L11: Algebraic Path Problems with applications to Internet Routing Lectures 7 and 8

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Recall our basic iterative algorithm

$$\begin{aligned} \mathbf{A}^{\langle 0 \rangle} &= \mathbf{I} \\ \mathbf{A}^{\langle k+1 \rangle} &= \mathbf{A}\mathbf{A}^{\langle k \rangle} \oplus \mathbf{I} \end{aligned}$$

A closer look ...

$$\begin{aligned} \mathbf{A}^{\langle k+1 \rangle}(i,j) &= \mathbf{I}(i,j) \oplus \bigoplus_{u} \mathbf{A}(i,u)\mathbf{A}^{\langle k \rangle}(u,j) \\ &= \mathbf{I}(i,j) \oplus \bigoplus_{(i,u)\in E} \mathbf{A}(i,u)\mathbf{A}^{\langle k \rangle}(u,j) \end{aligned}$$

This is the basis of distributed Bellman-Ford algorithms (as in RIP and BGP) — a node i computes routes to a destination j by applying its link weights to the routes learned from its immediate neighbors. It then makes these routes available to its neighbors and the process continues...

What if we start iteration in an arbitrary state **M**?

In a distributed environment the topology (captured here by **A**) can change and the state of the computation can start in an arbitrary state (with respect to a new **A**).

$$\begin{aligned} \mathbf{A}_{\mathbf{M}}^{\langle 0 \rangle} &= \mathbf{M} \\ \mathbf{A}_{\mathbf{M}}^{\langle k+1 \rangle} &= \mathbf{A}\mathbf{A}_{\mathbf{M}}^{\langle k \rangle} \oplus \mathbf{I} \end{aligned}$$

Theorem

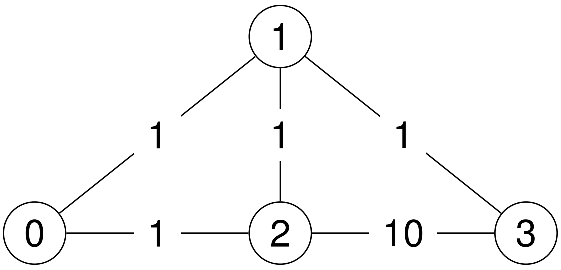
For $1 \leqslant k$,

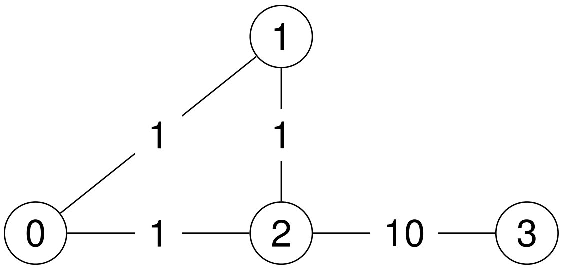
$$\mathbf{A}_{\mathbf{M}}^{\langle k \rangle} = \mathbf{A}^{k}\mathbf{M} \oplus \mathbf{A}^{(k-1)}$$

If **A** is q-stable and $q < k$, then

$$\mathbf{A}_{\mathbf{M}}^{\langle k \rangle} = \mathbf{A}^{k}\mathbf{M} \oplus \mathbf{A}^{*}$$

RIP-like example (see RFC 1058)





Adjacency matrix $\mathbf{A}_1$

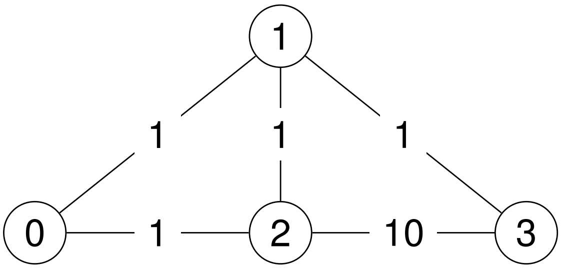
$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} \infty & 1 & 1 & \infty \\ 1 & \infty & 1 & 1 \\ 1 & 1 & \infty & 10 \\ \infty & 1 & 10 & \infty \end{array}\right] \end{array}$$

