VI. Approx. Algorithms: Randomisation and Rounding

Thomas Sauerwald

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UNIVERSITY OF CAMBRIDGE

Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Weighted Set Cover

MAX-CNF

Conclusion

Performance Ratios for Randomised Approximation Algorithms

Approximation Ratio

A randomised algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size n, the expected cost C of the returned solution and optimal cost C^* satisfy:

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An approximation scheme is an approximation algorithm, which given any input and $\epsilon > 0$, is a $(1+\epsilon)$-approximation algorithm.

- It is a polynomial-time approximation scheme (PTAS) if for any fixed $\epsilon > 0$, the runtime is polynomial in n. For example, $O(n^{2/\epsilon})$.
- It is a fully polynomial-time approximation scheme (FPTAS) if the runtime is polynomial in both $1/\epsilon$ and n. For example, $O((1/\epsilon)^2 \cdot n^3)$.

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Idea: What about assigning each variable uniformly and independently at random?

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Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

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Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised $8/7$-approximation algorithm.

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Any instance of MAX-3-CNF with at most 7 clauses is satisfiable.

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Follows from the previous Corollary.

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$$\mathbf{E}[\,Y\,] = \frac{1}{2} \cdot \mathbf{E}[\,Y \mid x_1 = 1\,] + \frac{1}{2} \cdot \mathbf{E}[\,Y \mid x_1 = 0\,].$$

Y is defined as in the previous proof.

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One of the two conditional expectations is at least $\mathbf{E}[\,Y\,]$!

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$$\mathbf{E}[\,Y\,] = \frac{1}{2} \cdot \mathbf{E}[\,Y \mid x_1 = 1\,] + \frac{1}{2} \cdot \mathbf{E}[\,Y \mid x_1 = 0\,].$$

Y is defined as in the previous proof.

One of the two conditional expectations is at least $\mathbf{E}[\,Y\,]$!

Algorithm: Assign x_1 so that the conditional expectation is maximized and recurse.

Expected Approximation Ratio

Theorem 35.6

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GREEDY-3-CNF(ϕ, n, m)

for $j = 1, 2, \ldots, n$
    Compute $\mathbf{E}[\,Y \mid x_1 = v_1 \ldots, x_{j-1} = v_{j-1}, x_j = 1\,]$
    Compute $\mathbf{E}[\,Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 0\,]$
    Let $x_j = v_j$ so that the conditional expectation is maximized
return the assignment $v_1, v_2, \ldots, v_n$

Analysis of GREEDY-3-CNF(ϕ, n, m)

Theorem

GREEDY-3-CNF(ϕ, n, m) is a polynomial-time $8/7$-approximation.

Analysis of GREEDY-3-CNF(ϕ, n, m)

This algorithm is deterministic.

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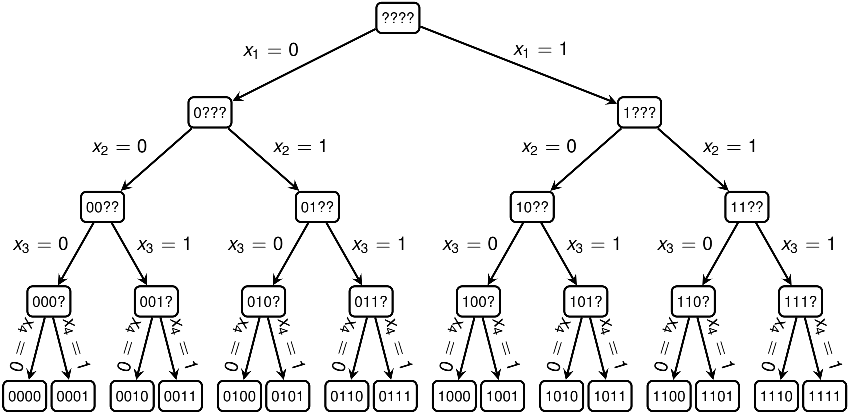
\end{aligned}$$

Run of GREEDY-3-CNF(φ, n, m)

$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



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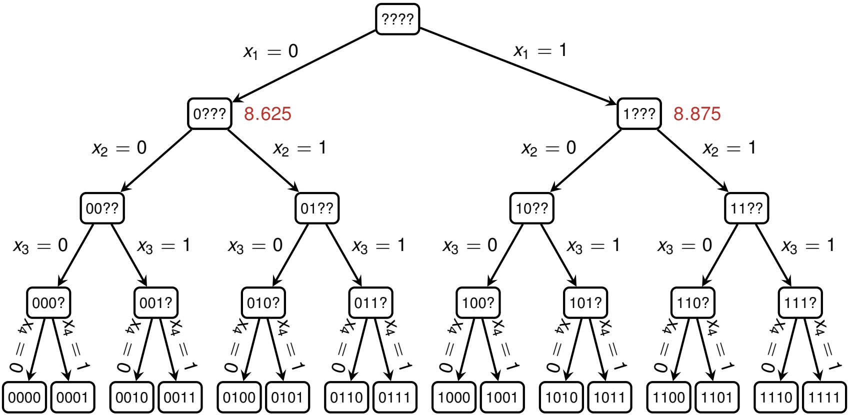
$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



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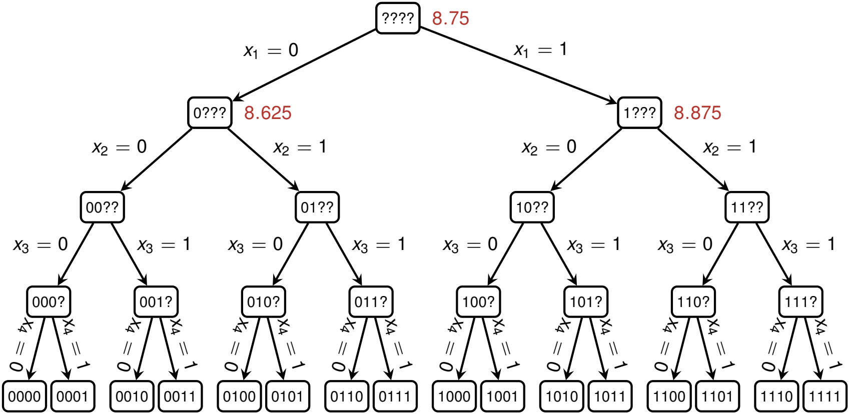

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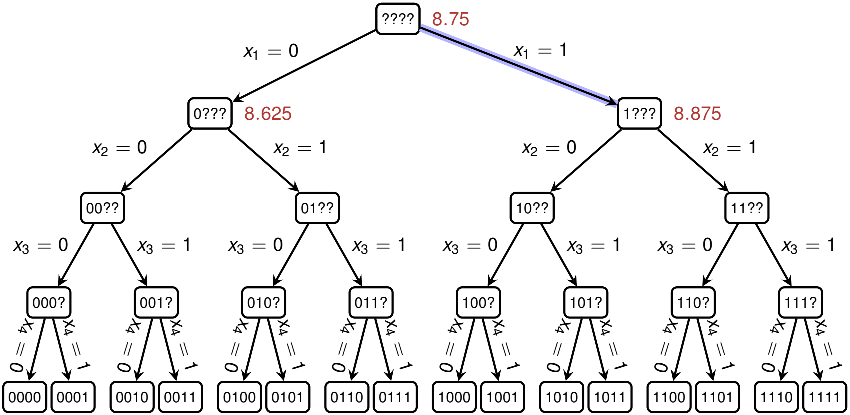

