THEOREM OF THE DAY

Euler's Identity *With τ and e the mathematical constants*

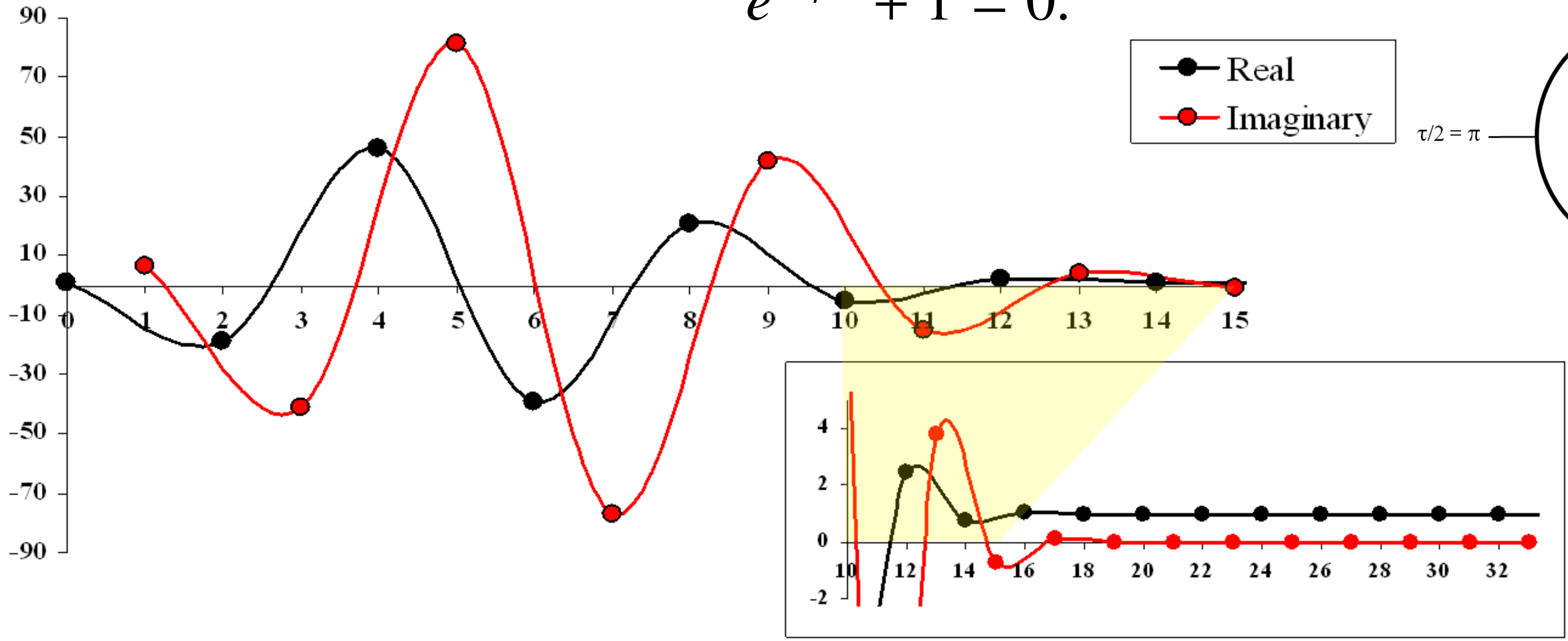
$\tau = 2\pi = 6.2831853071\ 7958647692\ 5286766559\ 0057683943\ 3879875021\ 1641949889\ 1846156328\ 1257241799\ 7256069650\ 6842341359\ldots$

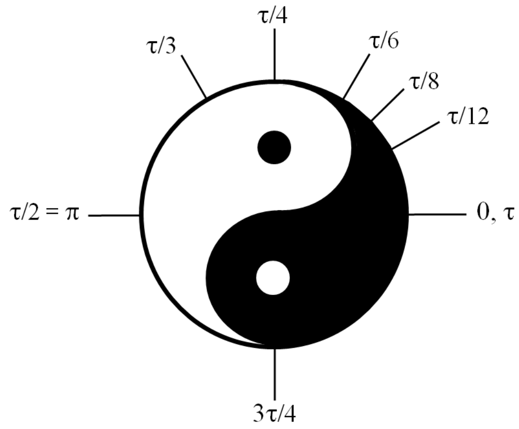
and

$e = 2.7182818284\ 5904523536\ 0287471352\ 6624977572\ 4709369995\ 9574966967\ 6277240766\ 3035354759\ 4571382178\ 5251664274\ldots$

(the first 100 places of decimal being given), and using i to denote $\sqrt{-1}$, we have

$$e^{i\tau/2} + 1 = 0.$$





Squaring both sides of $e^{i\tau/2} = -1$ gives $e^{i\tau} = 1$, encoding the defining fact that τ radians measures one full circumference. The calculation can be confirmed explicitly using the evaluation of e^z, for any complex number z, as an infinite sum: $e^z = 1 + z + z^2/2! + z^3/3! + z^4/4! + \ldots$. The even powers of $i = \sqrt{-1}$ alternate between 1 and -1, while the odd powers alternate between i and $-i$, so we get two separate sums, one with i's (the imaginary part) and one without (the real part). Both converge rapidly as shown in the two plots above: the real part to 1, the imaginary to 0. In the *limit* equality is attained, $e^{i\tau} = 1 + 0 \times i$, whence $e^{\tau i} = 1$. The value of $e^{i\tau/2}$ may be confirmed in the same way.

Combining as it does the six most fundamental constants of mathematics: 0, 1, 2, i, τ and e, the identity has an air of magic. J.H. Conway, in The Book of Numbers, traces the identity to Leonhard Euler's 1748 *Introductio*; certainly Euler deserves credit for the much more general formula $e^{i\theta} = \cos\theta + i\sin\theta$, from which the identity follows using $\theta = \tau/2$ radians (180°).

Web link: `fermatslasttheorem.blogspot.com/2006/02/eulers-identity.html`

Further reading: *Dr Euler's Fabulous Formula: Cures Many Mathematical Ills,* by Paul J. Nahin, Princeton University Press, 2006