L11: Algebraic Path Problems with applications to Internet Routing Lecture 9 Path Vector

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Lifted Product

Lifted product semigroup

Assume $(S,\ \bullet)$ is a semigroup. Let $\mathrm{lift}(S,\ \bullet) \equiv (\mathcal{P}_{\mathrm{fin}}(S),\ \hat{\bullet})$ where

$$X \hat{\bullet} Y \equiv \{x \bullet y \mid x \in X,\ y \in Y\}.$$

and $\mathcal{P}_{\mathrm{fin}}(S)$ is the set of finite subsets of S.

$$\{1,\ 3,\ 17\} \mathbin{\hat{+}} \{1,\ 3,\ 17\} = \{2,\ 4,\ 6,\ 18,\ 20,\ 34\}$$

Property management...

$$
\begin{array}{rcl}
\mathbb{AS}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \\
\mathbb{ID}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{ID}(S, \bullet) \;\; (\hat{\alpha} = \{\alpha\}) \\
\mathbb{AN}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{TRUE} \;\; (\omega = \{\}) \\
\mathbb{CM}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{CM}(S, \bullet) \\
\mathbb{SL}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{IL}(S, \bullet) \vee \mathbb{IR}(S, \bullet) \vee (\mathbb{IP}(S, \bullet) \;\wedge \mid S \mid = 2) \\
\mathbb{IP}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{SL}((S, \bullet)) \\
\mathbb{IL}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{IR}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{FALSE}
\end{array}
$$

union_lift

Assume $(S, \;\bullet, \;\overline{1})$ is a monoid.

Semiring?

$$\mathrm{union\_lift}(S, \;\bullet) \equiv (\mathcal{P}_{\mathrm{fin}}(S), \;\cup, \;\hat{\bullet}, \;\{\}, \;\{\overline{1}\})$$

Paths in a graph

Given a directed graph $G = (V, \ E)$.

A path in G

A path p in G is any sequence in V^*. Let ϵ represent the empty path.

Let $\cdot$ represent path concatenation.

$$\begin{array}{rcl}
[17,21] \cdot \epsilon & = & [17,21] \\
\epsilon \cdot [21,22] & = & [21,22] \\
[17,21,22] \cdot [33,55] & = & [17,21,22,33,55] \\
[21,22] \cdot [21,55] & = & [21,22,21,55]
\end{array}$$

Elementray (simple, loopless) paths in a graph

A path p is <u>elementary</u> if no node is repeated.

$$\mathrm{elem}(V) = \{p \in X \mid p \text{ is an elementary path}\}$$

Let $\odot$ represent path concatenation over $\mathrm{elem}(V) \uplus \{\bot\}$

$$\begin{array}{rcl}
\mathrm{inl}([17,21,22]) \odot \mathrm{inl}([33,55]) & = & \mathrm{inl}([17,21,22,33,55]) \\
\mathrm{inl}([21,22]) \odot \mathrm{inl}([21,55]) & = & \mathrm{inr}(\bot) \\
\mathrm{inl}[17,21] \odot \mathrm{inl}(\epsilon) & = & \mathrm{inl}([17,21]) \\
\mathrm{inl}[17,21] \odot \mathrm{inr}(\bot) & = & \mathrm{inr}(\bot)
\end{array}$$

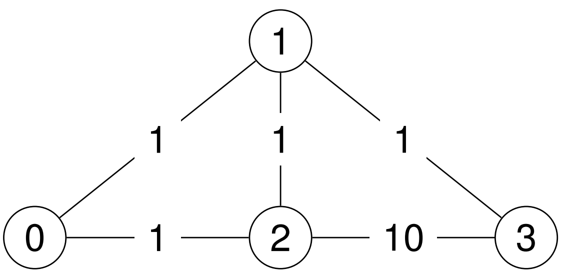
Elementray (simple, loopless) paths in a graph

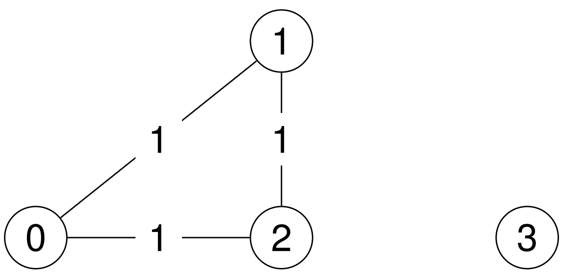
$$X \tilde{\odot} Y \equiv \{x \bullet y \mid x \in X,\ y \in Y,\ x \odot y \neq \mathrm{inr}(\perp)\}.$$

Semiring?

$$\mathrm{epaths}(V) \equiv (\mathcal{P}_{\mathrm{fin}}(\mathrm{elem}(V)),\ \cup,\ \tilde{\odot},\ \{\},\ \{\mathrm{inl}(\epsilon)\})$$

Recall state transition





First attempt doesn't work ...

