Conjunction

Conjunctive statements are of the form

$$\boxed{P \textbf{ and } Q}$$

or, in other words,

$$\boxed{\text{both } P \text{ and also } Q \text{ hold}}$$

or, in symbols,

$$\boxed{P \wedge Q} \qquad \text{or} \qquad \boxed{P \ \& \ Q}$$

The proof strategy for conjunction:

To prove a goal of the form

$$P \wedge Q$$

first prove P and subsequently prove Q (or vice versa).

Proof pattern:

In order to prove

$$P \wedge Q$$

1. Write: Firstly, we prove P. and provide a proof of P.
2. Write: Secondly, we prove Q. and provide a proof of Q.

Scratch work:

Before using the strategy

Assumptions	Goal
	$P \wedge Q$
$\vdots$	

After using the strategy

Assumptions	Goal		Assumptions	Goal
	P			Q
$\vdots$			$\vdots$	

When they are assumptions

Assumption $\quad\quad\quad\quad$ Goal

$$\begin{array}{c} \vdots \\ P \wedge Q \\ \vdots \\ P \,,\ Q \end{array} \qquad\qquad Q$$

The use of conjunctions:

To use an assumption of the form $P \wedge Q$,
treat it as two separate assumptions: P and Q.

Theorem 20 *For every integer* n, *we have that* $6 \mid n$ *iff* $2 \mid n$ *and* $3 \mid n$.

PROOF: $\forall$ int n. $6 \mid n \iff (2 \mid n \wedge 3 \mid n)$

Let n be an arbitrary integer.

RTP: $6 \mid n \iff (2 \mid n \wedge 3 \mid n)$

$(\Rightarrow)$ Assume $6 \mid n \iff n = 6 \cdot k$ for an int. k

RTP: $2 \mid n \wedge 3 \mid n$

RTP: $2 \mid n$

$\Updownarrow$

$n = 2i$ for an int. i

RTP: $3 \mid n$

$\Updownarrow$

$n = 3j$ for a j int.

By assumption
$n = 6k = 2(3k)$
and since $3k$ is an integer as so is k
we are done.

$n = 3(2k)$
and $2k$ int.
gives $n = 3j$ for an int. j.

$(\Leftarrow)$ $(2|n \wedge 3|n) \Longrightarrow 6|n$

Assume $(2|n \wedge 3|n)$

We have that $2|n \Leftrightarrow n = 2i$ for an int i

and that $3|n \Leftrightarrow n = 3j$ for an int j

RTP: $n = 6.k$ for an int k.

$n = 2i$ $\qquad$ $n = 3j$

$n^2 = 2i \cdot 3j = 6ij$

$n \equiv 0 \pmod 2$

$n \equiv 0 \pmod 3$

$n = 2i$ $\qquad$ $n = 3j$

$3n = 6i$ $\qquad$ $2n = 6j$

$n = 3n - 2n = 6i - 6j = 6(i - j)$

found k

Witness for divisibility by 6

What about

$$\left(2 \mid n \wedge 3 \mid n \wedge 5 \mid n\right) \overset{?}{\iff} 30 \mid n \quad ?$$

$$\left(a_1 \mid n \wedge a_2 \mid n \wedge \cdots \wedge a_\ell \mid n\right) \overset{?}{\iff} (a_1 \cdot a_2 \cdot \cdots \cdot a_\ell) \mid n \quad ?$$

$d \mid n \iff (\exists \text{int } k.\ n = k \cdot d)$

Existential quantification

Existential statements are of the form

> **there exists** an individual x in the universe of discourse for which the property $P(x)$ holds

or, in other words,

> **for some** individual x in the universe of discourse, the property $P(x)$ holds

