Logical Deduction
— Modus Ponens —

A main rule of *logical deduction* is that of *Modus Ponens*:

> From the statements P and $P \implies Q$,
> the statement Q follows.

or, in other words,

> If P and $P \implies Q$ hold then so does Q.

or, in symbols,

$$\frac{P \qquad P \implies Q}{Q}$$

The use of implications:

To use an assumption of the form $P \implies Q$, aim at establishing P.
Once this is done, by Modus Ponens, one can conclude Q and so further assume it.

Theorem 11 *Let P_1, P_2, and P_3 be statements. If $P_1 \implies P_2$ and $P_2 \implies P_3$ then $P_1 \implies P_3$.*

PROOF: Ass. $P_1 \Rightarrow P_2$, $P_2 \Rightarrow P_3$.

RTP $P_1 \Rightarrow P_3$.

Assume P_1.

By MP, P_1, $P_1 \Rightarrow P_2$, get P_2.

By MP, P_2, $P_2 \Rightarrow P_3$ get P_3.

Therefore $P_1 \Rightarrow P_3$. ⊠

Bi-implication

Some theorems can be written in the form

P is equivalent to Q

or, in other words,

P implies Q, and vice versa

or

Q implies P, and vice versa

or

P if, and only if, Q $\qquad$ P iff Q

or, in symbols,

$$P \iff Q$$

Proof pattern:

In order to prove that

$$P \iff Q$$

1. Write: $(\Longrightarrow)$ and give a proof of $P \implies Q$.
2. Write: $(\Longleftarrow)$ and give a proof of $Q \implies P$.

Proposition 12 *Suppose that* n *is an integer. Then,* n *is even iff* n^2 *is even.*

PROOF: ($\Longrightarrow$) Assume n even, i.e. $n = 2 \cdot i$ for an integer i. Therefore $n^2 = (2i)^2 = 2 \cdot (2i^2)$

so n^2 is even.

($\Longleftarrow$) ~~Assume n^2 is even, i.e. $n^2 = 2i \ldots$~~

We show the contrapositive, n is odd $\Rightarrow n^2$ odd.

But we have shown Prop 8 that

m, n odd $\Rightarrow m \cdot n$ odd. So we

know this $\boxtimes$

Divisibility and congruence

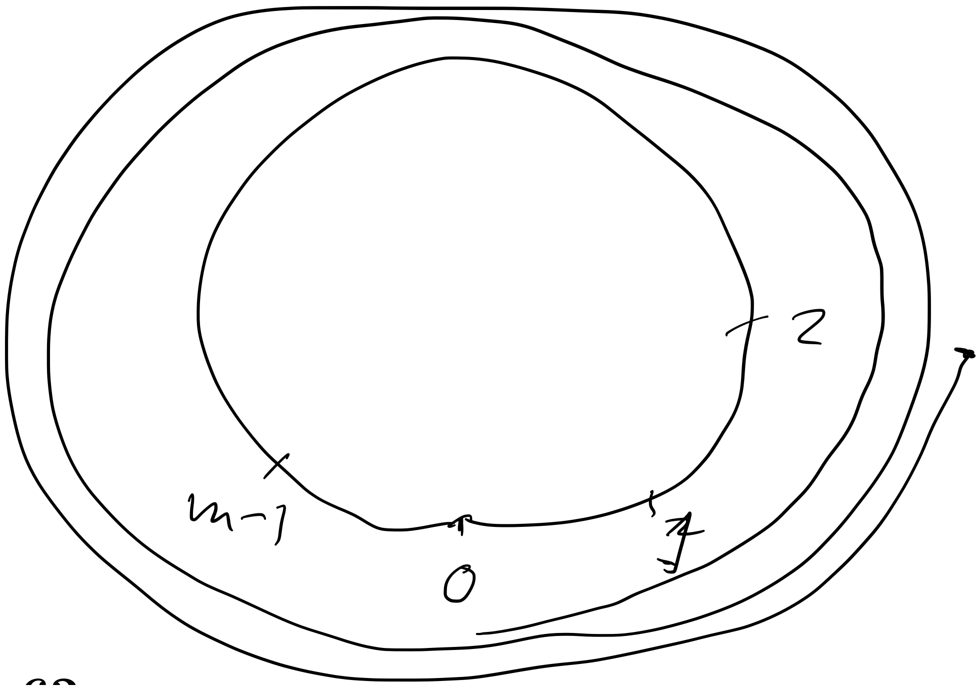
Definition 13 *Let d and n be integers. We say that d divides n, and write $d \mid n$, whenever there is an integer k such that $n = k \cdot d$.*