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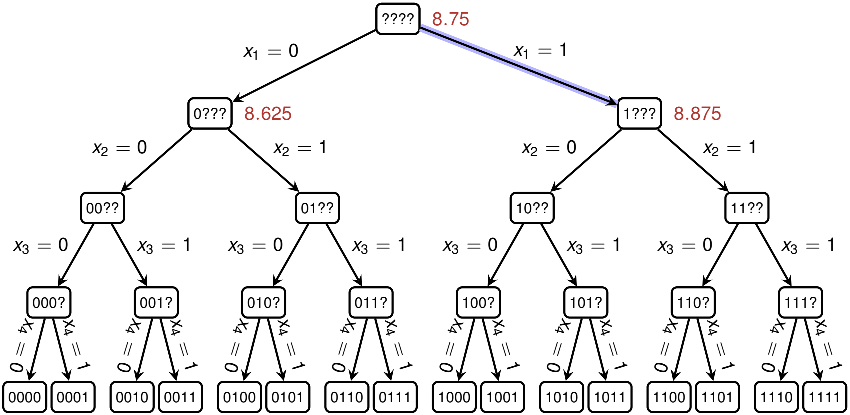
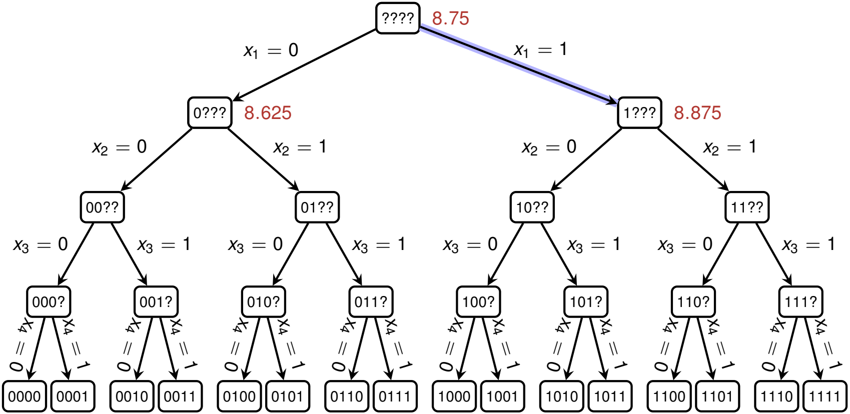


Run of GREEDY-3-CNF(φ, n, m)

$$1 \wedge 1 \wedge 1 \wedge (\overline{x_3} \vee x_4) \wedge 1 \wedge (\overline{x_2} \vee \overline{x_3}) \wedge (x_2 \vee x_3) \wedge (\overline{x_2} \vee x_3) \wedge 1 \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$$



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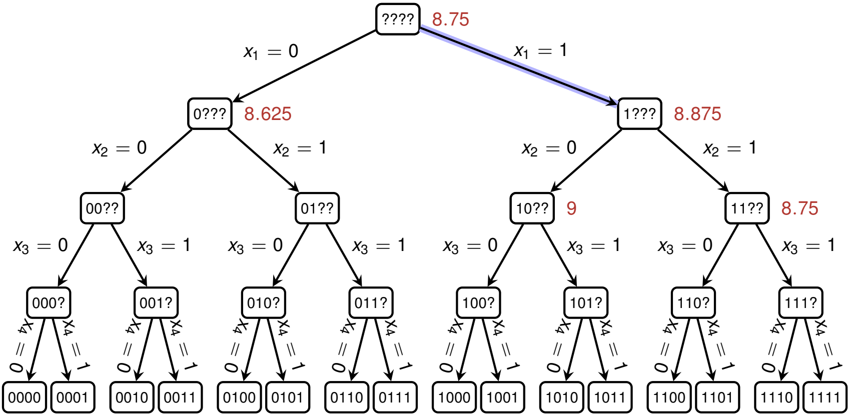

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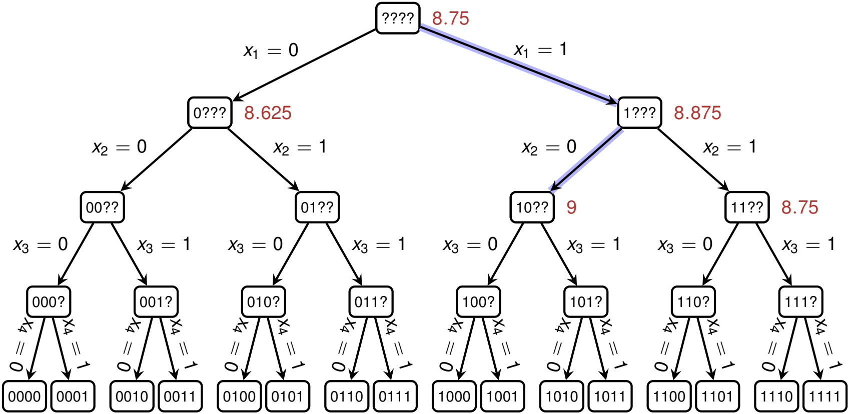

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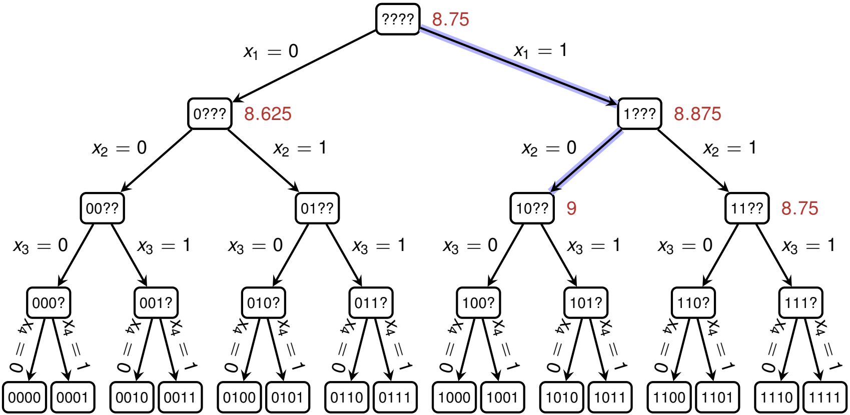

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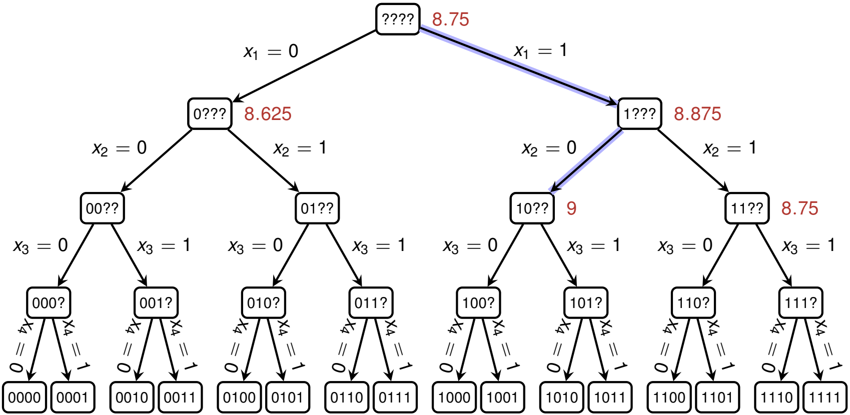

