Modelling data Types.

products $*$

functions $\to$

$$\frac{a : \alpha \quad b : \beta}{(a, b) : \alpha * \beta}$$

$$\frac{x : \alpha \vdash t : \beta}{\vdash \mathrm{fn}\ x \Rightarrow t : (\alpha \to \beta)}$$

Topic 3

Constructions on Domains

datatypes $<$ enumerated (non-recursive) / recursive.

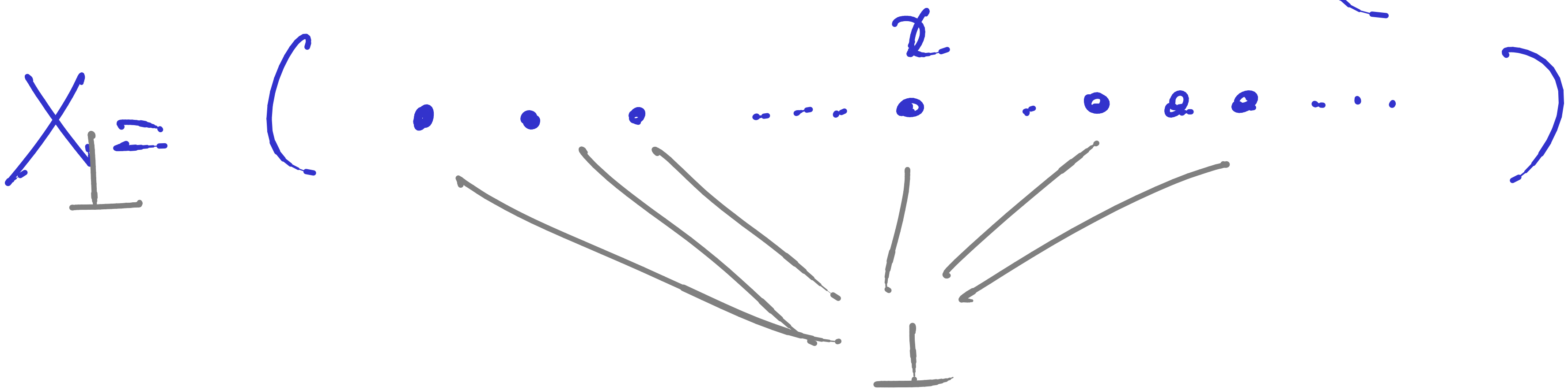
Discrete cpo's and flat domains

For any set X, the relation of equality

$$x \sqsubseteq x' \overset{\text{def}}{\Leftrightarrow} x = x' \qquad (x, x' \in X)$$

makes $(X, \sqsubseteq)$ into a cpo, called the discrete cpo with underlying set X.

$(x \in X)$



Discrete cpo's and flat domains

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Let $X_\perp \overset{\text{def}}{=} X \cup \{\perp\}$, where $\perp$ is some element not in X. Then

$$d \sqsubseteq d' \overset{\text{def}}{\Leftrightarrow} (d = d') \vee (d = \perp) \qquad (d, d' \in X_\perp)$$

makes $(X_\perp, \sqsubseteq)$ into a domain (with least element $\perp$), called the flat domain determined by X.

If $\underline{D}$ models a type α
and $\underline{E}$ models a type β
What is the domain (constructed from $\underline{D}$ and $\underline{E}$) that models $\alpha * \beta$?

$\underline{D} = (D, \sqsubseteq_D)$ $\qquad$ $\underline{E} = (E, \sqsubseteq_E)$

$$\underline{D} \times \underline{E} = (\underset{\overset{||\,\mathrm{def}}{\{(d,e) \mid d \in D, e \in E\}}}{D \times E}, \sqsubseteq_{\underline{D} \times \underline{E}})$$

Given (d_1, e_1) and (d_2, e_2) in $D \times E$

when

$$(d_1, e_1) \sqsubseteq_{D \times E} (d_2, e_2) \qquad ?$$

$\overset{\text{def}}{\Leftrightarrow}$

$$d_1 \sqsubseteq_D d_2 \text{ and } e_1 \sqsubseteq_E e_2 .$$

Check it: is a partial order. ✓

has a least element $(\bot_D, \bot_E)$ ✓

lubs of ω-chains.

Take an ω-chain in $\underline{D} \times \underline{E}$:

$$(d_0, e_0) \sqsubseteq (d_1, e_1) \sqsubseteq \cdots \sqsubseteq (d_n, e_n) \sqsubseteq \cdots \quad (n \in \mathbb{N})$$

Then

$$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots \quad \text{in } \underline{D}$$

$$e_0 \sqsubseteq e_1 \sqsubseteq \cdots \sqsubseteq e_n \sqsubseteq \cdots \quad \text{in } \underline{E}$$

We have

$$\bigsqcup_n d_n \quad \text{and} \quad \bigsqcup_n e_n$$

in $\underline{D}$ and in $\underline{E}$

and so

$$\left(\bigsqcup_n d_n, \bigsqcup_n e_n\right) \in D \times E$$

$(\bigsqcup_n d_n, \bigsqcup_n e_n)$ is a lub of (d_n, e_n) in $D \times E$.