Example 14 *The statement $2 \mid 4$ is true, while $4 \mid 2$ is not.*

Definition 15 *Fix a positive integer m. For integers a and b, we say that a is congruent to b modulo m, and write $a \equiv b \pmod{m}$, whenever $m \mid (a - b)$.*

Example 16

1. $18 \equiv 2 \pmod{4}$
2. $2 \equiv -2 \pmod{4}$
3. $18 \equiv -2 \pmod{4}$



Proposition 17 *For every integer n,*

1. *n is even if, and only if, $n \equiv 0 \pmod 2$, and*
2. *n is odd if, and only if, $n \equiv 1 \pmod 2$.*

PROOF:

The use of bi-implications:

To use an assumption of the form $P \iff Q$, use it as two separate assumptions $P \implies Q$ and $Q \implies P$.

Universal quantification

Universal statements are of the form

> **for all** individuals x of the universe of discourse, the property $P(x)$ holds

or, in other words,

> no matter what individual x in the universe of discourse one considers, the property $P(x)$ for it holds

or, in symbols,

$$\forall x.\, P(x)$$

Example 18

2. *For every positive real number* x*, if* x *is irrational then so is* $\sqrt{x}$*.*

3. *For every integer* n*, we have that* n *is even iff so is* n^2*.*

The main proof strategy for universal statements:

To prove a goal of the form

$$\forall x.\, P(x)$$

let x stand for an arbitrary individual and prove $P(x)$.

Proof pattern:

In order to prove that

$$\forall x. \, P(x)$$

1. Write: Let x be an arbitrary individual.

2. Show that $P(x)$ holds.

∀y. P(y)

α - conversion

Proof pattern:

In order to prove that

$$\forall x.\, P(x)$$

1. Write: Let x be an arbitrary individual.

   Warning: Make sure that the variable x is new (also referred to as fresh) in the proof! If for some reason the variable x is already being used in the proof to stand for something else, then you must use an unused variable, say y, to stand for the arbitrary individual, and prove $P(y)$.

2. Show that $P(x)$ holds.

Scratch work:

Before using the strategy

Assumptions	Goal
	$\forall x.\, P(x)$
$\vdots$	

After using the strategy

Assumptions	Goal
	$P(x)$ (for a new (or fresh) x)
$\vdots$	

The use of universal statements:

To use an assumption of the form $\forall x. \, P(x)$, you can plug in any value, say a, for x to conclude that $P(a)$ is true and so further assume it.

This rule is called *universal instantiation*.

Proposition 19 *Fix a positive integer* m. *For integers* a *and* b, *we have that* $a \equiv b \pmod{m}$ *if, and only if, for all positive integers* n, *we have that* $n \cdot a \equiv n \cdot b \pmod{n \cdot m}$.

PROOF: ($\Longrightarrow$) Assume $a \equiv b \pmod{m}$, i.e. $a - b = k \cdot m$ (1) for integer k. RTP: $\forall n$, n +ve integer, $n \cdot a \equiv n \cdot b \pmod{n \cdot m}$.

Let n be a +ve int. By (1), $n \cdot (a - b) = n \cdot k \cdot m$

i.e. $n \cdot a - n \cdot b = k \cdot (n \cdot m)$, so by defn. $n \cdot a \equiv n \cdot b \pmod{n \cdot m}$

($\Longleftarrow$) Assume $\forall n$, n +ve integer, $n \cdot a \equiv n \cdot b \pmod{n \cdot m}$

Use universal instantiation, taking $a = 1$,

$1 \cdot a \equiv 1 \cdot b \pmod{1 \cdot m}$

i.e. $a \equiv b \pmod{m}$ ∎

$\forall x \, \forall y \, (x = y \Rightarrow (y = z \Rightarrow x = z))$

$x \neq z \Rightarrow y \neq z$

Equality axioms

$P(y) \Leftrightarrow y = x$

Just for the record, here are the axioms for *equality*.

- Every individual is equal to itself.

$$\forall x.\ x = x$$

$\forall x \, \forall y.\ x = y \Rightarrow y = x$

- For any pair of equal individuals, if a property holds for one of them then it also holds for the other one.

$$\forall x. \forall y.\ x = y \implies (P(x) \implies P(y))$$

Liebniz

NB From these axioms one may deduce the usual intuitive properties of equality, such as

$$\forall x. \forall y.\ x = y \implies y = x$$

and

$$\forall x. \forall y. \forall z.\ x = y \implies (y = z \implies x = z) \quad .$$

However, in practice, you will not be required to formally do so; rather you may just use the properties of equality that you are already familiar with.