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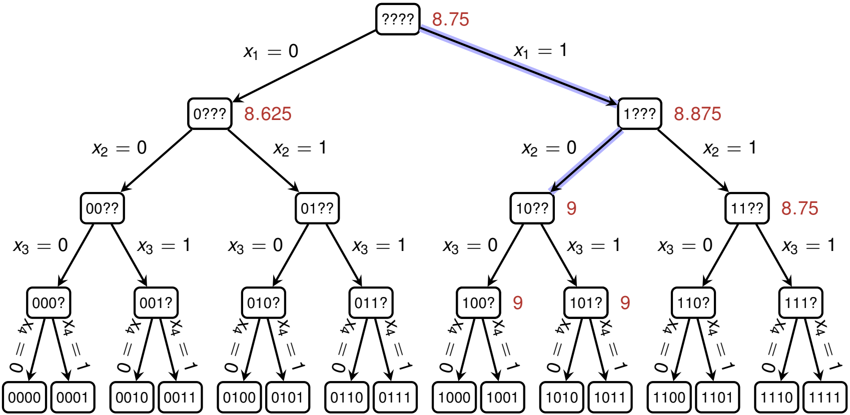

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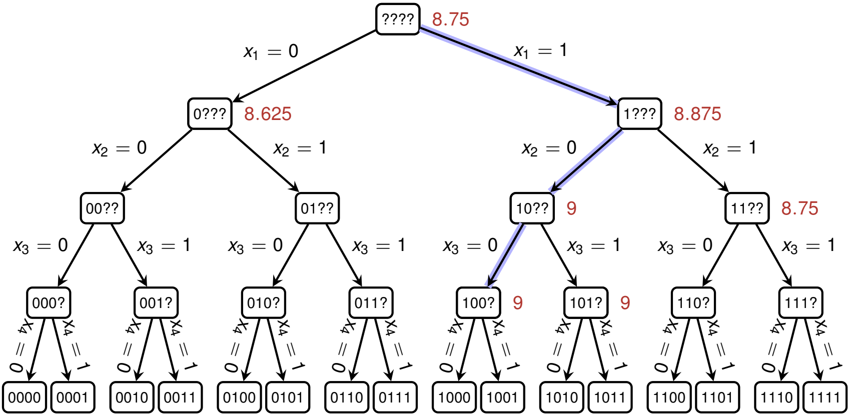
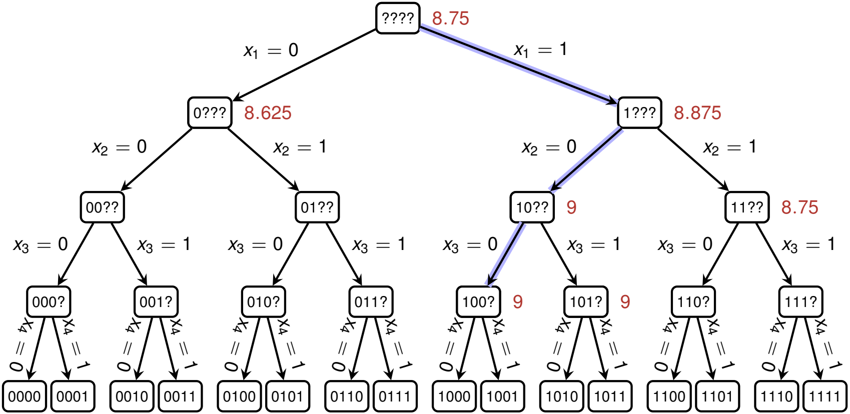


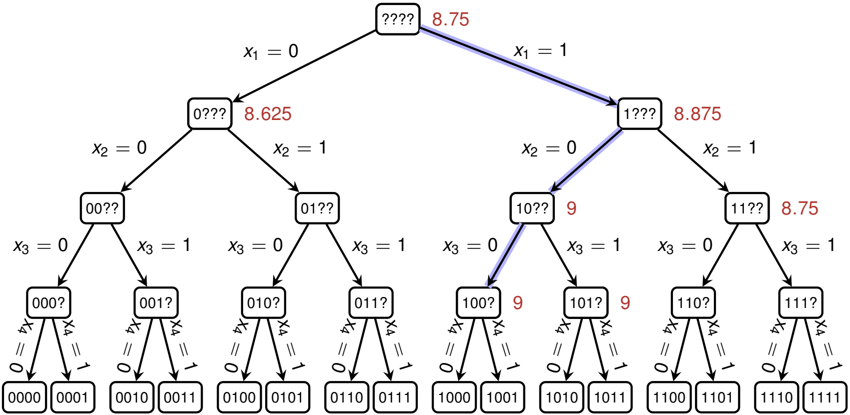
Run of GREEDY-3-CNF(φ, n, m)

$$1 \wedge 1 \wedge 1 \wedge \cancel{(\overline{x_3} \vee x_4)} \wedge 1 \wedge 1 \wedge (\cancel{x_3}) \wedge 1 \wedge 1 \wedge (\cancel{\overline{x_3}} \vee \overline{x_4})$$



Run of GREEDY-3-CNF(φ, n, m)

$1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 0 \wedge 1 \wedge 1 \wedge 1$



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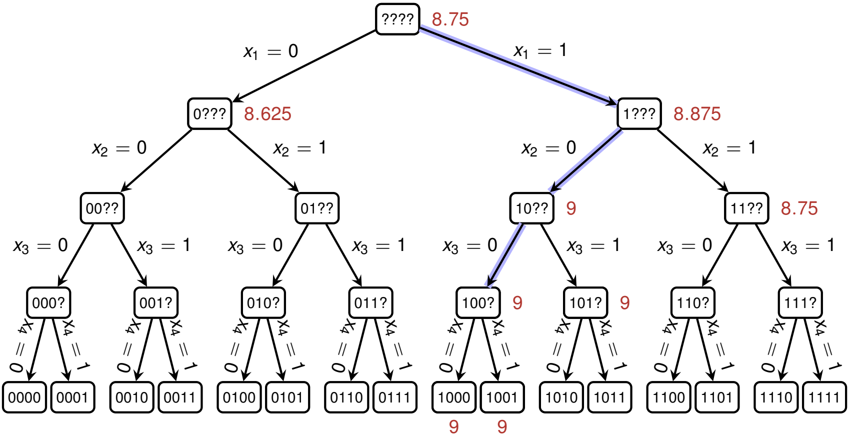

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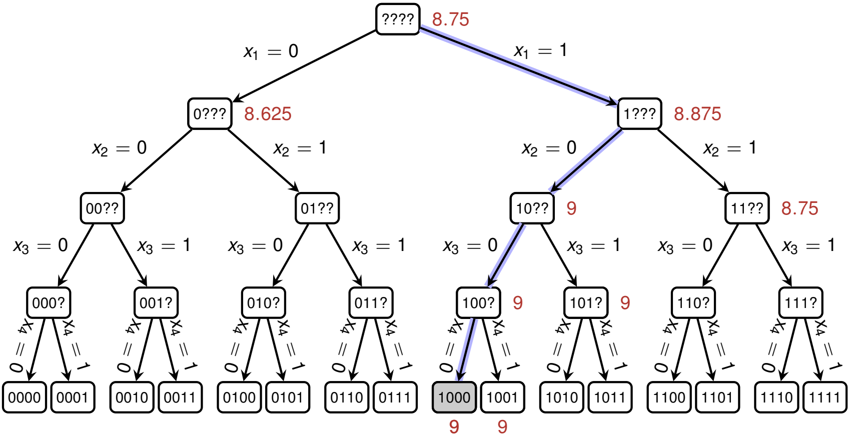

