Recall:

λ-Terms, M

are built up from a given, countable collection of

- variables $x, y, z, \ldots$

by two operations for forming λ-terms:

- λ-abstraction: $(\lambda x.M)$
  (where x is a variable and M is a λ-term)
- application: $(M\,M')$
  (where M and M' are λ-terms).

Some random examples of λ-terms:

$$x \quad (\lambda x.x) \quad ((\lambda y.(x\,y))x) \quad (\lambda y.((\lambda y.(x\,y))x))$$

β-Reduction

Recall that $\lambda x.M$ is intended to represent the function f such that $f(x) = M$ for all x. We can regard $\lambda x.M$ as a function on λ-terms via substitution: map each N to $M[N/x]$.

So the natural notion of computation for λ-terms is given by stepping from a

β-redex $\qquad (\lambda x.M)N$

to the corresponding

β-reduct $\qquad M[N/x]$

β-Conversion $M =_\beta N$

is the binary relation inductively generated by the rules:

$$\frac{M =_\alpha M'}{M =_\beta M'} \qquad \frac{M \to M'}{M =_\beta M'} \qquad \frac{M =_\beta M'}{M' =_\beta M}$$

$$\frac{M =_\beta M' \qquad M' =_\beta M''}{M =_\beta M''} \qquad \frac{M =_\beta M'}{\lambda x.M =_\beta \lambda x.M'}$$

$$\frac{M =_\beta M' \qquad N =_\beta N'}{M\,N =_\beta M'\,N'}$$

Church-Rosser Theorem

Theorem. $\twoheadrightarrow$ is confluent, that is, if $M_1 \twoheadleftarrow M \twoheadrightarrow M_2$, then there exists M' such that $M_1 \twoheadrightarrow M' \twoheadleftarrow M_2$.

[Proof omitted.]

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Corollary. Two show that two terms are β-convertible, it suffices to show that they both reduce to the same term. More precisely: $M_1 =_\beta M_2$ iff $\exists M\,(M_1 \twoheadrightarrow M \twoheadleftarrow M_2)$.

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Corollary. $M_1 =_\beta M_2$ iff $\exists M\,(M_1 \twoheadrightarrow M \twoheadleftarrow M_2)$.

Proof. $=_\beta$ satisfies the rules generating $\twoheadrightarrow$; so $M \twoheadrightarrow M'$ implies $M =_\beta M'$. Thus if $M_1 \twoheadrightarrow M \twoheadleftarrow M_2$, then $M_1 =_\beta M =_\beta M_2$ and so $M_1 =_\beta M_2$.

Conversely,

Church-Rosser Theorem

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Conversely, the relation $\{(M_1, M_2) \mid \exists M\,(M_1 \twoheadrightarrow M \twoheadleftarrow M_2)\}$ satisfies the rules generating $=_\beta$: the only difficult case is closure of the relation under transitivity and for this we use the Church-Rosser theorem: $M_1 \longrightarrow\!\!\!\!\!\rightarrow M \twoheadleftarrow\!\!\!\!\longleftarrow M_2 \longrightarrow\!\!\!\!\!\rightarrow M' \twoheadleftarrow\!\!\!\!\longleftarrow M_3$

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$$\begin{tikzcd}
M_1 \arrow[r, two heads] & M \arrow[dr, two heads] & M_2 \arrow[l, two heads] \arrow[r, two heads] \arrow[d, phantom, "\text{C-R}"] & M' \arrow[dl, two heads] & M_3 \arrow[l, two heads] \\
 & & M_2' & &
\end{tikzcd}$$

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Conversely, the relation $\{(M_1, M_2) \mid \exists M\,(M_1 \twoheadrightarrow M \twoheadleftarrow M_2)\}$ satisfies the rules generating $=_\beta$: the only difficult case is closure of the relation under transitivity and for this we use the Church-Rosser theorem. Hence $M_1 =_\beta M_2$ implies $\exists M\,(M_1 \twoheadrightarrow M' \twoheadleftarrow M_2)$.

β-Normal Forms

Definition. A λ-term N is in β-normal form (nf) if it contains no β-redexes (no sub-terms of the form $(\lambda x.M)M'$). M has β-nf N if $M =_\beta N$ with N a β-nf.

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Note that if N is a β-nf and $N \twoheadrightarrow N'$, then it must be that $N =_\alpha N'$ (why?).

