

Discrete Mathematics for Part I CST 2018/19

Sets Exercises

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- Suggested supervision schedule
 - On sets, relations, and partial functions: Basic (§§ 1.1, 2.1, 3.1) and core (§§ 1.2, 2.2, 3.2) exercises. Lectures 12–14 onwards.
 - On functions, bijections, and equivalence relations: Basic (§§ 4.1, 5.1, 6.1) and core (§§ 4.2, 5.2, 6.2) exercises. Lecture 16 onwards.
 - On surjections, injections, and images: Basic (§§ 7.1, 8.1, 9.1) and core (§§ 7.2, 8.2, 9.2) exercises. Lecture 17 onwards.
 - On countability: Basic (§ 10.1) and core (§ 10.2) exercises. Lecture 18 onwards.
- Suggested Easter-break work
 - 2018 Paper 2 Questions 7 (b), 8 (b) & (c), and 9 (b) & (c)
 - 2017 Paper 2 Questions 7 (b), 8 (b) & (c), and 9 (b) & (c)
 - 2016 Paper 2 Question 9 (b) & (c)
 - 2015 Paper 2 Questions 7 (c), 8 (c), and 9 (b) & (c)
 - 2014 Paper 2 Question 8
 - 2013 Paper 2 Question 5
 - 2011 Paper 2 Question 5
 - 2009 Paper 1 Question 4
 - 2008 Paper 2 Question 3
 - 2007 Paper 2 Question 5
 - 2006 Paper 2 Question 5

1 On sets

1.1 Basic exercises

1. Prove the following statements:

- Reflexivity: \forall sets $A. A \subseteq A$.
- Transitivity: \forall sets $A, B, C. (A \subseteq B \wedge B \subseteq C) \implies A \subseteq C$.
- Antisymmetry: \forall sets $A, B. (A \subseteq B \wedge B \subseteq A) \iff A = B$.

2. Prove the following statements:

- \forall set $S. \emptyset \subseteq S$.

(b) $\forall \text{ set } S. (\forall x. x \notin S) \iff S = \emptyset.$

3. Find the union and intersection of:

(a) $\{1, 2, 3, 4, 5\}$ and $\{-1, 1, 3, 5, 7\};$

(b) $\{x \in \mathbb{R} \mid x > 7\}$ and $\{x \in \mathbb{N} \mid x > 5\}.$

4. Find the product of $\{1, 2, 3, 4, 5\}$ and $\{-1, 1, 3, 5, 7\}.$

5. Let $I = \{2, 3, 4, 5\}$, and for each $i \in I$ let $A_i = \{i, i + 1, i - 1, 2 \cdot i\}.$

(a) List the elements of all the sets A_i for $i \in I.$

(b) Let $\{A_i \mid i \in I\}$ stand for $\{A_2, A_3, A_4, A_5\}.$ Find $\bigcup \{A_i \mid i \in I\}$ and $\bigcap \{A_i \mid i \in I\}.$

6. Find the disjoint union of $\{1, 2, 3, 4, 5\}$ and $\{-1, 1, 3, 5, 7\}.$

7. Let U be a set. For all $A, B \in \mathcal{P}(U)$ prove that

(a) $A^c = B \iff (A \cup B = U \wedge A \cap B = \emptyset),$

(b) $(A^c)^c = A,$ and

(c) the De Morgan's laws:

$$(A \cup B)^c = A^c \cap B^c \text{ and } (A \cap B)^c = A^c \cup B^c .$$

8. Establish the laws of the powerset Boolean algebra.

1.2 Core exercises

1. Either prove or disprove that, for all sets A and $B,$

(a) $A \subseteq B \implies \mathcal{P}(A) \subseteq \mathcal{P}(B),$

(b) $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B),$

(c) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B).$

(d) $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B),$

(e) $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B).$

2. Let U be a set. For all $A, B \in \mathcal{P}(U)$ prove that the following statements are equivalent.

(a) $A \cup B = B.$

(b) $A \subseteq B.$

(c) $A \cap B = A.$

(d) $B^c \subseteq A^c.$

3. For sets $A, B, C, D,$ either prove or disprove the following statements.

(a) $(A \subseteq B \wedge C \subseteq D) \implies A \times C \subseteq B \times D.$

(b) $(A \cup C) \times (B \cup D) \subseteq (A \times B) \cup (C \times D).$

(c) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D).$

(d) $A \times (B \cup D) \subseteq (A \times B) \cup (A \times D).$

(e) $(A \times B) \cup (A \times D) \subseteq A \times (B \cup D).$

4. Prove or disprove the following statements for all sets $A, B, C, D:$

- (a) $(A \subseteq B \wedge C \subseteq D) \implies A \uplus C \subseteq B \uplus D$,
- (b) $(A \cup B) \uplus C \subseteq (A \uplus C) \cup (B \uplus C)$,
- (c) $(A \uplus C) \cup (B \uplus C) \subseteq (A \cup B) \uplus C$,
- (d) $(A \cap B) \uplus C \subseteq (A \uplus C) \cap (B \uplus C)$,
- (e) $(A \uplus C) \cap (B \uplus C) \subseteq (A \cap B) \uplus C$.