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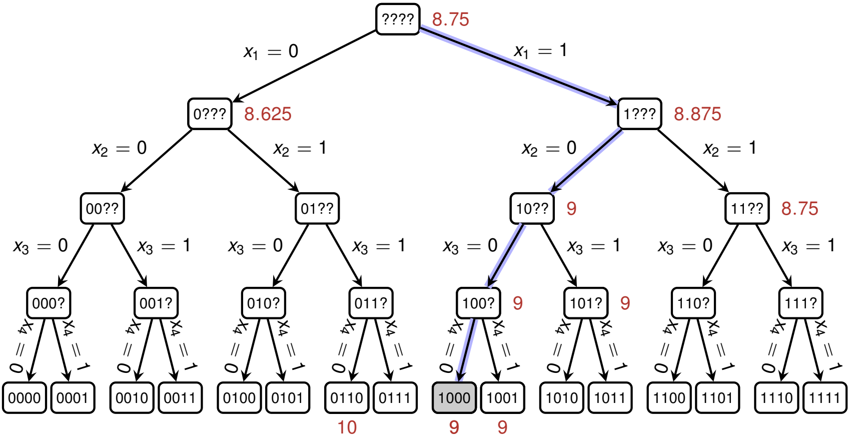

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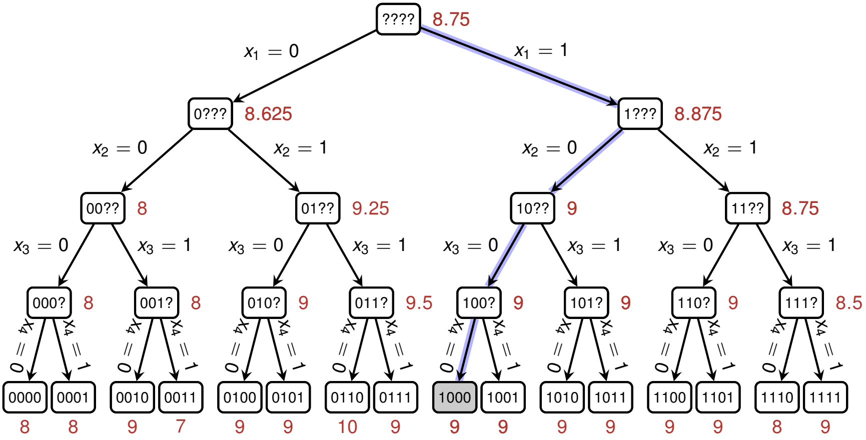

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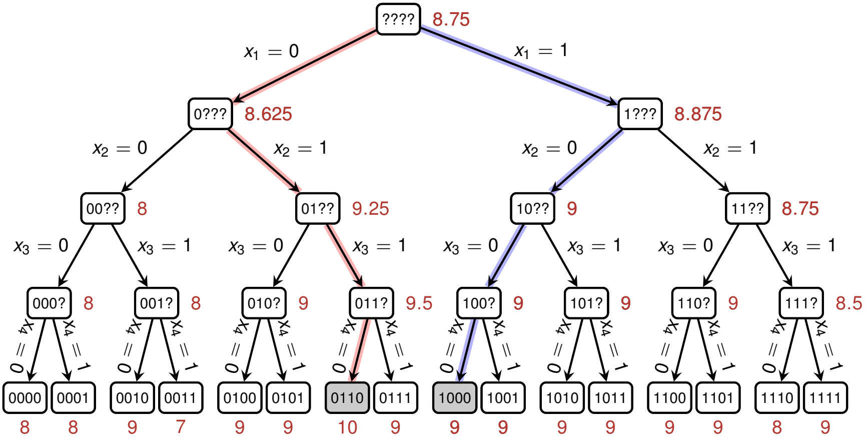

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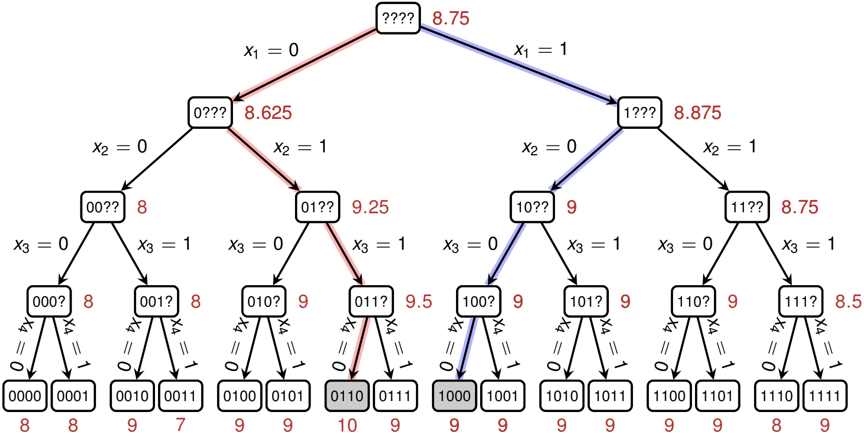

Returned solution satisfies 9 out of 10 clauses, but the formula is satisfiable.

MAX-3-CNF: Concluding Remarks

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

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GREEDY-3-CNF(ϕ, n, m) is a polynomial-time $8/7$-approximation.

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Theorem (Hastad'97)

For any $\epsilon > 0$, there is no polynomial time $8/7 - \epsilon$ approximation algorithm of MAX3-CNF unless P=NP.

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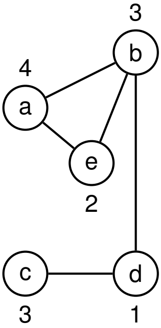
Essentially there is nothing smarter than just guessing!

Outline

The Weighted Vertex-Cover Problem

Vertex Cover Problem

- Given: Undirected, vertex-weighted graph $G = (V, E)$
- Goal: Find a minimum-weight subset $V' \subseteq V$ such that if $(u, v) \in E(G)$, then $u \in V'$ or $v \in V'$.



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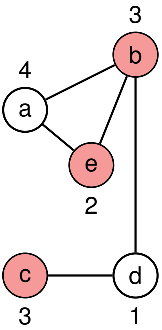

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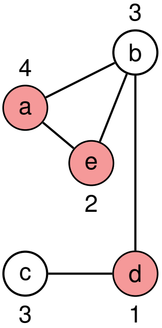

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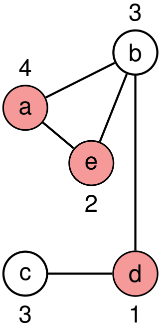

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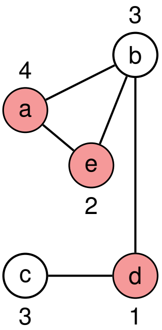

Applications:

The Weighted Vertex-Cover Problem

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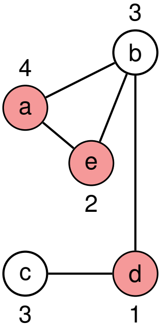

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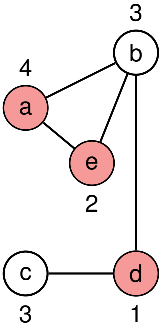

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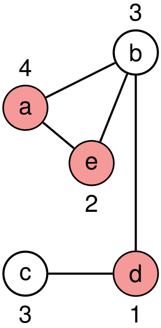

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- Weight of a vertex could be salary of a person
- Perform all tasks with the minimal amount of resources