or, in symbols,

$\boxed{\exists x.\, P(x)} \iff \boxed{\exists y.\, P(y)}$

Cf. fn $x \Rightarrow x+1$

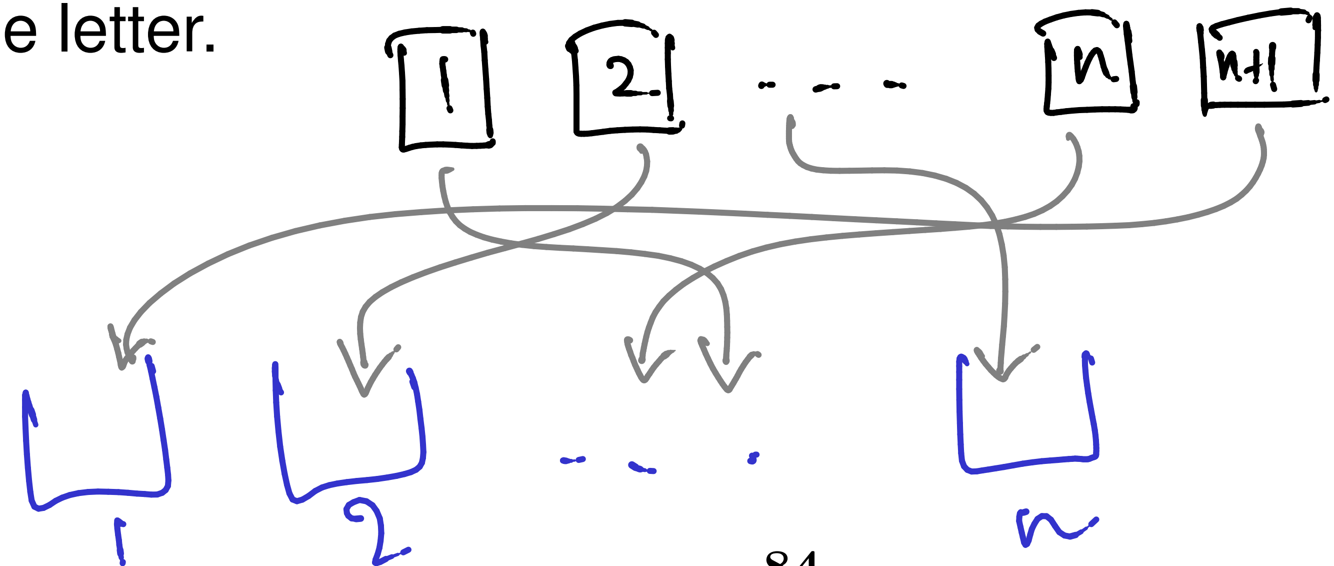
fn $v \Rightarrow v+1$

$\forall$ pos. int n

n+1 letters are put in n pigeonholes, say $1, 2, \ldots, n$ $\Longrightarrow$ $\exists\, i.\ 1 \le i \le n$ s.t. pigeonhole i has more than one letter.

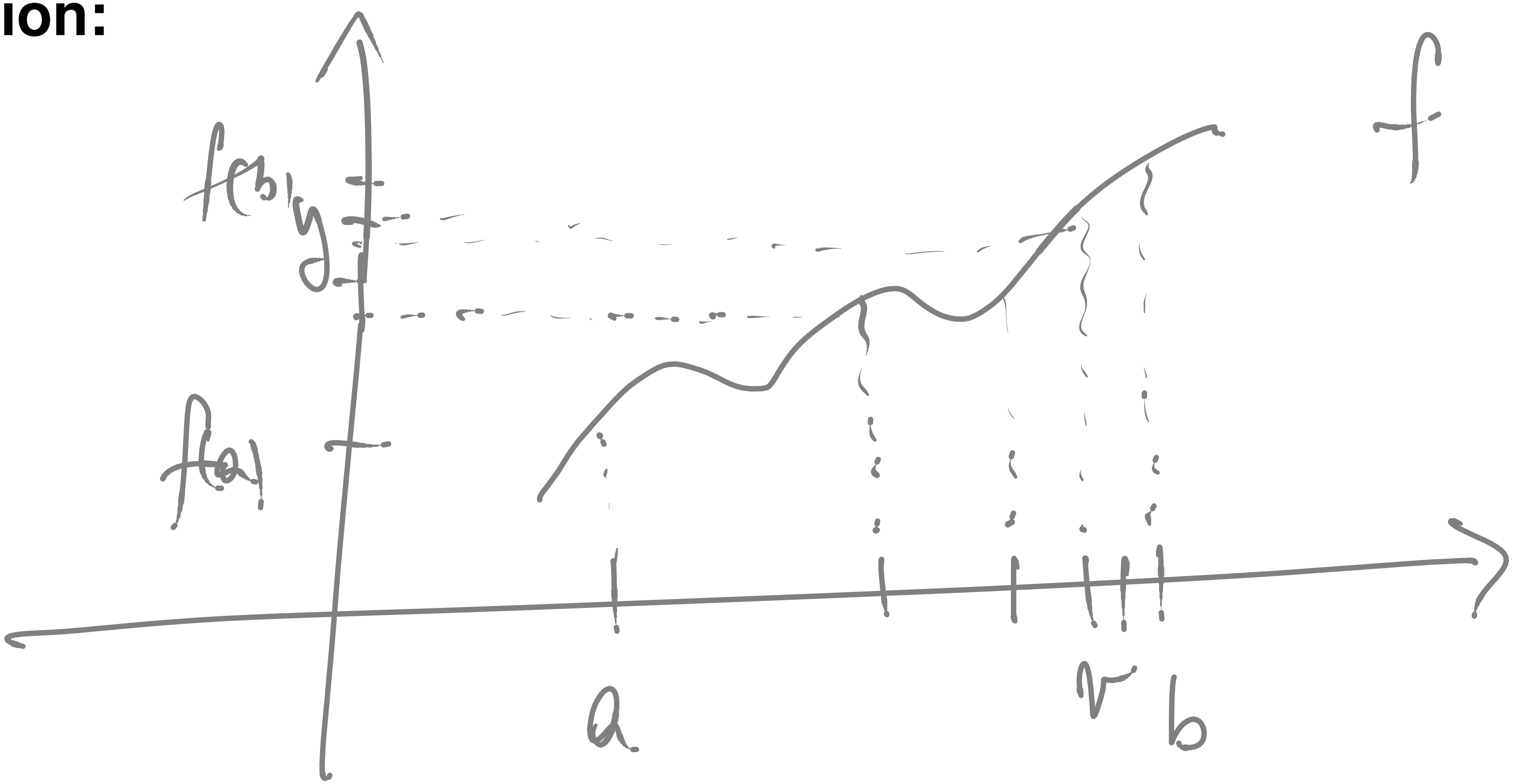
Example: The Pigeonhole Principle.