Hence if $N_1 =_\beta N_2$ with N_1 and N_2 both β-nfs, then $N_1 =_\alpha N_2$. (For if $N_1 =_\beta N_2$, then by Church-Rosser $N_1 \twoheadrightarrow M' \twoheadleftarrow N_2$ for some M', so $N_1 =_\alpha M' =_\alpha N_2$.)

So the β-nf of M is unique up to α-equivalence if it exists.

Non-termination

Some λ terms have no β-nf.

E.g. $\Omega \triangleq (\lambda x.x\,x)(\lambda x.x\,x)$ satisfies

- $\Omega \to (x\,x)[(\lambda x.x\,x)/x] = \Omega$,
- $\Omega \twoheadrightarrow M$ implies $\Omega =_\alpha M$.

So there is no β-nf N such that $\Omega =_\beta N$.

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So there is no β-nf N such that $\Omega =_\beta N$.

A term can possess both a β-nf and infinite chains of reduction from it.

E.g. $(\lambda x.y)\Omega \to y$, but also $(\lambda x.y)\Omega \to (\lambda x.y)\Omega \to \cdots$.

Non-termination

Normal-order reduction is a deterministic strategy for reducing λ-terms: reduce the "left-most, outer-most" redex first.

- left-most: reduce M before N in $M\,N$, and then
- outer-most: reduce $(\lambda x.M)N$ rather than either of M or N.

(cf. call-by-name evaluation).

Fact: normal-order reduction of M always reaches the β-nf of M if it possesses one.

$$\frac{M_1 =_\alpha M_1' \quad M_1' \rightarrow_{NOR} M_2' \quad M_2' =_\alpha M_2}{M_1 \rightarrow_{NOR} M_2}$$

$$\frac{M \rightarrow_{NOR} M'}{\lambda x. M \rightarrow_{NOR} \lambda x. M'}$$

$$\frac{M_1 \rightarrow_{NOR} M_1'}{M_1 M_2 \rightarrow_{NOR} M_1' M_2} \qquad \frac{}{(\lambda x. M) M' \rightarrow_{NOR} M[M'/x]}$$

$$\frac{M \rightarrow_{NOR} M'}{UM \rightarrow_{NOR} UM'} \quad \text{where} \quad \begin{cases} U ::= x \mid UN \\ N ::= \lambda x. N \mid U \end{cases}$$

β-normal forms

"neutral" forms

Lambda-Definable Functions

Encoding data in λ-calculus

Computation in λ-calculus is given by β-reduction. To relate this to register/Turing-machine computation, or to partial recursive functions, we first have to see how to encode numbers, pairs, lists, ... as λ-terms.

We will use the original encoding of numbers due to Church...

Church's numerals

$$\begin{aligned}
\underline{0} &\triangleq \lambda f\, x.x \\
\underline{1} &\triangleq \lambda f\, x.f\, x \\
\underline{2} &\triangleq \lambda f\, x.f(f\, x) \\
&\vdots \\
\underline{n} &\triangleq \lambda f\, x.\underbrace{f(\cdots(f}_{n \text{ times}}\, x)\cdots)
\end{aligned}$$

Notation: $\begin{cases} M^0 N & \triangleq N \\ M^1 N & \triangleq M\, N \\ M^{n+1} N & \triangleq M(M^n N) \end{cases}$

so we can write $\underline{n}$ as $\lambda f\, x.f^n x$ and we have $\boxed{\underline{n}\, M\, N =_\beta M^n\, N}$.

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$$\begin{aligned}
\underline{0} &\triangleq \lambda f\,x.x \\
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&\vdots \\
\underline{n} &\triangleq \lambda f\,x.\,\underbrace{f\,(\cdots(f}_{n \text{ times}}\,x)\cdots)
\end{aligned}$$

N.B. not ffx, which stands for $(ff)x$

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λ-Definable functions

Definition. $f \in \mathbb{N}^n \rightharpoonup \mathbb{N}$ is λ-definable if there is a closed λ-term F that represents it: for all $(x_1, \ldots, x_n) \in \mathbb{N}^n$ and $y \in \mathbb{N}$

- if $f(x_1, \ldots, x_n) = y$, then $F\,\underline{x_1} \cdots \underline{x_n} =_\beta \underline{y}$
- if $f(x_1, \ldots, x_n)\uparrow$, then $F\,\underline{x_1} \cdots \underline{x_n}$ has no β-nf.