5. For $\mathcal{F} \subseteq \mathcal{P}(A)$, let $\mathcal{U} = \{X \subseteq A \mid \forall S \in \mathcal{F}. S \subseteq X\} \subseteq \mathcal{P}(A)$. Prove that $\bigcup \mathcal{F} = \bigcap \mathcal{U}$. Analogously, define $\mathcal{L} \subseteq \mathcal{P}(A)$ such that $\bigcap \mathcal{F} = \bigcup \mathcal{L}$. Also prove this statement.
6. Prove that, for all collections of sets \mathcal{F} , it holds that

$$\forall \text{ set } U. \bigcup \mathcal{F} \subseteq U \iff (\forall X \in \mathcal{F}. X \subseteq U) \quad .$$

1.3 Optional advanced exercises

Prove that for all collections of sets \mathcal{F}_1 and \mathcal{F}_2 ,

$$(\bigcup \mathcal{F}_1) \cup (\bigcup \mathcal{F}_2) = \bigcup (\mathcal{F}_1 \cup \mathcal{F}_2) \quad .$$

State and prove the analogous property for intersections of non-empty collections of sets.

2 On relations

2.1 Basic exercises

1. Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, and $C = \{x, y, z\}$.
Let $R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\} : A \leftrightarrow B$ and $S = \{(b, x), (b, y), (c, y), (d, z)\} : B \leftrightarrow C$.
What is the composition $S \circ R : A \leftrightarrow C$?
2. Prove that relational composition is associative and has the identity relation as neutral element.
3. For a relation $R : A \leftrightarrow B$, let its *opposite*, or *dual*, $R^{\text{op}} : B \leftrightarrow A$ be defined by

$$b R^{\text{op}} a \iff a R b \quad .$$

For $R, S : A \leftrightarrow B$, prove that

- (a) $R \subseteq S \implies R^{\text{op}} \subseteq S^{\text{op}}$.
- (b) $(R \cap S)^{\text{op}} = R^{\text{op}} \cap S^{\text{op}}$.
- (c) $(R \cup S)^{\text{op}} = R^{\text{op}} \cup S^{\text{op}}$.

4. For a relation R on a set A , prove that R is antisymmetric iff $R \cap R^{\text{op}} \subseteq \text{id}_A$.

2.2 Core exercises

1. Let $\mathcal{F} \subseteq \mathcal{P}(A \times B)$ be a collection of relations from A to B . Prove that,
 - (a) for all $R : X \leftrightarrow A$,

$$(\bigcup \mathcal{F}) \circ R = \bigcup \{S \circ R \mid S \in \mathcal{F}\} : X \leftrightarrow B \quad ,$$
 and that,
 - (b) for all $R : B \leftrightarrow Y$,

$$R \circ (\bigcup \mathcal{F}) = \bigcup \{ R \circ S \mid S \in \mathcal{F} \} : A \leftrightarrow Y \ .$$

What happens in the case of big intersections?

- For a relation R on a set A , let

$$\mathcal{T}_R = \{ Q \subseteq A \times A \mid R \subseteq Q \wedge Q \text{ is transitive} \} \ .$$

For $R^{\circ+} = R \circ R^{\circ*}$, prove that (i) $R^{\circ+} \in \mathcal{T}_R$ and (ii) $R^{\circ+} \subseteq \bigcap \mathcal{T}_R$. Hence, $R^{\circ+} = \bigcap \mathcal{T}_R$.

3 On partial functions

3.1 Basic exercises

- Let $A_2 = \{1, 2\}$ and $A_3 = \{a, b, c\}$. List the elements of the four sets $(A_i \Rightarrow A_j)$ for $i, j \in \{2, 3\}$.
- Prove that a relation $R : A \leftrightarrow B$ is a partial function iff $R \circ R^{\text{op}} \subseteq \text{id}_B$.
- Prove that the identity relation is a partial function, and that the composition of partial functions yields a partial function.

