, then $\exists!\ j = 1, 2, \ldots, n$ with $x_j = y_j + 1$ and $x_i = y_i$ for $j \neq i$.

Key Lemma

Consider a MERGER[n]. Then if the inputs $x_1, \ldots, x_{n/2}$ and $x_{n/2+1}, \ldots, x_n$ have the step property, then so does the output $y_1, \ldots, y_n$.

Proof (by induction on n being a power of 2)

- Case $n = 2$ is clear, since MERGER[2] is a single balancer

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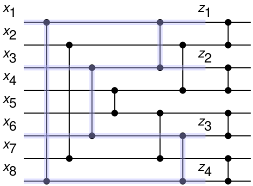

Proof (by induction on n being a power of 2)

- Case $n = 2$ is clear, since MERGER[2] is a single balancer
- $n > 2$: Let $z_1, \ldots, z_{n/2}$ and $z'_1, \ldots, z'_{n/2}$ be the outputs of the MERGER[$n/2$] subnetworks

Correctness of the Bitonic Counting Network (non-examinable)

Facts

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have the step property. Then:

1. We have $\sum_{i=1}^{n/2} x_{2i-1} = \left\lceil \frac{1}{2} \sum_{i=1}^{n} x_i \right\rceil$, and $\sum_{i=1}^{n/2} x_{2i} = \left\lfloor \frac{1}{2} \sum_{i=1}^{n} x_i \right\rfloor$
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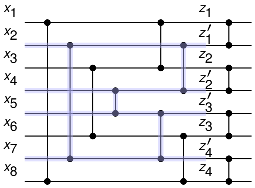

Proof (by induction on n being a power of 2)

- Case $n = 2$ is clear, since MERGER[2] is a single balancer
- $n > 2$: Let $z_1, \ldots, z_{n/2}$ and $z_1', \ldots, z_{n/2}'$ be the outputs of the MERGER[$n/2$] subnetworks

Correctness of the Bitonic Counting Network (non-examinable)

Facts

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have the step property. Then:

1. We have $\sum_{i=1}^{n/2} x_{2i-1} = \lceil \frac{1}{2} \sum_{i=1}^{n} x_i \rceil$, and $\sum_{i=1}^{n/2} x_{2i} = \lfloor \frac{1}{2} \sum_{i=1}^{n} x_i \rfloor$
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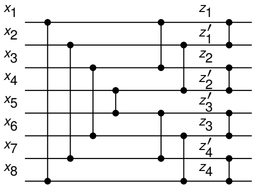
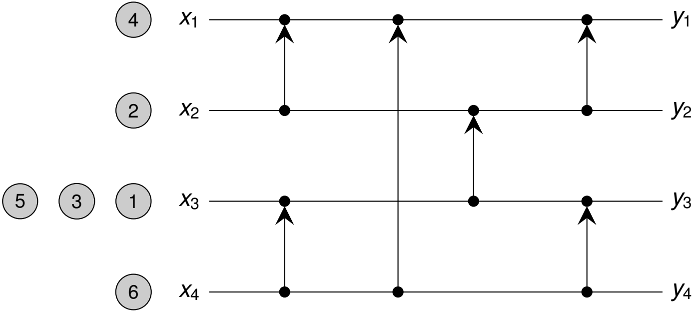


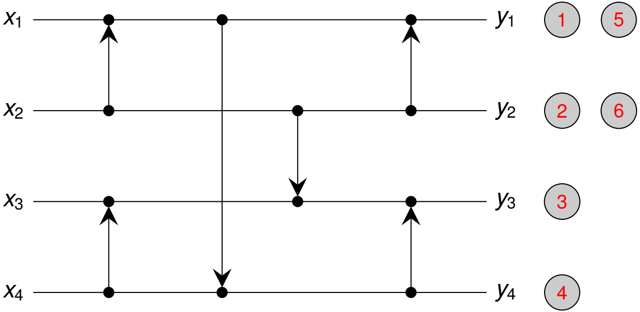
Proof (by induction on n being a power of 2)