For example, addition is λ-definable because it is represented by $P \triangleq \lambda x_1\,x_2.\lambda f\,x.\,x_1\,f\,(x_2\,f\,x)$:

$$\begin{aligned} P\,\underline{m}\,\underline{n} &=_\beta \lambda f\,x.\,\underline{m}\,f\,(\underline{n}\,f\,x) \\ &=_\beta \lambda f\,x.\,\underline{m}\,f\,(f^n x) \\ &=_\beta \lambda f\,x.\,f^m(f^n x) \\ &= \lambda f\,x.f^{m+n} x \\ &= \underline{m+n} \end{aligned}$$

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$$
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P\,\underline{m}\,\underline{n} &=_\beta \lambda f\, x .\, \underline{m}\, f\,(\underline{n}\, f\, x) \\
&=_\beta \lambda f\, x .\, \underline{m}\, f\,(f^n x) \\
&=_\beta \lambda f\, x .\, f^m (f^n x) \\
&= \lambda f\, x . f^{m+n} x \\
&= \underline{m+n}
\end{aligned}
$$

can prove this equality by induction on n

Computable = λ-definable

Theorem. A partial function is computable if and only if it is λ-definable.

We already know that

$$\begin{array}{ll} & \text{Register Machine computable} \\ = & \text{Turing computable} \\ = & \text{partial recursive.} \end{array}$$

Using this, we break the theorem into two parts:

- every partial recursive function is λ-definable
- λ-definable functions are RM computable

λ-Definable functions

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- if $f(x_1, \ldots, x_n) = y$, then $F\,\underline{x_1} \cdots \underline{x_n} =_\beta \underline{y}$
- if $f(x_1, \ldots, x_n)\uparrow$, then $F\,\underline{x_1} \cdots \underline{x_n}$ has no β-nf.

This condition can make it quite tricky to find a λ-term representing a non-total function.

For now, we concentrate on total functions. First, let us see why the elements of **PRIM** (primitive recursive functions) are λ-definable.

Basic functions

- Projection functions, $\mathtt{proj}_i^n \in \mathbb{N}^n \to \mathbb{N}$:

$$\mathtt{proj}_i^n(x_1, \ldots, x_n) \triangleq x_i$$

- Constant functions with value $\mathbf{0}$, $\mathtt{zero}^n \in \mathbb{N}^n \to \mathbb{N}$:

$$\mathtt{zero}^n(x_1, \ldots, x_n) \triangleq \mathbf{0}$$

- Successor function, $\mathtt{succ} \in \mathbb{N} \to \mathbb{N}$:

$$\mathtt{succ}(x) \triangleq x + 1$$

Basic functions are representable

- $\mathtt{proj}_i^n \in \mathbb{N}^n \to \mathbb{N}$ is represented by $\lambda x_1 \ldots x_n.x_i$
- $\mathtt{zero}^n \in \mathbb{N}^n \to \mathbb{N}$ is represented by $\lambda x_1 \ldots x_n.\underline{0}$
- $\mathtt{succ} \in \mathbb{N} \to \mathbb{N}$ is represented by

$$\mathbf{Succ} \triangleq \lambda x_1\, f\, x.f\,(x_1\, f\, x)$$

since

$$\begin{aligned} \mathbf{Succ}\,\underline{n} &=_\beta \lambda f\, x.\, f\,(\underline{n}\, f\, x) \\ &=_\beta \lambda f\, x.\, f\,(f^n\, x) \\ &= \lambda f\, x.\, f^{n+1}\, x \\ &= \underline{n+1} \end{aligned}$$

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- $\texttt{proj}_i^n \in \mathbb{N}^n \to \mathbb{N}$ is represented by $\lambda x_1 \ldots x_n . x_i$
- $\texttt{zero}^n \in \mathbb{N}^n \to \mathbb{N}$ is represented by $\lambda x_1 \ldots x_n . \underline{0}$
- $\texttt{succ} \in \mathbb{N} \to \mathbb{N}$ is represented by

$$\mathbf{Succ} \triangleq \lambda x_1\, f\, x . f\, (x_1\, f\, x)$$

since

$$\begin{aligned} \mathbf{Succ}\, \underline{n} &=_\beta \lambda f\, x . f\, (\underline{n}\, f\, x) \\ &=_\beta \lambda f\, x . f\, (f^n\, x) \\ &= \lambda f\, x . f^{n+1}\, x \\ &= \underline{n+1} \end{aligned}$$

($\lambda x_1 f x . x_1 f (f x)$ also represents Succ)