Addendum

Theorem. *If a linear program (in slack form) has an optimum solution, one of them occurs at a vertex.*

Proof. First, we will slightly rearrange the slack form so that it becomes a matrix-vector multiplication: $A \cdot x = b$, where x has dimension $n+m$ and A has dimension $n \times (n+m)$ (this also means that this matrix A is slightly different from the matrix A on slide 20).

Suppose now x is an optimal solution that is not a vertex. That means x can be written as a strict convex combination, $x = \lambda \cdot y + (1-\lambda) \cdot z$ where y and z are feasible and $\lambda \in (0,1)$. Since the feasible region is convex, this implies that there exists a direction d, $d \neq (0,\ldots,0)$, such that $x+d$ and $x-d$ are both feasible solutions, i.e.,

$$A(x+d) = b, A(x-d) = b \qquad \text{and} \qquad (x+d) \geqslant 0, (x-d) \geqslant 0.$$

Since $A(x+d) = b$ and $Ax = b$, we have

$$Ad = 0.$$

Next consider $c^T d$ (here c is the original vector of coefficients of the objective function appended by m zeros).

We may assume that $c^T d \geqslant 0$ (otherwise, we simply replace d by $-d$). We continue by a case distinction:

- **Case 1:** There exists a coordinate $j \in \{1,\ldots,n+m\}$ so that $d_j < 0$. Consider the function $\lambda \mapsto x + \lambda \cdot d$, and let $\lambda' > 0$ be the smallest value such that a new coordinate of $x + \lambda' \cdot d$ becomes zero. We will now verify the following three properties: (i) $x + \lambda' \cdot d$ is a feasible solution, (ii) the objective value of $x + \lambda' \cdot d$ is at least as high as the one of x and (iii) the number of zero coordinates in $x + \lambda' \cdot d$ is at least by one higher than in x.

  (i) First, since $Ad = 0$ we have $A(x + \lambda' d) = Ax + \lambda' Ad = b + 0 = b$. Further, since $x \geqslant 0$ and $x + d \geqslant 0$, we also have that $x + \lambda' d \geqslant 0$ thanks to the definition of λ'. Therefore $x + \lambda' d$ is a feasible solution.

  (ii) We have $c^T(x+\lambda' d) = c^T x + \underbrace{\lambda'}_{>0} \cdot \underbrace{c^T d}_{\geqslant 0} \geqslant c^T x$, and hence $x + \lambda' d$ must be also an optimal solution.

  (iii) By definition of $\lambda' > 0$, we know that at least one coordinate in $x + \lambda' d$ is zero which has not been zero in x. We now further show that if a coordinate is zero in x, it remains zero in $x + \lambda' d$. Suppose we have $x_i = 0$ for a coordinate $i \in \{1,\ldots,m+n\}$. Since $x + d \geqslant 0$ and $x - d \geqslant 0$, we conclude that $d_i = 0$.

- **Case 2:** For all $j \in \{1, \ldots, n+m\}$ we have $d_j \geqslant 0$. This implies that $x + \lambda \cdot d$ is feasible for any $\lambda \geqslant 0$. If we have $c^T d > 0$, then

  $$c^T(x + \lambda d) = c^T x + \lambda \cdot \underbrace{c^T d}_{>0},$$

  and we obtain a better solution for any $\lambda > 0$ (in fact, by letting $\lambda \to \infty$ we even conclude that there is no optimal solution).

  This leaves the case $c^T d = 0$. However, in this case we can just replace d by $-d$ still satisfying $c^T(-d) \geqslant 0$, but also having a direction which has at least one negative coordinate (since $d \neq (0, \ldots, 0)$). Hence we can apply Case 1.

In conclusion, starting with an arbitrary optimal solution x which is not a vertex, we can either replace it by another optimal solution which has one more zero coordinate (Case 1) or we obtain a contradiction to the optimality of x (Case 2). Since any feasible solution has $m + n$ coordinates, it follows that after at most $m + n$ iterations, we either reach an optimal solution that is a vertex, or, we reach an optimal solution and direction where Case 2 applies, in which case we obtain a contradiction. $\square$