$$\# U = n \Rightarrow \# \mathcal{P}(U) = 2^n$$

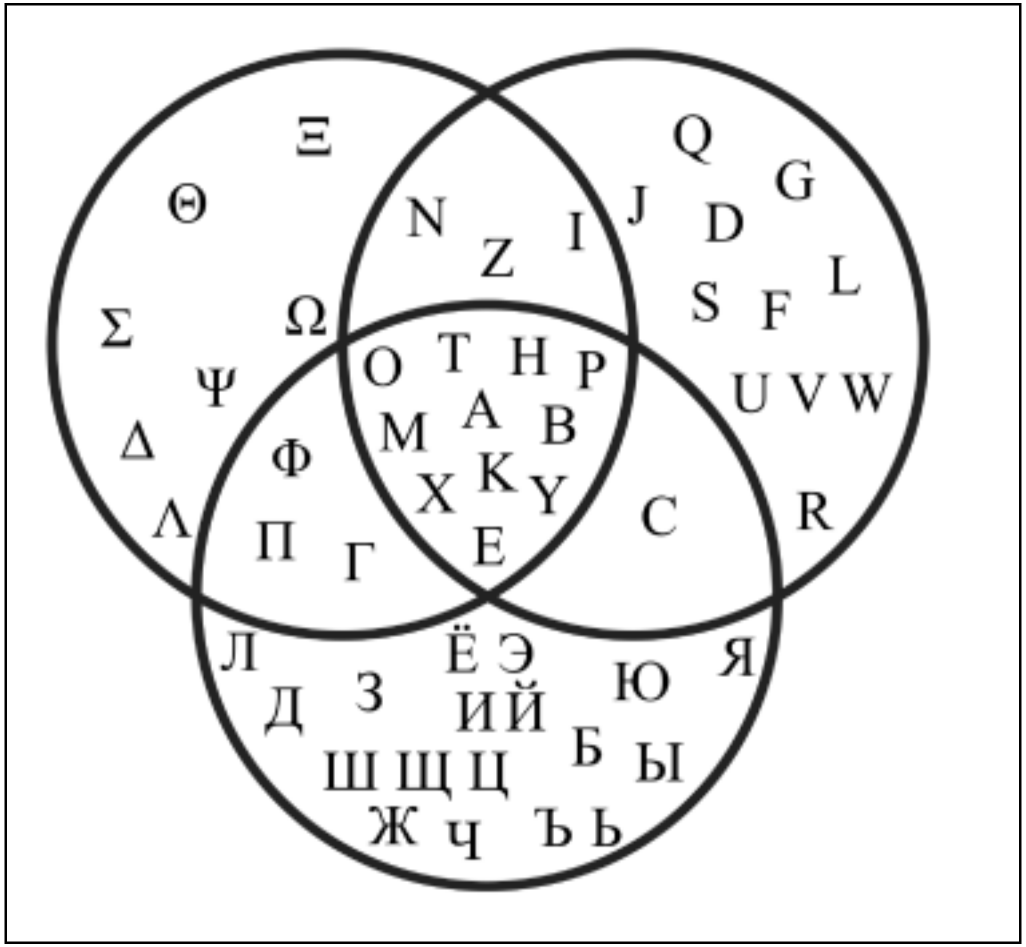
Powerset axiom

For any set, there is a set consisting of all its subsets.

$$\mathcal{P}(U) = \{ X \mid X \subseteq U \}$$

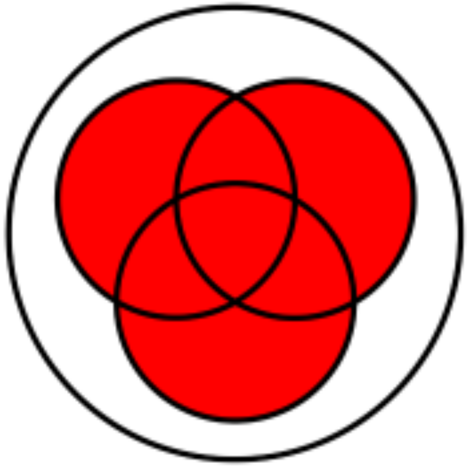
$$\forall X.\ X \in \mathcal{P}(U) \iff X \subseteq U \quad .$$