The Greedy Approach from (Unweighted) Vertex Cover

APPROX-VERTEX-COVER(G)

```
C = ∅
E′ = G.E
while E′ ≠ ∅
    let (u, v) be an arbitrary edge of E′
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    remove from E′ every edge incident on either u or v
return C
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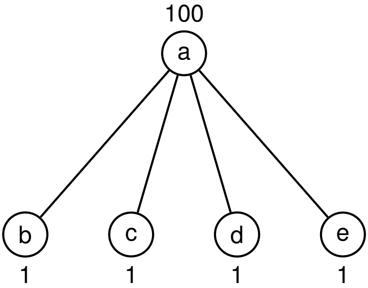

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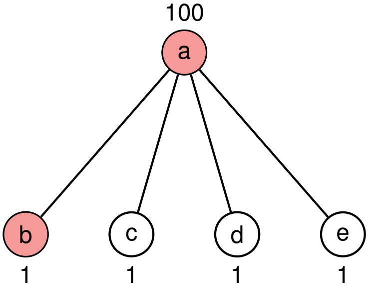

Computed solution has weight 101

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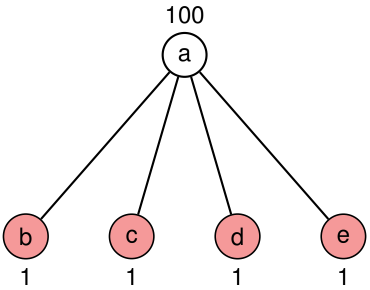

Optimal solution has weight 4

Invoking an (Integer) Linear Program

Idea: Round the solution of an associated linear program.

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0-1 Integer Program

$$
\begin{array}{llcll}
\text{minimize} & \sum_{v \in V} w(v)x(v) & & & \\
\text{subject to} & x(u) + x(v) & \geq & 1 & \text{for each } (u,v) \in E \\
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optimum is a lower bound on the optimal weight of a minimum weight-cover.

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Rounding Rule: if $x(v) \geq 1/2$ then round up, otherwise round down.

The Algorithm

APPROX-MIN-WEIGHT-VC(G, w)

$C = \emptyset$
compute $\bar{x}$, an optimal solution to the linear program
for each $v \in V$
    if $\bar{x}(v) \geq 1/2$
        $C = C \cup \{v\}$
return C

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APPROX-MIN-WEIGHT-VC is a polynomial-time 2-approximation algorithm for the minimum-weight vertex-cover problem.

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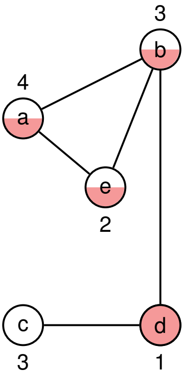
is polynomial-time because we can solve the linear program in polynomial time

Example of APPROX-MIN-WEIGHT-VC

$\overline{x}(a) = \overline{x}(b) = \overline{x}(e) = \frac{1}{2}, \overline{x}(d) = 1, \overline{x}(c) = 0$



fractional solution of LP
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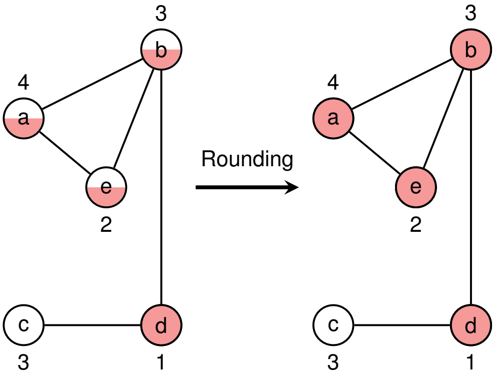
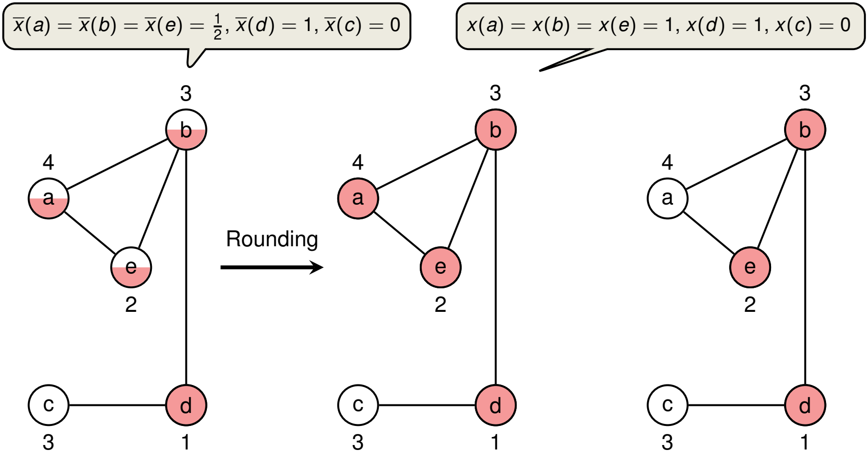


fractional solution of LP with weight $= 5.5$

rounded solution of LP with weight $= 10$

Example of APPROX-MIN-WEIGHT-VC



fractional solution of LP
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rounded solution of LP
with weight $= 10$

optimal solution
with weight $= 6$

Approximation Ratio

Proof (Approximation Ratio is 2 and Correctness):

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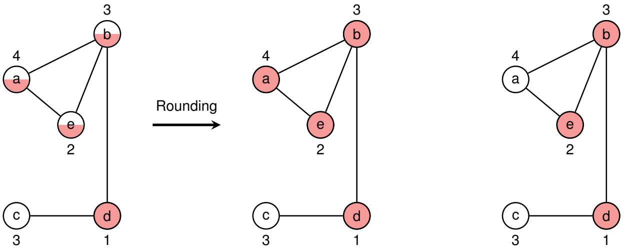

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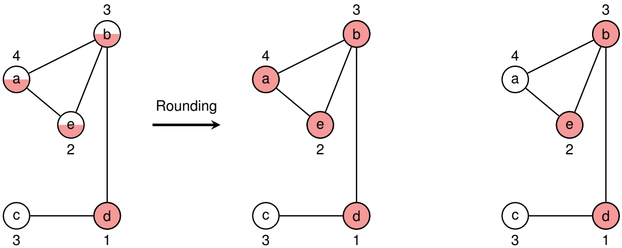

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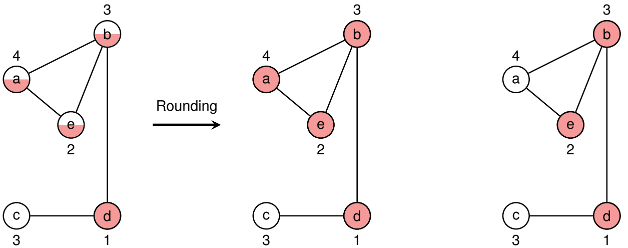

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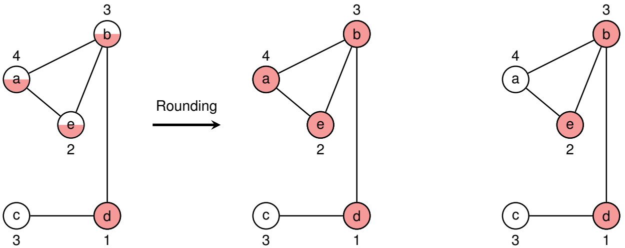

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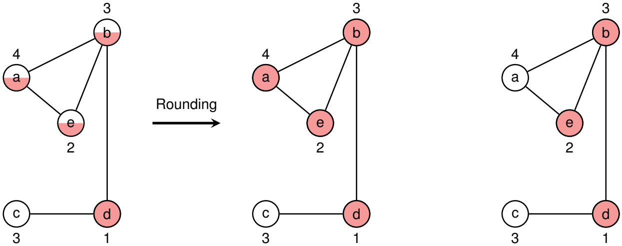

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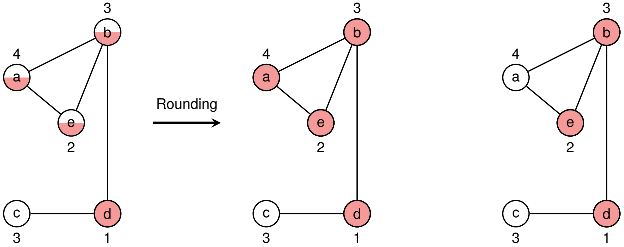