① It is an upper bound.

Because

$$(d_i, e_i) \sqsubseteq (\bigsqcup_n d_n, \bigsqcup_n e_n)$$

as $\quad d_i \sqsubseteq \bigsqcup_n d_n$ and $e_i \sqsubseteq \bigsqcup_n e_n$

$\forall i \in \mathbb{N}$.

② It is least

Because let $(u, v) \in D \times E$ be above every (d_i, e_i). Then, u is above every d_i in D and v is above every e_i in E. So

Sol: $\bigsqcup_n d_n \sqsubseteq u$ in D $\qquad \bigsqcup_n e_n \sqsubseteq v$ in E

and thus

$$\left(\bigsqcup_n d_n, \bigsqcup_n e_n\right) \sqsubseteq (u, v).$$

Binary product of cpo's and domains

The product of two cpo's $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \mathbin{\&} d_2 \in D_2\}$$

and partial order $\sqsubseteq$ defined by

$$(d_1, d_2) \sqsubseteq (d_1', d_2') \overset{\text{def}}{\Leftrightarrow} d_1 \sqsubseteq_1 d_1' \mathbin{\&} d_2 \sqsubseteq_2 d_2' \ .$$

$$\frac{(x_1, x_2) \sqsubseteq (y_1, y_2)}{x_1 \sqsubseteq_1 y_1 \qquad x_2 \sqsubseteq_2 y_2}$$

Lubs of chains are calculated componentwise:

$$\bigsqcup_{n\geq 0}(d_{1,n}, d_{2,n}) = (\bigsqcup_{i\geq 0} d_{1,i}, \bigsqcup_{j\geq 0} d_{2,j}) \ .$$

If $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ are domains so is $(D_1 \times D_2, \sqsubseteq)$ and $\perp_{D_1 \times D_2} = (\perp_{D_1}, \perp_{D_2})$.

Continuous functions of two arguments

Proposition. *Let* D, E, F *be cpo's. A function* $f : (D \times E) \to F$ *is monotone if and only if it is monotone in each argument separately:*

$$\forall d, d' \in D, e \in E.\, d \sqsubseteq d' \;\Rightarrow\; f(d,e) \sqsubseteq f(d',e)$$

$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \;\Rightarrow\; f(d,e) \sqsubseteq f(d,e').$$

Moreover, it is continuous if and only if it preserves lubs of chains in each argument separately:

$$f(\bigsqcup_{m \geq 0} d_m\,,\, e) = \bigsqcup_{m \geq 0} f(d_m, e)$$

$$f(d\,,\, \bigsqcup_{n \geq 0} e_n) = \bigsqcup_{n \geq 0} f(d, e_n).$$

Suppose $f : D \times E \to F$ is continuous:

$$(d,e) \sqsubseteq (d',e') \Rightarrow f(d,e) \sqsubseteq f(d',e')$$

$$f\left(\bigsqcup_n (d_n, e_n)\right) = \bigsqcup_n f(d_n, e_n).$$

$\Downarrow$

f is continuous in each argument:

$\forall e \in E.\ f(-,e) : D \to F : x \mapsto f(x,e)$

$\forall d \in D.\ f(d,-) : E \to F : y \mapsto f(d,y)$

continuous.

$f(-,e)$ monotone

$$d \sqsubseteq d' \overset{?}{\Rightarrow} f(d,e) \sqsubseteq f(d',e)$$

$$d \sqsubseteq d',\ e \sqsubseteq e \Rightarrow (d,e) \sqsubseteq (d',e)$$

$$\Rightarrow f(d,e) \sqsubseteq f(d',e) \quad \checkmark$$

$f(-,e)$ preserves lubs

preservation of lubs in 2 arguments.

$$f\left(\bigsqcup_n d_n, e\right) \overset{?}{=} \bigsqcup_n f(d_n,e)$$

$$f\left(\bigsqcup_n d_n, e\right) = f\left(\bigsqcup_n d_n, \bigsqcup_n e\right) = \bigsqcup_n f(d_n, e)$$

- A couple of derived rules:

$$\frac{x \sqsubseteq x' \qquad y \sqsubseteq y'}{f(x,y) \sqsubseteq f(x',y')} \quad (f \text{ monotone})$$

$$\frac{}{f(\bigsqcup_m x_m, \bigsqcup_n y_n) = \bigsqcup_k f(x_k, y_k)}$$

Function cpo's and domains

Given cpo's $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$$(D \to E) \stackrel{\text{def}}{=} \{f \mid f : D \to E \text{ is a } \textit{continuous} \text{ function}\}$$

and partial order: $f \sqsubseteq f' \stackrel{\text{def}}{\Leftrightarrow} \forall d \in D \,.\, f(d) \sqsubseteq_E f'(d)$.

Given $\underline{D}$ modelling α and $\underline{E}$ modelling β

What domain models $(\alpha \to \beta)$?