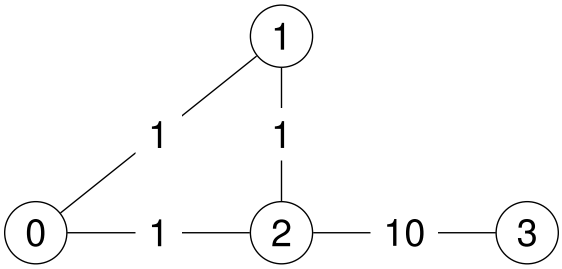
Adjacency matrix $\mathbf{A}_2$

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} \infty & 1 & 1 & \infty \\ 1 & \infty & 1 & \infty \\ 1 & 1 & \infty & 10 \\ \infty & \infty & 10 & \infty \end{array}\right] \end{array}$$

Using $\mathrm{AddZero}(\infty,\ (\mathbb{N},\ \min,\ +))$ but ignoring inl and inr

RIP-like example — counting to convergence (2)





The solution $\mathbf{A}_1^*$

$$
\begin{array}{c c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{array}\right]
\end{array}
$$

The solution $\mathbf{A}_2^*$

$$
\begin{array}{c c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 11 \\ 1 & 0 & 1 & 11 \\ 1 & 1 & 0 & 10 \\ 11 & 11 & 10 & 0 \end{array}\right]
\end{array}
$$

RIP-like example — counting to convergence (3)

The scenario: we arrived at $\mathbf{A}_1^*$, but then links $\{(1,3),\ (3,1)\}$ fail. So we start iterating using the new matrix $\mathbf{A}_2$.

Let $\mathbf{B}_K$ represent $\mathbf{A}_{2\mathbf{M}}^{\langle k \rangle}$, where $\mathbf{M} = \mathbf{A}_1^*$.

RIP-like example — counting to convergence (4)

$$\mathbf{B}_0 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{array}\right] \end{array} \qquad \mathbf{B}_3 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 0 & 4 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array}$$

$$\mathbf{B}_1 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 2 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array} \qquad \mathbf{B}_4 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 5 \\ 1 & 0 & 1 & 5 \\ 1 & 1 & 0 & 5 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array}$$

$$\mathbf{B}_2 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 3 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array} \qquad \mathbf{B}_5 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 6 \\ 1 & 0 & 1 & 6 \\ 1 & 1 & 0 & 6 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array}$$

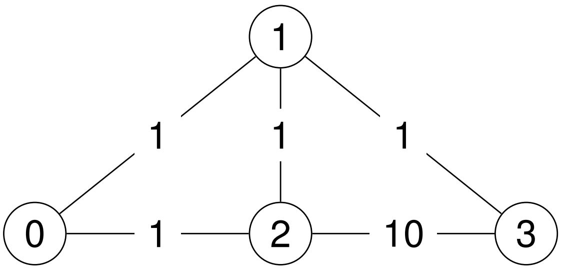
RIP-like example — counting to convergence (5)

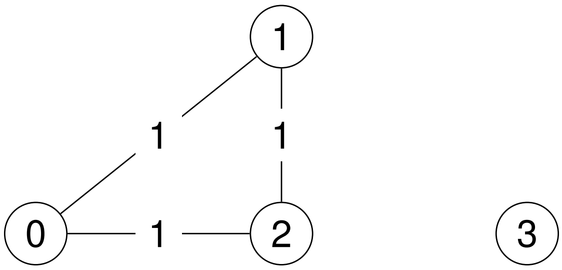
$$\mathbf{B}_6 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 7 \\ 1 & 0 & 1 & 7 \\ 1 & 1 & 0 & 7 \\ 2 & 1 & 2 & 0 \end{array}\right] \end{array} \qquad \mathbf{B}_9 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 10 \\ 1 & 0 & 1 & 10 \\ 1 & 1 & 0 & 10 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array}$$

$$\mathbf{B}_7 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 8 \\ 1 & 0 & 1 & 8 \\ 1 & 1 & 0 & 8 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array} \qquad \mathbf{B}_{10} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 11 \\ 1 & 0 & 1 & 11 \\ 1 & 1 & 0 & 10 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array}$$

$$\mathbf{B}_8 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 9 \\ 1 & 0 & 1 & 9 \\ 1 & 1 & 0 & 9 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array}$$

