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With `apply`-style proofs we are reasoning *backwards*

Decomposing complex goals into simpler goals

Using a backward reading of rules and theorems

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Note there are *two* different kinds of implication at play here!

The implication in the logic we are reasoning about: $\phi \to \psi$

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Isabelle uses a logic rather than informal English for this purpose

The "fat arrow" $\Longrightarrow$ replaces the English "if-then" in Isabelle

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The meta-level universal quantifier ($\bigwedge$) replaces the English "for-every" in Isabelle

Thus the Natural Deduction rule above would be rendered as

$$\bigwedge \Gamma\ \phi\ \psi. \quad \Gamma, \phi \vdash \psi \Longrightarrow \Gamma \vdash \phi \to \psi$$

when embedded in HOL