$(\underline{D} \to \underline{E})$ The domain given by:

- underlying set

$$\{ f : D \to E \mid f \text{ is continuous} \}$$

- partial order:

$$f \sqsubseteq g \overset{\text{def}}{\Longleftrightarrow} \forall x \in D .\ f(x) \sqsubseteq_{\underline{E}} g(x)$$

- least element: $\bot = (\lambda x \cdot \bot_{\underline{E}})$

Given an ω-chain of continuous functions from $\underline{D}$ to $\underline{E}$:

$$f_0 \sqsubseteq f_1 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \cdots \quad (n \in \omega)$$

Find an upper bound; that is, a continuous function $f : \underline{D} \to \underline{E}$ s.t

$$f_n \sqsubseteq f.$$

$\Updownarrow$ $\forall x \in D.\ f_n(x) \sqsubseteq f(x)$

$\sqsubseteq$

Guaranteed by Taking

$$f(x) \overset{\text{def}}{=} \bigsqcup_n f_n(x)$$

only well defined if $f_n(x)$ is an ω-chain.

Indeed

$f_0(x) \sqsubseteq f_1(x) \sqsubseteq \cdots \sqsubseteq f_n(x) \sqsubseteq \cdots$

in $\underline{E}$

Let us define

$$f =^{df} \lambda x. \bigsqcup_n f_n(x)$$

We need show it is <u>continuous</u>.

(1) $x \sqsubseteq y \overset{?}{\Rightarrow} f(x) \sqsubseteq f(y)$

$\Downarrow$ — f_n monotone fn.

$f_n(x) \sqsubseteq f_n(y)$

$\Downarrow$ — by lemma

$$\bigsqcup_n f_n(x) \sqsubseteq \bigsqcup_n f_n(y)$$

$$\begin{array}{ccc} \| & & \| \\ f(x) & & f(y) \end{array}$$

(2) for all ω-chains d_n in $\underline{D}$:

$$f\left(\bigsqcup_i d_i\right) \stackrel{?}{=} \bigsqcup_i f(d_i)$$

$$f\left(\bigsqcup_i d_i\right) = \bigsqcup_n f_n\left(\bigsqcup_i d_i\right)$$

$$\bigsqcup_n f_n\left(\bigsqcup_i d_i\right) = \bigsqcup_n \bigsqcup_i f_n(d_i) \quad \leadsto f_n \text{ preserve lubs of } \omega\text{-chains}$$

$$\bigsqcup_i f(d_i) = \bigsqcup_i \bigsqcup_n f_n(d_i)$$

$$\bigsqcup_n \bigsqcup_i f_n(d_i) = \bigsqcup_i \bigsqcup_n f_n(d_i) \quad \text{diag. Lemma}$$

By construction, $f_n \sqsubseteq f$ $\forall n \in \mathbb{N}$.

<u>Claim</u> $f =^{\mathrm{df}} \lambda x. \bigsqcup_n f_n(x)$

is least amongst all upper bounds.

Let g be an upper bound of $f_n (n \in \omega)$; that is,

$f_n \sqsubseteq g$

$\Downarrow$

$\forall x \;\; f_n(x) \sqsubseteq g(x)$

$\Downarrow$

$f(x) = \bigsqcup_n f_n(x) \sqsubseteq g(x) \implies f \sqsubseteq g.$ $\boxtimes$

$\forall x$

Function cpo's and domains

Given cpo's $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

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and partial order: $f \sqsubseteq f' \stackrel{\text{def}}{\Leftrightarrow} \forall d \in D \,.\, f(d) \sqsubseteq_E f'(d)$.

- A derived rule:

$$\frac{f \sqsubseteq_{(D \to E)} g \qquad x \sqsubseteq_D y}{f(x) \sqsubseteq g(y)}$$

Lubs of chains are calculated 'argumentwise' (using lubs in E):

$$\bigsqcup_{n\geq 0} f_n \;=\; \lambda d \in D.\, \bigsqcup_{n\geq 0} f_n(d)\;\; .$$

- A derived rule:

$$\frac{}{(\bigsqcup_n f_n)(\bigsqcup_m x_m) = \bigsqcup_k f_k(x_k)}$$

If E is a domain, then so is $D \to E$ and $\bot_{D\to E}(d) = \bot_E$, all $d \in D$.

Continuity of composition

For cpo's D, E, F, the composition function

$$\circ : \big((E \to F) \times (D \to E)\big) \longrightarrow (D \to F)$$

defined by setting, for all $f \in (D \to E)$ and $g \in (E \to F)$,

$$g \circ f \;=\; \lambda d \in D.\, g\big(f(d)\big)$$

is continuous.

Continuity of the fixpoint operator

Let D be a domain.

By Tarski's Fixed Point Theorem we know that each continuous function $f \in (D \to D)$ possesses a least fixed point, $\mathit{fix}(f) \in D$.

$\bigsqcup_n f^n(\bot)$

Proposition. *The function*

$$\mathit{fix} : (D \to D) \to D$$

$$f \longmapsto \mathit{fix}(f)$$

is continuous.