RIP-like example — counting to infinity (1)





The solution $\mathbf{A}_1^*$

$$\begin{array}{cc} & \begin{array}{cccc}0&1&2&3\end{array}\\ \begin{array}{c}0\\1\\2\\3\end{array} & \begin{bmatrix}0&1&1&2\\1&0&1&1\\1&1&0&2\\2&1&2&0\end{bmatrix}\end{array}$$

The solution $\mathbf{A}_3^*$

$$\begin{array}{cc} & \begin{array}{cccc}0&1&2&3\end{array}\\ \begin{array}{c}0\\1\\2\\3\end{array} & \begin{bmatrix}0&1&1&\infty\\1&0&1&\infty\\1&1&0&\infty\\\infty&\infty&\infty&0\end{bmatrix}\end{array}$$

Now let $\mathbf{B}_K$ represent $\mathbf{A_3}_{\mathbf{M}}^{\langle k\rangle}$, where $\mathbf{M} = \mathbf{A}_1^*$.

RIP-like example — counting to infinity (2)

$$\mathbf{B}_0 = \begin{array}{cc} & \begin{array}{cccc}0&1&2&3\end{array}\\ \begin{array}{c}0\\1\\2\\3\end{array} & \begin{bmatrix}0&1&1&2\\1&0&1&1\\1&1&0&2\\2&1&2&0\end{bmatrix}\end{array}$$

$$\mathbf{B}_1 = \begin{array}{cc} & \begin{array}{cccc}0&1&2&3\end{array}\\ \begin{array}{c}0\\1\\2\\3\end{array} & \begin{bmatrix}0&1&1&2\\1&0&1&3\\1&1&0&2\\\infty&\infty&\infty&0\end{bmatrix}\end{array}$$

$$\mathbf{B}_2 = \begin{array}{cc} & \begin{array}{cccc}0&1&2&3\end{array}\\ \begin{array}{c}0\\1\\2\\3\end{array} & \begin{bmatrix}0&1&1&3\\1&0&1&3\\1&1&0&3\\\infty&\infty&\infty&0\end{bmatrix}\end{array}$$

$$\vdots \quad \vdots \quad \vdots$$

$$\mathbf{B}_{376} = \begin{array}{cc} & \begin{array}{cccc}0&1&2&3\end{array}\\ \begin{array}{c}0\\1\\2\\3\end{array} & \begin{bmatrix}0&1&1&377\\1&0&1&377\\1&1&0&377\\\infty&\infty&\infty&0\end{bmatrix}\end{array}$$

$$\vdots \quad \vdots \quad \vdots$$

$$\mathbf{B}_{998} = \begin{array}{cc} & \begin{array}{cccc}0&1&2&3\end{array}\\ \begin{array}{c}0\\1\\2\\3\end{array} & \begin{bmatrix}0&1&1&999\\1&0&1&999\\1&1&0&999\\\infty&\infty&\infty&0\end{bmatrix}\end{array}$$

$$\vdots \quad \vdots \quad \vdots$$

RIP-like example — What's going on?

Recall

$$\mathbf{A}_{\mathbf{M}}^{\langle k \rangle}(i,\ j) = \mathbf{A}^k\mathbf{M}(i,\ j) \oplus \mathbf{A}^*(i,\ j)$$

- $\mathbf{A}^*$(i, j) may be arrived at very quickly
- but $\mathbf{A}^k\mathbf{M}(i,\ j)$ may be better until a very large value of k is reached (counting to convergence)
- or it may always be better (counting to infinity).

Solutions?

- RIP: $\infty = 16$

Other solutions?

The Border Gateway Protocol (BGP)

BGP exchanges metrics **and** paths. It avoids counting to infinity by throwing away routes that have a loop in the path.

The plan ...