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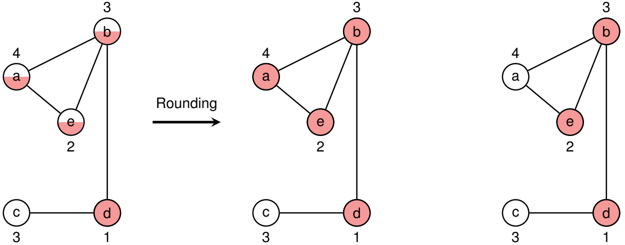

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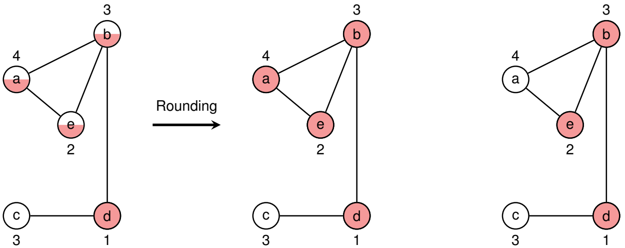

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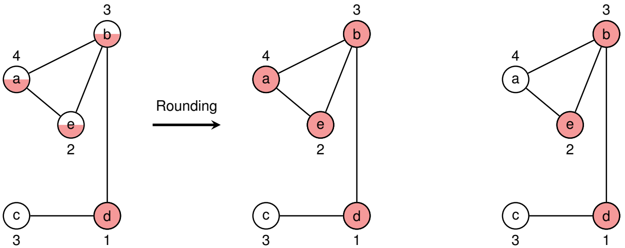

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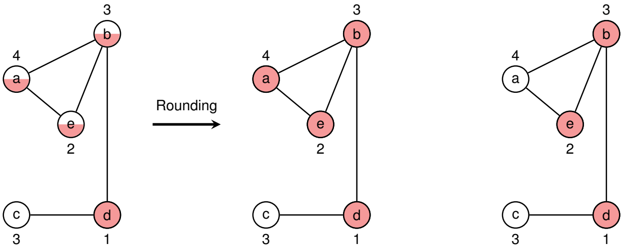

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Proof (Approximation Ratio is 2 and Correctness):

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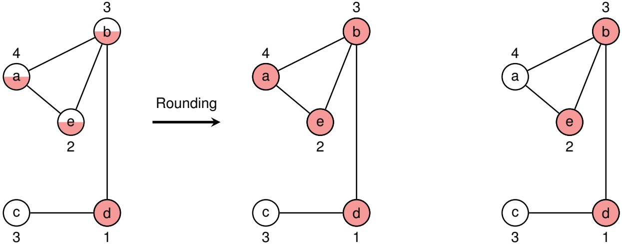

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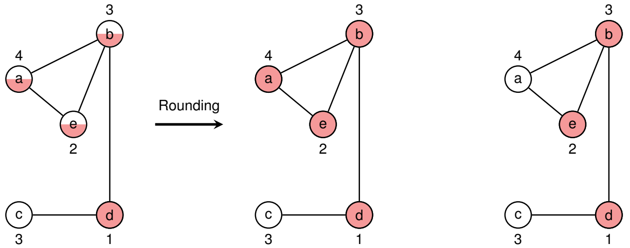

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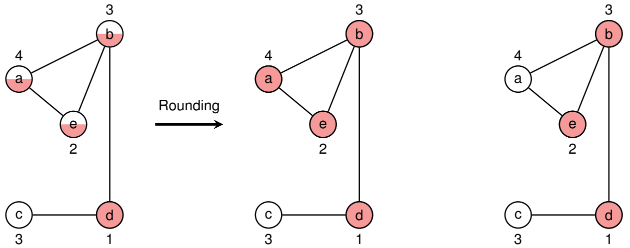

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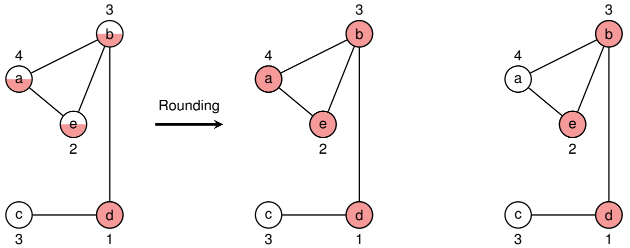

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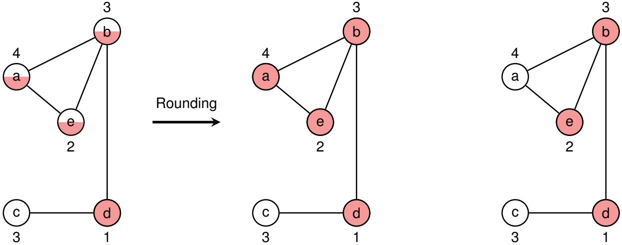

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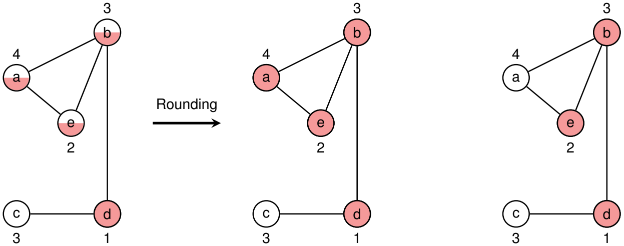

Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Weighted Set Cover

MAX-CNF

Conclusion

The Weighted Set-Covering Problem

Set Cover Problem

- Given: set X and a family of subsets $\mathcal{F}$, and a cost function $c : \mathcal{F} \to \mathbb{R}^+$
- Goal: Find a minimum-cost subset $\mathcal{C} \subseteq \mathcal{F}$

$$\text{s.t.} \qquad X = \bigcup_{S \in \mathcal{C}} S.$$

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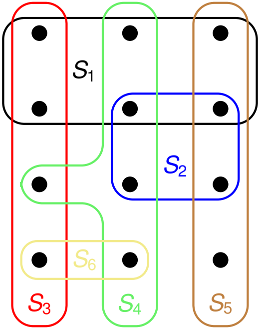

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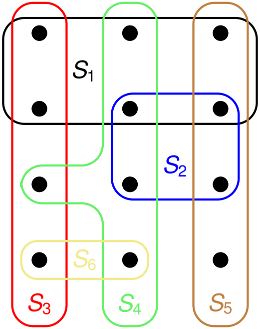

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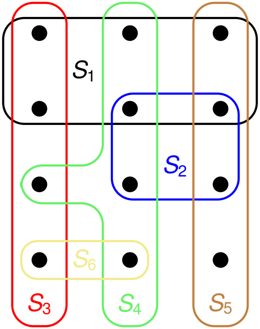

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Remarks:

- generalisation of the weighted vertex-cover problem
- models resource allocation problems

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0-1 Integer Program

$$\begin{array}{llll} \text{minimize} & \displaystyle\sum_{S \in \mathcal{F}} c(S) y(S) & & \\ \text{subject to} & \displaystyle\sum_{S \in \mathcal{F}\colon x \in S} y(S) \geq 1 & & \text{for each } x \in X \\ & y(S) \in \{0, 1\} & & \text{for each } S \in \mathcal{F} \end{array}$$

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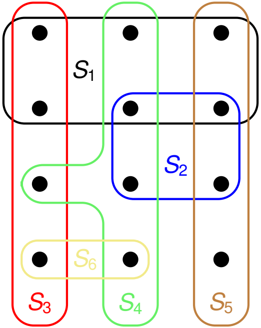

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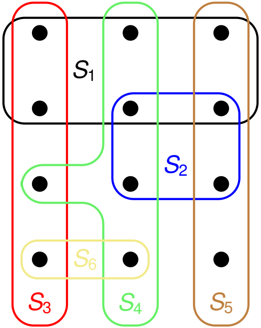

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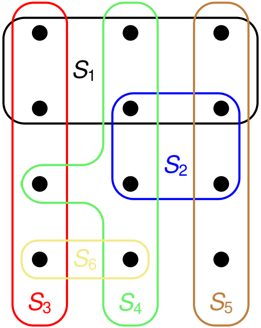

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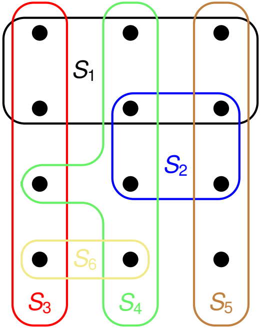

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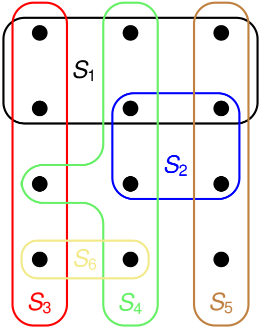

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Even worse: If all y's were below $1/2$, we would not even return a valid cover!