3.2 Core exercises

- Show that $(\text{PFun}(A, B), \subseteq)$ is a partial order.
- Show that the intersection of a non-empty collection of partial functions in $\text{PFun}(A, B)$ is a partial function in $\text{PFun}(A, B)$.
- Show that the union of two partial functions in $\text{PFun}(A, B)$ is a relation that need not be a partial function; but that for $f, g \in \text{PFun}(A, B)$ such that $f \subseteq h \supseteq g$ for some $h \in \text{PFun}(A, B)$, the union $f \cup g$ is a partial function in $\text{PFun}(A, B)$.

4 On functions

4.1 Basic exercises

- Let $A_2 = \{1, 2\}$ and $A_3 = \{a, b, c\}$. List the elements of the four sets $(A_i \Rightarrow A_j)$ for $i, j \in \{2, 3\}$.
- A relation $R : A \leftrightarrow B$ is said to be total whenever $\forall a \in A. \exists b \in B. a R b$. Prove that this is equivalent to $\text{id}_A \subseteq R^{\text{op}} \circ R$.

Conclude that a relation $R : A \leftrightarrow B$ is a function iff $R \circ R^{\text{op}} \subseteq \text{id}_B$ and $\text{id}_A \subseteq R^{\text{op}} \circ R$.

- Prove that the identity partial function is a function, and that the composition of functions yields a function.

4.2 Core exercises

- Find endofunctions $f, g : A \rightarrow A$ such that $f \circ g \neq g \circ f$. Prove your claim.
- Let $\chi : \mathcal{P}(U) \rightarrow (U \Rightarrow [2])$ be the function mapping subsets S of U to their characteristic (or indicator) functions $\chi_S : U \rightarrow [2]$.

(a) Prove that, for all $x \in U$,

- $\chi_{A \cup B}(x) = (\chi_A(x) \text{ OR } \chi_B(x)) = \max(\chi_A(x), \chi_B(x))$,
- $\chi_{A \cap B}(x) = (\chi_A(x) \text{ AND } \chi_B(x)) = \min(\chi_A(x), \chi_B(x))$,

- $\chi_{A^c}(x) = \text{NOT}(\chi_A(x)) = (1 - \chi_A(x))$.
- (b) For what construction $A?B$ on sets A and B it holds that

$$\chi_{A?B}(x) = (\chi_A(x) \text{ XOR } \chi_B(x)) = (\chi_A(x) +_2 \chi_B(x))$$

for all $x \in U$? Prove your claim.

4.3 Optional advanced exercises

Consider a set A together with an element $a \in A$ and an endofunction $f : A \rightarrow A$.

Say that a relation $R \subseteq \mathbb{N} \times A$ is (a, f) -closed whenever

$$(0, a) \in R \quad \text{and} \quad \forall (n, x) \in \mathbb{N} \times A. (n, x) \in R \implies (n+1, f(x)) \in R .$$

Define the relation $F \subseteq \mathbb{N} \times A$ as

$$F = \bigcap \{ R \subseteq \mathbb{N} \times A \mid R \text{ is } (a, f)\text{-closed} \} .$$

- Prove that the relation F is (a, f) -closed.
- Prove that the relation F is total; that is, $\forall n \in \mathbb{N}. \exists y \in A. (n, y) \in F$.
- Prove that the relation F is a (total) function $\mathbb{N} \rightarrow A$; that is,

$$\forall n \in \mathbb{N}. \exists! y \in A. (n, y) \in F .$$

Hint: Proceed by induction. Observe that, in view of the previous item, to show that $\exists! y \in A. (\ell, y) \in F$ it suffices to exhibit an (a, f) -closed relation R_ℓ such that $\exists! y \in A. (\ell, y) \in R_\ell$. (Why?) For instance, as the relation $R_0 = \{ (m, y) \in \mathbb{N} \times A \mid m = 0 \implies y = a \}$ is (a, f) -closed one has that $(0, y) \in F \implies (0, y) \in R_0 \implies y = a$.

- Show that if h is a function $\mathbb{N} \rightarrow A$ such that $h(0) = a$ and $\forall n \in \mathbb{N}. h(n+1) = f(h(n))$ then $h = F$.