Starting from $(\mathbb{N},\ min,\ +)$ we will attempt to construct a semiring (using our lexicographic operators) that has elements of the form $(d,\ X)$, where d is a shortest-path metric and X is a set of paths. Then, by succsessive refinements, we will arrive at a BGP-like solution.

A useful construction: The Lifted Product

Lifted product semigroup

Assume $(S, \bullet)$ is a semigroup. Let $\mathrm{lift}(S, \bullet) \equiv (\mathrm{fin}(2^S), \hat{\bullet})$ where

$$X \hat{\bullet} Y = \{x \bullet y \mid x \in X, \ y \in Y\}.$$

$$\{1, 3, 17\} \hat{+} \{1, 3, 17\} = \{2, 4, 6, 18, 20, 34\}$$

Some rules (remember $2 \leqslant |S|$)

$$\begin{array}{rcl}
\mathbb{AS}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \\
\mathbb{ID}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{ID}(S, \bullet) \ \ (\hat{\alpha} = \{\alpha\}) \\
\mathbb{AN}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{TRUE} \ \ (\omega = \{\}) \\
\mathbb{CM}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{CM}(S, \bullet) \\
\mathbb{SL}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{IL}(S, \bullet) \vee \mathbb{IR}(S, \bullet) \vee (\mathbb{IP}(S, \bullet) \ \wedge \mid S \mid = 2) \\
\mathbb{IP}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{SL}((S, \bullet)) \\
\mathbb{IL}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{IR}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{FALSE}
\end{array}$$

Turn the Crank ...

$$\begin{array}{ll} & \mathbb{IP}(\text{lift}(\text{lift}(\{t,\ f\},\ \wedge)) \\ \Leftrightarrow & \mathbb{SL}(\text{lift}(\{t,\ f\},\ \wedge) \\ \Leftrightarrow & \mathbb{IL}(\{t,\ f\},\ \wedge) \vee \mathbb{IR}(\{t,\ f\},\ \wedge) \vee (\mathbb{IP}(\{t,\ f\},\ \wedge)\ \wedge \mid \{t,\ f\} \mid = 2) \\ \Leftrightarrow & \mathbb{FALSE} \vee \mathbb{FALSE} \vee (\mathbb{TRUE} \wedge \mathbb{TRUE}) \\ \Leftrightarrow & \mathbb{TRUE} \end{array}$$

Note

This kind of calculation become more interesting as we introduce more complex constructors and consider more complex properties ...

Let's use lift to construct bi-semigroups ...

Union with lifted product

Assume $(S, \bullet)$ is a semigroup. Let

$$\text{union\_lift}(S,\ \bullet) \equiv (\mathcal{P}_{\text{fin}}(S),\ \cup,\ \hat{\bullet})$$

where

$$X \hat{\bullet} Y = \{x \bullet y \mid x \in X,\ y \in Y\},$$

and $X, Y \in \mathcal{P}_{\text{fin}}(S)$, the set of finite subsets of S.

- $\{\}$ is the $\overline{0}$
- if $\bullet$ has idenity α, then $\{\alpha\}$ is the $\overline{1}$
- a non-empty sequence will be written as $[s_0,\ s_1,\ \cdots,\ s_k]$
- when does it have an annihilator for $\cup$?
- we really need some inference rules here ...

$\text{paths}(E)$ over graph $G = (V,\ E)$

$$\text{paths}(E) \equiv \text{union\_lift}(E^*,\ \cdot)$$

where $\cdot$ is sequence concatenation.

spwp

For our graph $G = (V,\ E)$, we will build a "shortest paths with paths" semiring

$$\text{spwp} \equiv \text{AddZero}(\overline{0},\ (\mathbb{N},\ \min,\ +)\ \vec{\times}\ \text{paths}(E))$$

Given an arc $(i,\ j)$ we will give it a weight of the form

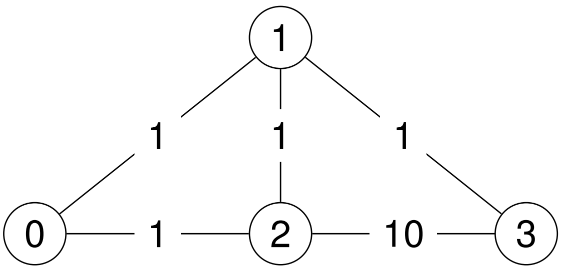
$$A(i,\ j) = \text{inl}(n,\ \{[(i,\ j)]\})$$