- Case $n = 2$ is clear, since MERGER[2] is a single balancer
- $n > 2$: Let $z_1, \ldots, z_{n/2}$ and $z'_1, \ldots, z'_{n/2}$ be the outputs of the MERGER[$n/2$] subnetworks
- IH $\Rightarrow$ $z_1, \ldots, z_{n/2}$ and $z'_1, \ldots, z'_{n/2}$ have the step property
- Let $Z := \sum_{i=1}^{n/2} z_i$ and $Z' := \sum_{i=1}^{n/2} z'_i$
- Claim: $|Z - Z'| \leq 1$ (since $Z' = \lfloor \frac{1}{2} \sum_{i=1}^{n/2} x_i \rfloor + \lceil \frac{1}{2} \sum_{i=n/2+1}^{n} x_i \rceil$)
- Case 1: If $Z = Z'$, then F2 implies the output of MERGER[n] is $y_i = z_{1+\lfloor (i-1)/2 \rfloor}$ ✓
- Case 2: If $|Z - Z'| = 1$, F3 implies $z_i = z'_i$ for $i = 1, \ldots, n/2$ except a unique j with $z_j \neq z'_j$.
  Balancer between z_j and z'_j will ensure that the step property holds.

Bitonic Counting Network in Action (Asychnronous Execution)

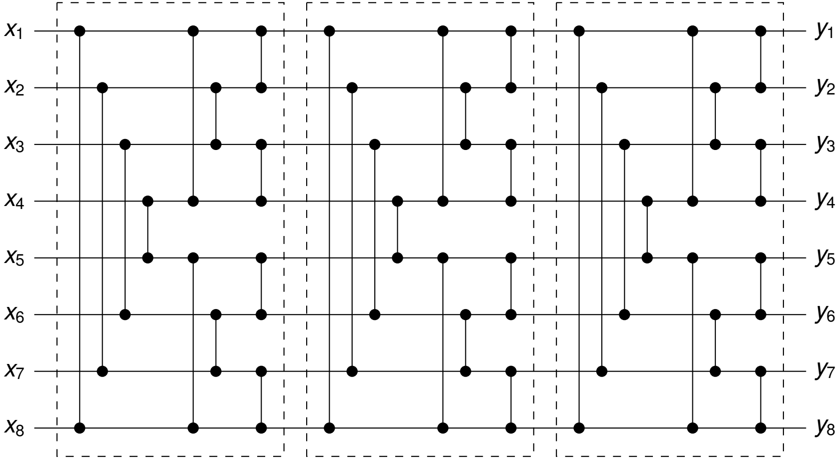


Bitonic Counting Network in Action (Asychnronous Execution)



Counting can be done as follows:
Add **local counter** to each output wire i, to assign consecutive numbers $i, i + n, i + 2 \cdot n, \ldots$

A Periodic Counting Network [Aspnes, Herlihy, Shavit, JACM 1994]



Consists of $\log n$ BLOCK[n] networks each of which has depth $\log n$

From Counting to Sorting

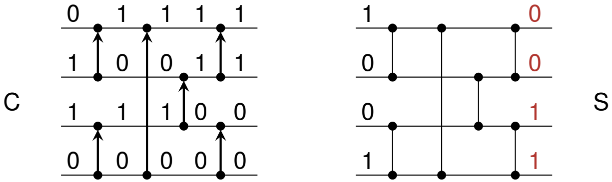
The converse is not true!

Counting vs. Sorting

If a network is a counting network, then it is also a sorting network.

Proof.

- Let C be a counting network, and S be the corresponding sorting network
- Consider an input sequence $a_1, a_2, \ldots, a_n \in \{0,1\}^n$ to S
- Define an input $x_1, x_2, \ldots, x_n \in \{0,1\}^n$ to C by $x_i = 1$ iff $a_i = 0$.
- C is a counting network $\Rightarrow$ all ones will be routed to the lower wires
- S corresponds to C $\Rightarrow$ all zeros will be routed to the lower wires
- By the Zero-One Principle, S is a sorting network. $\square$



Exercise: Consider a network which is a sorting network, but not a counting network.
Hint: Try to find a simple network with 4 wires that corresponds to a basic sequential sorting algorithm.