Venn diagrams[a]

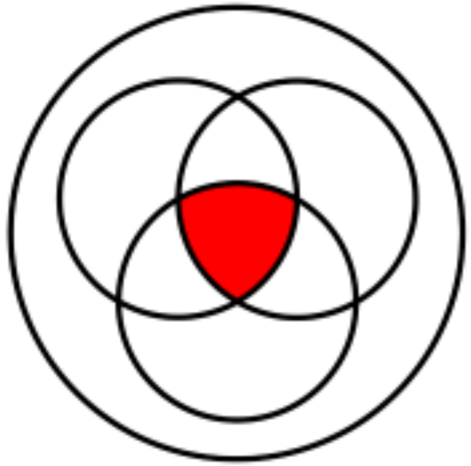


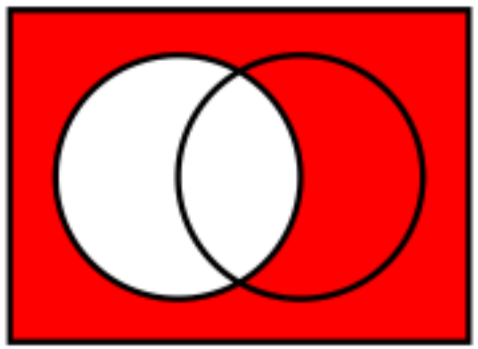
[a] From http://en.wikipedia.org/wiki/Intersection_(set_theory) .

Union



Intersection





Complement

N.B $\emptyset \in \mathcal{P}(\mathcal{U})$ $\mathcal{U} \in \mathcal{P}(\mathcal{U})$

The powerset Boolean algebra

$$\big(\quad \mathcal{P}(U) \,, \quad \emptyset \,, \quad U \,, \quad \underset{\underset{\vee}{\wr}}{\cup} \,, \quad \underset{\underset{\wedge}{\wr}}{\cap} \,, \quad \underset{\underset{\neg}{\wr}}{(\cdot)^c} \quad \big)$$

For all $A, B \in \mathcal{P}(U)$,

$$A \cup B \;=\; \{ x \in U \mid x \in A \,\vee\, x \in B \} \;\in \mathcal{P}(U)$$

$$A \cap B \;=\; \{ x \in U \mid x \in A \,\wedge\, x \in B \} \;\in \mathcal{P}(U)$$

$$A^c \;=\; \{ x \in U \mid \neg(x \in A) \} \;\in \mathcal{P}(U)$$

- The union operation $\cup$ and the intersection operation $\cap$ are associative, commutative, and idempotent.

$$(A \cup B) \cup C = A \cup (B \cup C) \ , \quad A \cup B = B \cup A \ , \quad A \cup A = A$$

$$(A \cap B) \cap C = A \cap (B \cap C) \ , \quad A \cap B = B \cap A \ , \quad A \cap A = A$$

- The union operation $\cup$ and the intersection operation $\cap$ are associative, commutative, and idempotent.

$$(A \cup B) \cup C = A \cup (B \cup C) \;, \quad A \cup B = B \cup A \;, \quad A \cup A = A$$

$$(A \cap B) \cap C = A \cap (B \cap C) \;, \quad A \cap B = B \cap A \;, \quad A \cap A = A$$

- The *empty set* $\emptyset$ is a neutral element for $\cup$ and the *universal set* $\mathcal{U}$ is a neutral element for $\cap$.