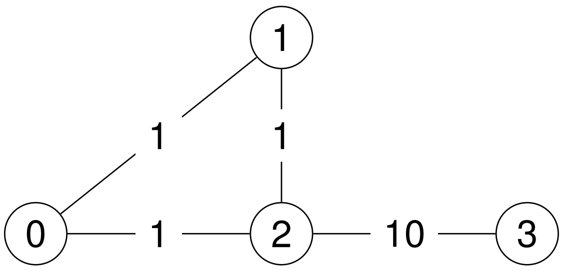
for some $n \in \mathbb{N}$. However, for ease of reading we will write

$$A(i,\ j) = (n,\ \{[(i,\ j)]\})$$

and so on.

Recall : RIP-like counting to convergence





The solution $\mathbf{A}_1^*$

$$\begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

The solution $\mathbf{A}_2^*$

$$\begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 11 \\ 1 & 0 & 1 & 11 \\ 1 & 1 & 0 & 10 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

Imagine that we have augmented this example with paths.

$\mathbf{B}_0$

Let $\mathbf{B}_0 = \mathbf{A}_1^*$ be the solution for the (path augmented) matrix $\mathbf{A}_1$:

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \left[\begin{array}{cccc} (0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (2, \{[(0,1),(1,3)]\}) \\ (1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & (1, \{[(1,3)]\}) \\ (1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & (2, \{[(2,1),(1,3)]\}) \\ (2, \{[(3,1),(1,0)]\}) & (1, \{[(3,1)]\}) & (2, \{[(3,1),(1,2)]\}) & (0, \{\epsilon\}) \end{array}\right] \end{array}$$

$\mathbf{B}_1$

Now, arcs $(1,\ 3)$ and $(3,\ 1)$ vanish! With the new (path-augmented) matrix $\mathbf{A}_2$, we start the iteration in the "old state" $\mathbf{B}_0$. After one iteration we have

$$\begin{array}{lcll} \mathbf{B}_1(0,\ 3) & = & (2, \{[(0,1),(1,3)]\}) & \\ \mathbf{B}_1(1,\ 3) & = & (3, \{[(1,0),(0,1),(1,3)],[(1,2),(2,1),(1,3)]\}) & \\ \mathbf{B}_1(2,\ 3) & = & (2, \{[(2,1),(1,3)]\}) & \\ \mathbf{B}_1(3,\ 0) & = & (11, \{[(3,2),(2,0)]\}) & \text{(stable!)} \\ \mathbf{B}_1(3,\ 1) & = & (11, \{[(3,2),(2,1)]\}) & \text{(stable!)} \\ \mathbf{B}_1(3,\ 2) & = & (10, \{[(3,2)]\}) & \text{(stable!)} \end{array}$$

$\mathbf{B}_2$

After another iteration we have

$$
\begin{aligned}
\mathbf{B}_2(0,\ 3) &= (3, \{[(0,2),(2,1),(1,3)]\}) \\
\mathbf{B}_2(1,\ 3) &= (3, \{[(1,0),(0,1),(1,3)], [(1,2),(2,1),(1,3)]\}) \\
\mathbf{B}_2(2,\ 3) &= (3, \{[(2,0),(0,1),(1,3)]\})
\end{aligned}
$$