Thus, for every set A together with an element $a \in A$ and an endofunction $f : A \rightarrow A$ there exists a unique function $F : \mathbb{N} \rightarrow A$, typically said to be *inductively defined*, satisfying the recurrence relation

$$F(n) = \begin{cases} a & , \text{ for } n = 0 \\ f(F(n-1)) & , \text{ for } n \geq 1 \end{cases}$$

5 On bijections

5.1 Basic exercises

- Give examples of functions that have (i) none, (ii) exactly one, and (iii) more than one retraction.
 - Give examples of functions that have (i) none, (ii) exactly one, and (iii) more than one section.
- Let n be an integer.
 - How many sections are there for the absolute-value map $[-n..n] \rightarrow [0..n] : x \mapsto |x|$?
 - How many retractions are there for the exponential map $[0..n] \rightarrow [0..2^n] : x \mapsto 2^x$?
- Give an example of two sets A and B and a function $f : A \rightarrow B$ satisfying both:
 - there is a retraction for f , and
 - there is no section for f .

Explain how you know that f has these two properties.

- Prove that the identity function is a bijection, and that the composition of bijections yields a bijection.
- For $f : A \rightarrow B$, prove that if there are $g, h : B \rightarrow A$ such that $g \circ f = \text{id}_A$ and $f \circ h = \text{id}_B$ then $g = h$. Conclude as a corollary that, whenever it exists, the inverse of a function is unique.

5.2 Core exercises

1. We say that two functions $s : A \rightarrow B$ and $r : B \rightarrow A$ are a *section-retraction* pair whenever $r \circ s = \text{id}_A$; and that a function $e : B \rightarrow B$ is an *idempotent* whenever $e \circ e = e$.
 - (a) Show that if $s : A \rightarrow B$ and $r : B \rightarrow A$ are a section-retraction pair then the composite $s \circ r : B \rightarrow B$ is an idempotent.
 - (b) Prove that for every idempotent $e : B \rightarrow B$ there exists a set A and a section-retraction pair $s : A \rightarrow B$ and $r : B \rightarrow A$ such that $s \circ r = e$.
 - (c) Let $p : C \rightarrow D$ and $q : D \rightarrow C$ be functions such that $p \circ q \circ p = p$. Can you conclude that
 - $p \circ q$ is idempotent? If so, how?
 - $q \circ p$ is idempotent? If so, how?
2. Prove the isomorphisms of the *Calculus of Bijections, I*.
3. Prove that, for all $m, n \in \mathbb{N}$,
 - (a) $\mathcal{P}([n]) \cong [2^n]$
 - (b) $[m] \times [n] \cong [m \cdot n]$
 - (c) $[m] \uplus [n] \cong [m + n]$
 - (d) $([m] \rightrightarrows [n]) \cong [(n + 1)^m]$
 - (e) $([m] \Rightarrow [n]) \cong [n^m]$
 - (f) $\text{Bij}([n], [n]) \cong [n!]$

6 On equivalence relations

6.1 Basic exercises

1. For a relation R on a set A , prove that
 - R is reflexive iff $\text{id}_A \subseteq R$,
 - R is symmetric iff $R \subseteq R^{\text{op}}$,
 - R is transitive iff $R \circ R \subseteq R$.
2. Prove that the isomorphism relation \cong between sets is an equivalence relation.
3. Prove that the identity relation id_A on a set A is an equivalence relation and that $A /_{\text{id}_A} \cong A$.
4. Show that, for a positive integer m , the relation \equiv_m on \mathbb{Z} given by

$$x \equiv_m y \iff x \equiv y \pmod{m} .$$

is an equivalence relation.

5. Show that the relation \equiv on $\mathbb{Z} \times \mathbb{N}^+$ given by

$$(a, b) \equiv (x, y) \iff a \cdot y = x \cdot b$$

is an equivalence relation.

6. Let B be a subset of a set A . Define the relation E on $\mathcal{P}(A)$ by

$$(X, Y) \in E \iff X \cap B = Y \cap B .$$

Show that E is an equivalence relation.

6.2 Core exercises

- Let E_1 and E_2 be two equivalence relations on a set A . Either prove or disprove the following statements.
 - $E_1 \cup E_2$ is an equivalence relation on A .
 - $E_1 \cap E_2$ is an equivalence relation on A .
- For an equivalence relation E on a set A , show that $[a_1]_E = [a_2]_E$ iff $a_1 E a_2$, where $[a]_E = \{x \in A \mid x E a\}$.
- For a function $f : A \rightarrow B$ define a relation \equiv_f on A by the rule

$$a \equiv_f a' \iff f(a) = f(a')$$

for all $a, a' \in A$.

- Show that for every function $f : A \rightarrow B$, the relation \equiv_f is an equivalence on A .
- Prove that every equivalence relation E on a set A is equal to \equiv_q for q the quotient function $A \twoheadrightarrow A/_E : a \mapsto [a]_E$.
- Prove that for every surjection $f : A \twoheadrightarrow B$,

$$B \cong (A/_{\equiv_f}) .$$

7 On surjections

7.1 Basic exercises

- Give three examples of functions that are surjective and three examples of functions that are not.
- Prove that the identity function is a surjection, and that the composition of surjections yields a surjection.