using

$$\mathrm{AddZero}(\infty,\ (\mathbb{N},\ \min,\ +)\ \vec{\times}\ \mathrm{epaths}(V))$$

with an adjacency matrix $\mathbf{A}$ such that if $(i,j) \in E$ then $\mathbf{A}(i,j) = \mathrm{inl}(d,\ \{[i]\})$ for some $d \in \mathbb{N}$ and $\mathbf{A}(i,j) = \mathrm{inr}(\infty)$ otherwise.

$$\vdots \quad \vdots \quad \vdots$$

$$\mathbf{B}_{998} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} (0, \{\epsilon\}) & (1, \{[0,1]\}) & (1, \{[0,2]\}) & (999, \{\}) \\ (1, \{[1,0]\}) & (0, \{\epsilon\}) & (1, \{[1,2]\}) & (999, \{\}) \\ (1, \{[2,0]\}) & (1, \{[2,1]\}) & (0, \{\epsilon\}) & (999, \{\}) \\ \infty & \infty & \infty & (0, \{\epsilon\}) \end{bmatrix} \end{array}$$

$$\vdots \quad \vdots \quad \vdots$$

Solution?

Note this is a bit of a "hack" — can we do better?

Let's redefine the multiplication $\otimes$ of

$$\mathrm{AddZero}(\infty,\ (\mathbb{N},\ \min,\ +)\ \vec{\times}\ \mathrm{epaths}(V))$$

as follows:

$$\begin{array}{rcl} x \otimes' \mathrm{inr}(\infty) & = & \mathrm{inr}(\infty) \\ \mathrm{inr}(\infty) \otimes' x & = & \mathrm{inr}(\infty) \\ \mathrm{inl}(d_1,\ X) \otimes' \mathrm{inl}(d_1,\ Y) & = & \begin{cases} \mathrm{inr}(\infty) & \text{if } X \tilde{\odot} Y = \{\} \\ \mathrm{inl}(d_1 + d_2,\ X \tilde{\odot} Y) & \text{otherwise} \end{cases} \end{array}$$

Starting in an arbitrary state?

$\mathbf{B}_0$

$$
\begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3
\end{array}
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\left[\begin{array}{cccc}
(0, \{\epsilon\}) & (1, \{[0, 1]\}) & (1, \{[0, 2]\}) & (2, \{[0, 1, 3]\}) \\
(1, \{[1, 0]\}) & (0, \{\epsilon\}) & (1, \{[1, 2]\}) & (1, \{[1, 3]\}) \\
(1, \{[2, 0]\}) & (1, \{[2, 1]\}) & (0, \{\epsilon\}) & (2, \{[2, 1, 3]\}) \\
(2, \{[3, 1, 0]\}) & (1, \{[3, 1]\}) & (2, \{[3, 1, 2]\}) & (0, \{\epsilon\})
\end{array}\right]
\end{array}
$$

$\mathbf{B}_1$

$$
\begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3
\end{array}
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\left[\begin{array}{cccc}
(0, \{\epsilon\}) & (1, \{[0, 1]\}) & (1, \{[0, 2]\}) & (2, \{[0, 1, 3]\}) \\
(1, \{[1, 0]\}) & (0, \{\epsilon\}) & (1, \{[1, 2]\}) & \infty \\
(1, \{[2, 0]\}) & (1, \{[2, 1]\}) & (0, \{\epsilon\}) & (2, \{[2, 1, 3]\}) \\
\infty & \infty & \infty & (0, \{\epsilon\})
\end{array}\right]
\end{array}
$$

Starting in an arbitrary state?

$\mathbf{B}_2$

$$
\begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3
\end{array}
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\left[\begin{array}{cccc}
(0, \{\epsilon\}) & (1, \{[0, 1]\}) & (1, \{[0, 2]\}) & (3, \{[0, 2, 1, 3]\}) \\
(1, \{[1, 0]\}) & (0, \{\epsilon\}) & (1, \{[1, 2]\}) & \infty \\
(1, \{[2, 0]\}) & (1, \{[2, 1]\}) & (0, \{\epsilon\}) & (3, \{[2, 0, 1, 3]\}) \\
\infty & \infty & \infty & (0, \{\epsilon\})
\end{array}\right]
\end{array}
$$

$\mathbf{B}_3$

$$
\begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3
\end{array}
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\left[\begin{array}{cccc}
(0, \{\epsilon\}) & (1, \{[0, 1]\}) & (1, \{[0, 2]\}) & \infty \\
(1, \{[1, 0]\}) & (0, \{\epsilon\}) & (1, \{[1, 2]\}) & \infty \\
(1, \{[2, 0]\}) & (1, \{[2, 1]\}) & (0, \{\epsilon\}) & \infty \\
\infty & \infty & \infty & (0, \{\epsilon\})
\end{array}\right]
\end{array}
$$