$\mathbf{B}_3$

After another iteration we have

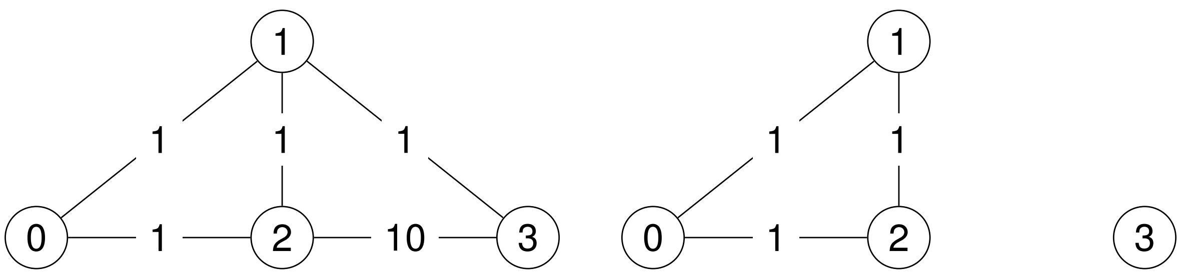
$$
\begin{aligned}
\mathbf{B}_3(0,\ 3) &= (4, \{[(0,1),(1,0),(0,1),(1,3)], \\
&\qquad [(0,1),(1,2),(2,1),(1,3)], \\
&\qquad [(0,2),(2,0),(0,1),(1,3)]\}) \\
\mathbf{B}_3(1,\ 3) &= (4, \{[(1,0),(0,2),(2,1),(1,3)], \\
&\qquad [(1,2),(2,0),(0,1),(1,3)]\}) \\
\mathbf{B}_3(2,\ 3) &= (4, \{[(2,0),(0,2),(2,1),(1,3)], \\
&\qquad [(2,1),(1,0),(0,1),(1,3)], \\
&\qquad [(2,1),(1,2),(2,1),(1,3)]\})
\end{aligned}
$$

$\mathbf{B}_{11}$

Finally, the problem is resolved at the 11-th iteration:

$$\begin{array}{cc}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc}
(0, \{\epsilon\}) & (1, \{[(0, 1)]\}) & (1, \{[(0, 2)]\}) & (11, \{[(0, 2), (\\
(1, \{[(1, 0)]\}) & (0, \{\epsilon\}) & (1, \{[(1, 2)]\}) & (11, \{[(1, 2), (\\
(1, \{[(2, 0)]\}) & (1, \{[(2, 1)]\}) & (0, \{\epsilon\}) & (10, \{[(2, 3 \\
(11, \{[(3, 2), (2, 0)]\}) & (11, \{[(3, 2), (2, 1)]\}) & (10, \{[(3, 2)]\}) & (0, \{\epsilon\})
\end{array} \right.
\end{array}$$

Recall RIP-like counting to infinity



It should be clear that we have not yet reached our goal. This example will lead to sets of ever longer and longer paths and never terminate.
How do we fix this?

Reductions

If $(S, \oplus, \otimes)$ is a semiring and r is a function from S to S, then r is a reduction if for all a and b in S

1. $r(a) = r(r(a))$
2. $r(a \oplus b) = r(r(a) \oplus b) = r(a \oplus r(b))$
3. $r(a \otimes b) = r(r(a) \otimes b) = r(a \otimes r(b))$

Note that if either operation has an identity, then the first axioms is not needed. For example,

$$r(a) = r(a \oplus \overline{0}) = r(r(a) \oplus \overline{0}) = r(r(a))$$

Reduce operation

If $(S, \oplus, \otimes)$ is semiring and r is a reduction, then let $\mathrm{red}_r(S) = (S_r, \oplus_r, \otimes_r)$ where

1. $S_r = \{s \in S \mid r(s) = s\}$
2. $x \oplus_r y = r(x \oplus y)$
3. $x \otimes_r y = r(x \otimes y)$

Is the result always semiring?

Elementary paths reduction

Recall $\text{paths}(E)$

$$\text{paths}(E) \equiv \text{union\_lift}(E^*, \cdot)$$

where $\cdot$ is sequence concatenation.