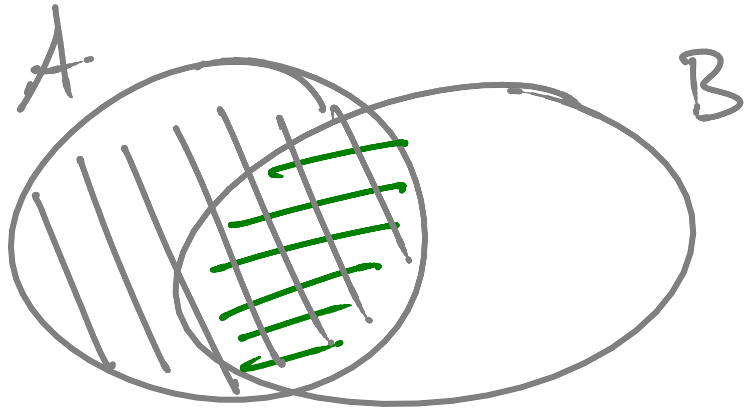
$$\emptyset \cup A \;=\; A \;=\; \mathcal{U} \cap A$$

- The empty set $\emptyset$ is an annihilator for $\cap$ and the universal set $\mathfrak{U}$ is an annihilator for $\cup$.

$$\emptyset \cap A = \emptyset$$

$$\mathfrak{U} \cup A = \mathfrak{U}$$

- The empty set $\emptyset$ is an annihilator for $\cap$ and the universal set $\mathcal{U}$ is an annihilator for $\cup$.



$$\emptyset \cap A = \emptyset$$

$$\mathcal{U} \cup A = \mathcal{U}$$

- With respect to each other, the union operation $\cup$ and the intersection operation $\cap$ are distributive and absorptive.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \ , \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cup (A \cap B) = A = A \cap (A \cup B)$$

$$A \cup (A \cap B) \stackrel{?}{=} A$$

We need show

($\subseteq$) $A \cup (A \cap B) \subseteq A$

For all x. $x \in A \cup (A \cap B) \Rightarrow x \in A$.

$(x \in A) \vee (x \in A \cap B)$

Let x be arbitrary. Assume $x \in A \cup (A \cap B)$

RTP: $x \in A$

Lemma: For all sets S, T, $S \subseteq S \cup T$

($\supseteq$) $A \subseteq A \cup (A \cap B)$

By Lemma.

Show $x \in A$ under assumption $(x \in A) \vee (x \in A \cap B)$

We use the assumption by cases:

(1) If $x \in A$ we are done

(2) If $x \in A \cap B \iff x \in A \wedge x \in B$

so $x \in A$ and we are done. $\boxtimes$

- The complement operation $(\cdot)^{\mathrm{c}}$ satisfies complementation laws.

$$A \cup A^{\mathrm{c}} = \mathcal{U} \ , \quad A \cap A^{\mathrm{c}} = \emptyset$$

TRANSITIVITY: $(P \subseteq Q \wedge Q \subseteq R) \Rightarrow (P \subseteq R)$

Proposition 85 *Let* U *be a set and let* $A, B \in \mathcal{P}(U)$.

1. $\forall X \in \mathcal{P}(U).\ A \cup B \subseteq X \iff (A \subseteq X \wedge B \subseteq X)$.
2. $\forall X \in \mathcal{P}(U).\ X \subseteq A \cap B \iff (X \subseteq A \wedge X \subseteq B)$.

PROOF:

Let $X \in \mathcal{P}(U) \iff X \subseteq U$.

Lemma: $A \subseteq A \cup B$, $B \subseteq A \cup B$

$(\Longrightarrow)$ Assume $A \cup B \subseteq X$

RTP: $A \subseteq X$

Know $A \subseteq A \cup B$

By assumption $A \cup B \subseteq X$

By transitivity of $\subseteq$ we are done.

RTP $B \subseteq X$

Analogous.

($\Leftarrow$) Assume $A \subseteq X \wedge B \subseteq X$

RTP: $A \cup B \subseteq X \iff (\forall x.\ x \in A \cup B \Rightarrow x \in X.)$

Let x be arbitrary. Assume $x \in A \cup B$

RTP $x \in X$.

$$x \in A \cup B \iff [x \in A \vee x \in B] \quad (*)$$

We use $(*)$ by cases:

1. $x \in A$

$\Longrightarrow x \in X$ because $A \subseteq X$ by assump.