7.2 Core exercises

From surjections $A \twoheadrightarrow B$ and $X \twoheadrightarrow Y$ define, and prove surjective, functions $A \times X \twoheadrightarrow B \times Y$ and $A \uplus X \twoheadrightarrow B \uplus Y$.

8 On injections

8.1 Basic exercises

- Give three examples of functions that are injective and three of functions that are not.
- Prove that the identity function is an injection, and that the composition of injections yields an injection.

8.2 Core exercises

From injections $A \hookrightarrow B$ and $X \hookrightarrow Y$ define, and prove injective, functions $A \times X \hookrightarrow B \times Y$ and $A \uplus X \hookrightarrow B \uplus Y$.

9 On images

9.1 Basic exercises

1. What is the direct image of \mathbb{N} under the integer square-root relation $R_2 = \{(m, n) \mid m = n^2\} : \mathbb{N} \dashrightarrow \mathbb{Z}$? And the inverse image of \mathbb{N} ?

2. For a relation $R : A \dashrightarrow B$, show that

(a) $\vec{R}(X) = \bigcup_{x \in X} \vec{R}(\{x\})$ for all $X \subseteq A$, and

(b) $\overleftarrow{R}(Y) = \{a \in A \mid \vec{R}(\{a\}) \subseteq Y\}$ for all $Y \subseteq B$.

9.2 Core exercises

1. For $X \subseteq A$, prove that the direct image $\vec{f}(X) \subseteq B$ under an injective function $f : A \rightarrow B$ is in bijection with X ; that is, $X \cong \vec{f}(X)$.

2. Prove that for a surjective function $f : A \rightarrow B$, the direct image function $\vec{f} : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ is surjective.

3. Show that, by inverse image,

every map $A \rightarrow B$ induces a Boolean algebra map $\mathcal{P}(B) \rightarrow \mathcal{P}(A)$.

That is, for every function $f : A \rightarrow B$,

- $\overleftarrow{f}(\emptyset) = \emptyset$
- $\overleftarrow{f}(X \cup Y) = \overleftarrow{f}(X) \cup \overleftarrow{f}(Y)$
- $\overleftarrow{f}(B) = A$
- $\overleftarrow{f}(X \cap Y) = \overleftarrow{f}(X) \cap \overleftarrow{f}(Y)$
- $\overleftarrow{f}(X^c) = (\overleftarrow{f}(X))^c$

for all $X, Y \subseteq B$.

9.3 Optional advanced exercises

For a relation $R : A \dashrightarrow B$, prove that

(a) $\vec{R}(\bigcup \mathcal{F}) = \bigcup \{\vec{R}(X) \mid X \in \mathcal{F}\} \in \mathcal{P}(B)$ for all $\mathcal{F} \in \mathcal{P}(\mathcal{P}(A))$, and

(b) $\overleftarrow{R}(\bigcap \mathcal{G}) = \bigcap \{\overleftarrow{R}(Y) \mid Y \in \mathcal{G}\} \in \mathcal{P}(A)$ for all $\mathcal{G} \in \mathcal{P}(\mathcal{P}(B))$.

10 On countability

10.1 Basic exercises

Prove that:

- (a) Every finite set is countable.
- (b) \mathbb{N} , \mathbb{Z} , \mathbb{Q} are countable sets.

10.2 Core exercises

1. For an infinite set S , prove that if there is a surjection $\mathbb{N} \rightarrow S$ then there is a bijection $\mathbb{N} \rightarrow S$.
2. Prove that:
 - (a) Every subset of a countable set is countable.
 - (b) The product and disjoint union of countable sets is countable.
3. For an infinite set S , prove that the following are equivalent:
 - (a) There is a bijection $\mathbb{N} \rightarrow S$.
 - (b) There is an injection $S \rightarrow \mathbb{N}$.
 - (c) There is a surjection $\mathbb{N} \rightarrow S$.
4. For a set X , prove that there is no injection $\mathcal{P}(X) \rightarrow X$.

10.3 Optional advance exercises

Prove that if X and A are countable sets then so are A^* , $\mathcal{P}_{\text{fin}}(A)$, and $(X \Rightarrow_{\text{fin}} A)$.

11 On indexed sets

11.1 Optional advanced exercises

Prove the isomorphisms of the *Calculus of Bijections, II*.