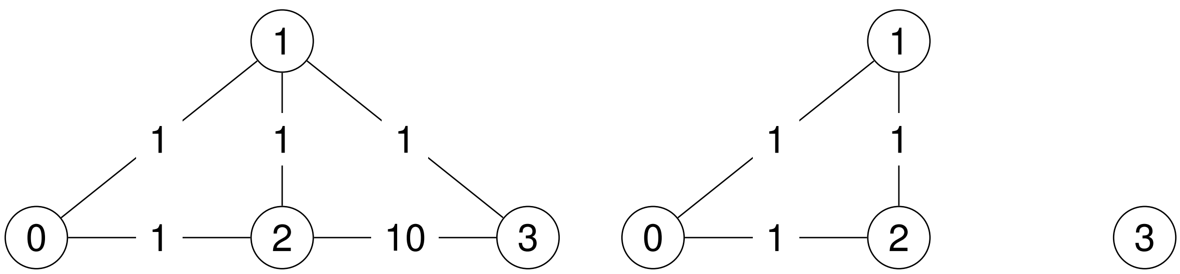
A path p is elementary if no node is repeated. Define the reduction

$$r(X) = \{p \in X \mid p \text{ is an elementary path}\}$$

Semiring of Elementary Paths

$$\text{epaths}(E) = \text{red}_r(\text{paths}(E))$$

Starting in an arbitrary state?



Starting in an arbitrary state? No!

using

$$\text{AddZero}(\infty,\ (\mathbb{N},\ \min,\ +)\ \vec{\times}\ \text{epaths}(E))$$

$$\begin{array}{c}
\vdots \quad \vdots \quad \vdots \\
\mathbf{B}_{998} \quad = \quad \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} (0, \{\epsilon\}) & (1, \{[(0, 1)]\}) & (1, \{[(0, 2)]\}) & (999, \{\}) \\ (1, \{[(1, 0)]\}) & (0, \{\epsilon\}) & (1, \{[(1, 2)]\}) & (999, \{\}) \\ (1, \{[(2, 0)]\}) & (1, \{[(2, 1)]\}) & (0, \{\epsilon\}) & (999, \{\}) \\ \infty & \infty & \infty & (0, \{\epsilon\}) \end{array} \right] \end{array} \\
\vdots \quad \vdots \quad \vdots
\end{array}$$

Solution: use another reduction!

$$\begin{array}{rcl}
r(\text{inr}(\infty)) & = & \text{inr}(\infty) \\
r(\text{inl})(s, W) & = & \begin{cases} \text{inr}(\infty) & \text{if } W = \{\} \\ \text{inl}(s, W) & \text{otherwise} \end{cases}
\end{array}$$

Now use this instead

$$\text{red}_r(\text{AddZero}(\infty,\ (\mathbb{N},\ \min,\ +)\ \vec{\times}\ \text{epaths}(E)))$$

Starting in an arbitrary state?

$\mathbf{B}_0$ and $\mathbf{B}_1$

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} (0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (2, \{[(0,1),(1,3)]\}) \\ (1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & (1, \{[(1,3)]\}) \\ (1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & (2, \{[(2,1),(1,3)]\}) \\ (2, \{[(3,1),(1,0)]\}) & (1, \{[(3,1)]\}) & (2, \{[(3,1),(1,2)]\}) & (0, \{\epsilon\}) \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} (0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (2, \{[(0,1),(1,3)]\}) \\ (1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & \infty \\ (1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & (2, \{[(2,1),(1,3)]\}) \\ \infty & \infty & \infty & (0, \{\epsilon\}) \end{bmatrix} \end{array}$$

Starting in an arbitrary state?

$\mathbf{B}_2$ and $\mathbf{B}_3$

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} (0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (3, \{[(0,2),(2,1),(1,3)]\}) \\ (1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & \infty \\ (1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & (3, \{[(2,0),(0,1),(1,3)]\}) \\ \infty & \infty & \infty & (0, \{\epsilon\}) \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} (0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & \infty \\ (1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & \infty \\ (1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & \infty \\ \infty & \infty & \infty & (0, \{\epsilon\}) \end{bmatrix} \end{array}$$

Homework 3 (due 6 November)

- Prove

$$\mathbb{SL}(\text{lift}(S, \bullet)) \quad \Leftrightarrow \quad \mathbb{IL}(S, \bullet) \vee \mathbb{IR}(S, \bullet) \vee (\mathbb{IP}(S, \bullet) \ \wedge \mid S \mid = 2)$$

- Characterise exactly when $\text{union\_lift}(S,\ \bullet)$ is a semiring.