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- Let $\mathcal{C} \subseteq \mathcal{F}$ be a random set with each set S being included independently with probability $y(S)$.
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- Therefore, $\mathbf{E}[\,\bar{y}(S)\,] = y(S)$.

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WEIGHTED SET COVER-LP$(X, \mathcal{F}, c)$

1: compute y, an optimal solution to the linear program
2: $\mathcal{C} = \emptyset$
3: **repeat** $2 \ln n$ times
4: **for** each $S \in \mathcal{F}$
5: let $\mathcal{C} = \mathcal{C} \cup \{S\}$ with probability $y(S)$
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compute y, an optimal solution to the linear program
C = ∅
repeat 2 ln n times
    for each S ∈ F
        let C = C ∪ {S} with probability y(S)
return C
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clearly runs in polynomial-time!

Analysis of WEIGHTED SET COVER-LP

Theorem

- With probability at least $1 - \frac{1}{n}$, the returned set $\mathcal{C}$ is a valid cover of X.
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Typical Approach for Designing Approximation Algorithms based on LPs

Outline

MAX-CNF

Recall:

MAX-3-CNF Satisfiability

- Given: 3-CNF formula, e.g.: $(x_1 \vee x_3 \vee \overline{x_4}) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge \cdots$
- Goal: Find an assignment of the variables that satisfies as many clauses as possible.

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Why study this generalised problem?

- Allowing arbitrary clause lengths makes the problem more interesting (we will see that simply guessing is not the best!)
- a nice concluding example where we can practice previously learned approaches

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Assign each variable true or false uniformly and independently at random.

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- As before, let $Y := \sum_{i=1}^{m} Y_i$ be the number of satisfied clauses. Then,

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- In the corresponding LP each $\in \{0, 1\}$ is replaced by $\in [0, 1]$
- Let (y^*, z^*) be the optimal solution of the LP
- Obtain an integer solution y through randomised rounding of y^*

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Lemma

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$$= 1 - \left(1 - \frac{\sum_{j=1}^{\ell} y_j^*}{\ell}\right)^{\ell} \geq 1 - \left(1 - \frac{z_i^*}{\ell}\right)^{\ell}.$$

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For any clause i of length ℓ,

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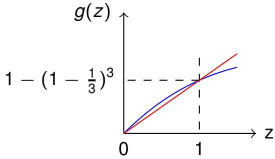

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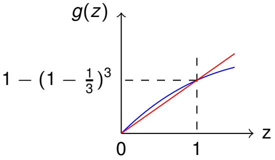

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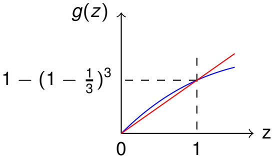

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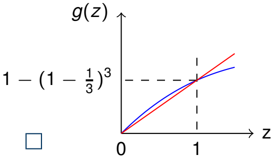