2. $x \in B$

$\Longrightarrow x \in X$ because $B \subseteq X$ by assump.

Corollary 86 *Let* $\mathcal{U}$ *be a set and let* $A, B, C \in \mathcal{P}(\mathcal{U})$.

1. $$C = A \cup B$$

iff

$$[A \subseteq C \wedge B \subseteq C]$$

$$\wedge$$

$$[\forall X \in \mathcal{P}(\mathcal{U}).\ (A \subseteq X \ \wedge\ B \subseteq X) \implies C \subseteq X]$$

2. $$C = A \cap B$$

iff

$$[C \subseteq A \wedge C \subseteq B]$$

$$\wedge$$

$$[\forall X \in \mathcal{P}(\mathcal{U}).\ (X \subseteq A \ \wedge\ X \subseteq B) \implies X \subseteq C]$$

PROOF PRINCIPLE

TO SHOW A SET C IS A UNION/INTERSECTION OF TWO SETS A AND B.

Sets and logic

$\mathcal{P}(\mathbb{U})$	$\{\,\mathbf{false}\,,\ \mathbf{true}\,\}$
$\emptyset$	$\mathbf{false}$
$\mathbb{U}$	$\mathbf{true}$
$\cup$	$\vee$
$\cap$	$\wedge$
$(\cdot)^c$	$\neg(\cdot)$

? $\leftsquigarrow$ $\exists$

$\leftsquigarrow$ $\forall$

NB $x \in \{a, a\} \iff (x = a) \vee (x = a) \iff (x = a)$

Pairing axiom

For every a and b, there is a set with a and b as its only elements.

$$\{a, b\}$$

defined by

$$\forall x.\, x \in \{a, b\} \iff (x = a \vee x = b)$$

NB The set $\{a, a\}$ is abbreviated as $\{a\}$, and referred to as a *singleton*.

Examples:

- $\#\{\emptyset\} = 1$
- $\#\{\{\emptyset\}\} = 1$
- $\#\{\emptyset, \{\emptyset\}\} = 2$

Fundamental property

$$\langle a, b\rangle = \langle a', b'\rangle \Rightarrow (a = a' \wedge b = b')$$

Ordered pairing

For every pair a and b, the set

$$\{\{a\},\{a,b\}\}$$

is abbreviated as

$$\langle a, b\rangle$$

and referred to as an *ordered pair*.

Proposition 87 (Fundamental property of ordered pairing)
For all a, b, x, y,

$$\langle a, b\rangle = \langle x, y\rangle \iff (a = x \land b = y) \quad .$$

PROOF: Let a, b, x, y be arbitrary.

($\Rightarrow$) Assume $\langle a, b\rangle = \langle x, y\rangle$; that is

$$\{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\}$$

RTP: $a = x \land b = y$.

Case $a = b$: Then we have $\{\{a\}\} = \{\{x\}, \{x, y\}\}$

So $\{a\} = \{x\}$ and $\{a\} = \{x, y\}$

$\Downarrow$ $a = x$

$\Downarrow$ $b = a = x = y$

Case $a \neq b$: Assumption

$$\{\{a\},\{a,b\}\} = \{\{x\},\{x,y\}\}$$

RTP: $a = x \wedge b = y$

We have $\{a,b\} \in \{\{x\},\{x,y\}\}$

So $(\{a,b\} = \{x\} \vee \{a,b\} = \{x,y\})$

But $\{a,b\} \neq \{x\}$ so we have $\{a,b\} \overset{(*)}{=} \{x,y\}$

Also $x \neq y$ otherwise $2 = \#\{a,b\} = \#\{x,y\} = 1$

By $(*)$:

Case (1) $a = x \wedge b = y$ so we are done

Case (2) $a = y \wedge b = x$

finish the proof arguing that this cannot be the case!

$\boxtimes$