Let n be a positive integer. If $n + 1$ letters are put in n pigeonholes then there will be a pigeonhole with more than one letter.



Theorem 21 (Intermediate value theorem) *Let f be a real-valued continuous function on an interval $[a, b]$. For every y in between $f(a)$ and $f(b)$, there exists v in between a and b such that $f(v) = y$.*

Intuition:



The main proof strategy for existential statements:

To prove a goal of the form

$$\exists x.\ P(x)$$

find a ***witness*** for the existential statement; that is, a value of x, say w, for which you think $P(x)$ will be true, and show that indeed $P(w)$, i.e. the predicate $P(x)$ instantiated with the value w, holds.

Definition

Proof pattern:

In order to prove

$$\exists x.\, P(x)$$

1. Write: $\text{Let } w = \ldots$ (the witness you decided on).
2. Provide a proof of $P(w)$.

Scratch work:

Before using the strategy

Assumptions	Goal
	$\exists x.\, P(x)$
$\vdots$	

After using the strategy

Assumptions	Goals
	$P(w)$
$\vdots$	

definition

$w = \ldots$ (the witness you decided on)

Proposition 22 *For every positive integer* k*, there exist natural numbers* i *and* j *such that* $4 \cdot k = i^2 - j^2$*.*

PROOF: $\forall$ pos. int k.

$\exists$ nat i. $\exists$ nat j. $4k = i^2 - j^2$.

Let k be an arbitrary pos. int.

RTP: $\exists$ nat i. $\exists$ nat j. $4k = i^2 - j^2$

Consider witness $w = \ldots$

Consider witness $v = \ldots$

We check $4k = w^2 - v^2$.

k	$4k$	$i^2 - j^2$	i	j
1	4	$2^2 - 0^2$	2	0
2	8	$3^2 - 1^2$		
3	12	$\vdots$		
4	16		$k+1$	$k-1$

idea/intuition

WRITE IT NICELY!

Assumptions

Goals

$\exists x. P(x)$

$P(x_0)$

The use of existential statements:

To use an assumption of the form $\exists x. \, P(x)$, introduce a new variable x_0 into the proof to stand for some individual for which the property $P(x)$ holds. This means that you can now assume $P(x_0)$ true.

Theorem 24 *For all integers* l, m, n, *if* $l \mid m$ *and* $m \mid n$ *then* $l \mid n$.

PROOF: $\forall$ int. l, m, n.

$(l \mid m \wedge m \mid n) \Rightarrow l \mid n$.

Assume l, m, n are arbitrary integers.

Assume $l \mid m \iff (\exists \text{ int. } i.\ m = i \cdot l)$

and $m \mid n \iff (\exists \text{ int. } j.\ n = j \cdot m)$

RTP: $\exists$ int. k. $n = k \cdot l$

Let $w = \ldots$

Check $n = w \cdot l$

By assuption we have

$\exists i \in \mathbb{Z}.\ m = i \cdot \ell$

So let i_0 be such that $\boxed{m = i_0 \cdot \ell}$

By assumption we have

$\exists j \in \mathbb{Z}.\ n = j \cdot m$

So let j_0 be such that $\boxed{n = j_0 \cdot m}$

Therefore $n = j_0 \cdot m = (j_0 \cdot i_0) \cdot \ell$

For witness $w = j_0 \cdot i_0$ we have

$n \cdot w = \ell$

$\boxtimes$