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LP solution at least as good as optimum

Approach 3: Hybrid Algorithm

Summary

- Approach 1 (Guessing) achieves better guarantee on longer clauses
- Approach 2 (Rounding) achieves better guarantee on shorter clauses

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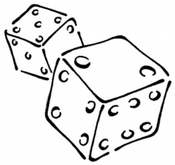
HYBRID-MAX-CNF(φ, n, m)

1: Let $b \in \{0, 1\}$ be the flip of a fair coin
2: **If** $b = 0$ **then** perform random guessing
3: **If** $b = 1$ **then** perform randomised rounding
4: **return** the computed solution



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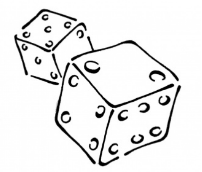

Algorithm sets each variable x_i to TRUE with prob. $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot y_i^*$.
Note, however, that variables are **not** independently assigned!

Analysis of Hybrid Algorithm

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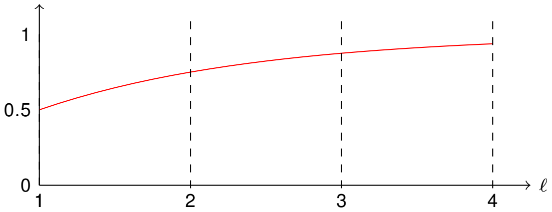

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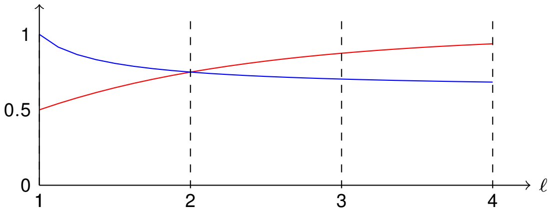

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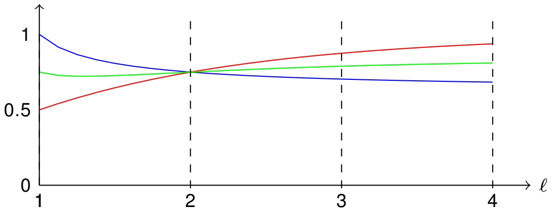

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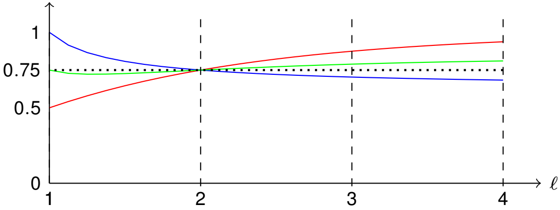

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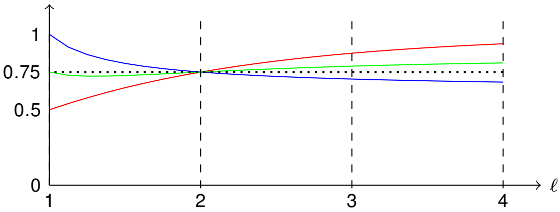
- $\Rightarrow$ HYBRID-MAX-CNF(φ, n, m) satisfies it with prob. at least $3/4 \cdot z_i^*$ □



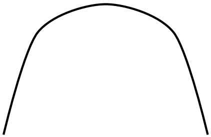
MAX-CNF Conclusion

Summary

- Since $\alpha_2 = \beta_2 = 3/4$, we cannot achieve a better approximation ratio than $4/3$ by combining Algorithm 1 & 2 in a different way
- The $4/3$-approximation algorithm can be easily derandomised
  - Idea: use the conditional expectation trick for both Algorithm 1 & 2 and output the better solution
- The $4/3$-approximation algorithm applies unchanged to a weighted version of MAX-CNF, where each clause has a non-negative weight
- Even MAX-2-CNF (every clause has length 2) is NP-hard!

Outline

Spectrum of Approximations



FPTAS

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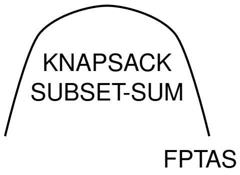

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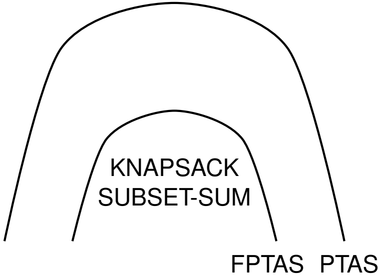

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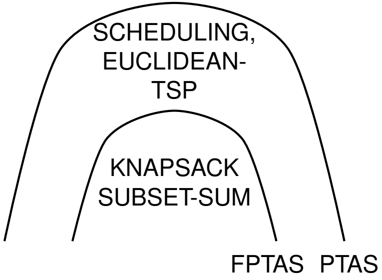

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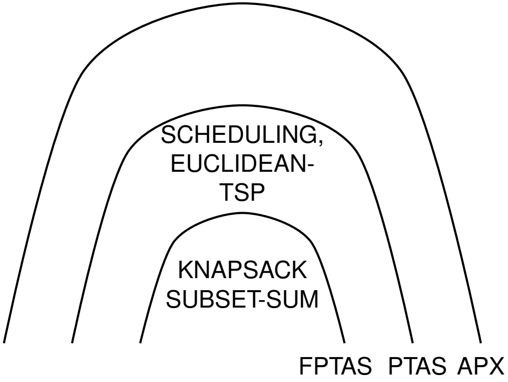

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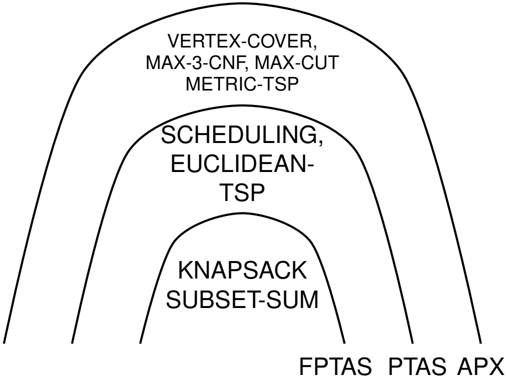

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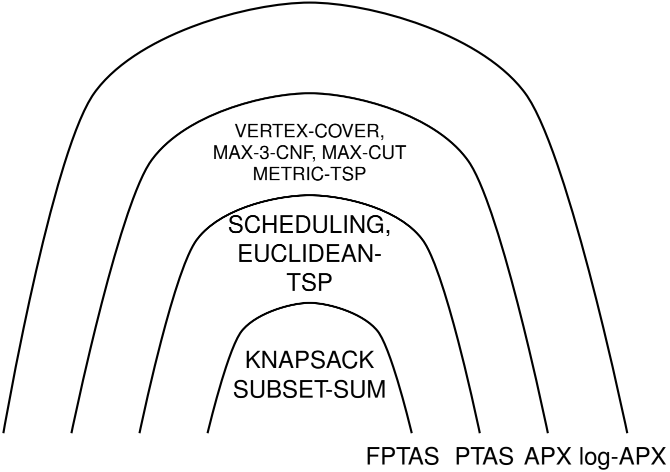

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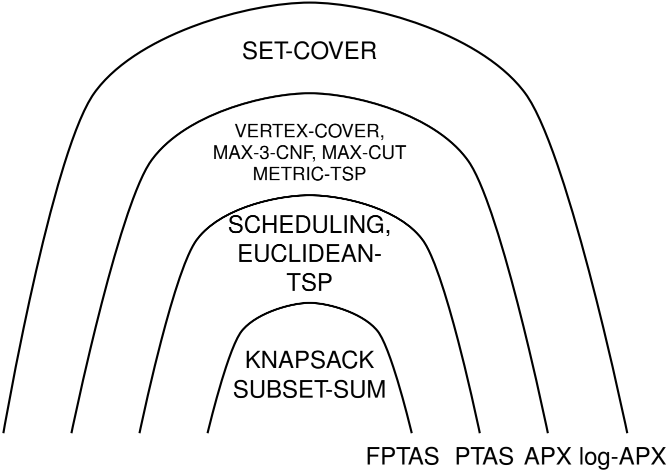

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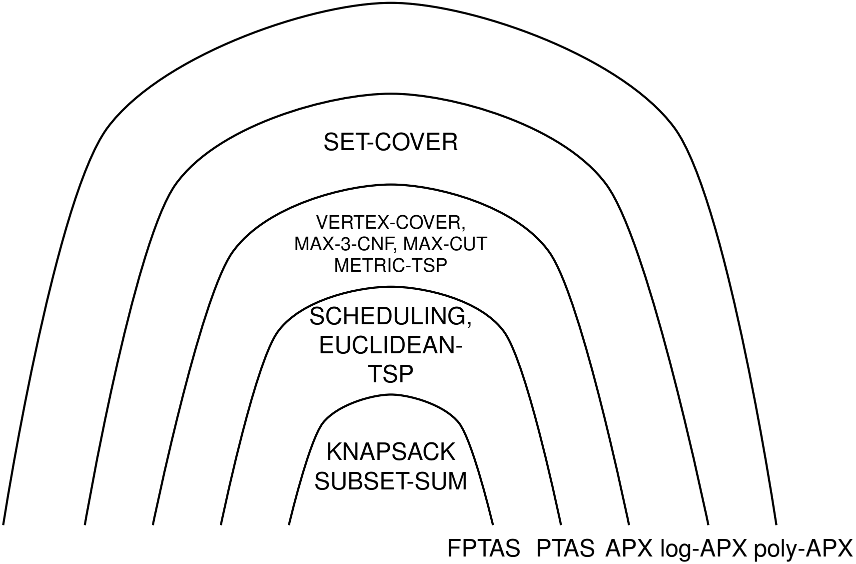

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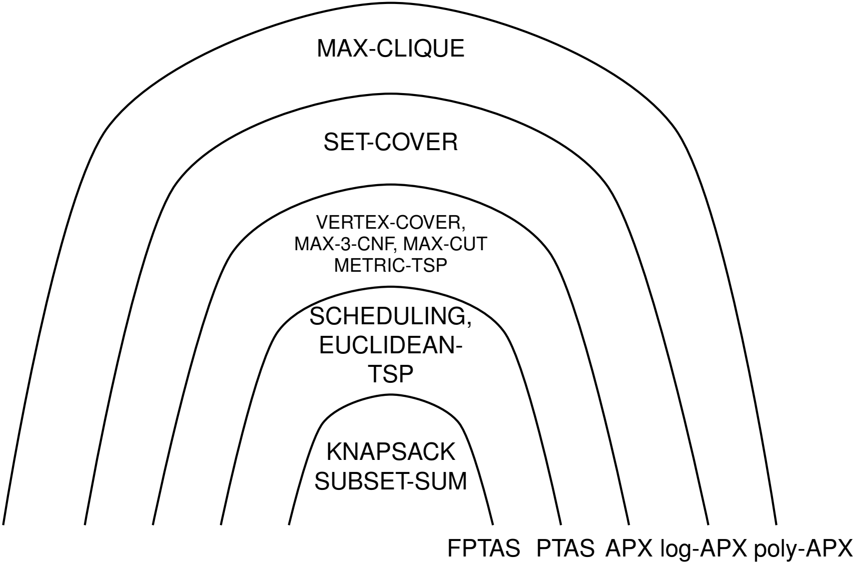

Thank you and Best Wishes for the Exam!