VI. Approximation Algorithms: Travelling Salesman Problem

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UNIVERSITY OF CAMBRIDGE

Outline

Introduction

General TSP

Metric TSP

The Traveling Salesman Problem (TSP)

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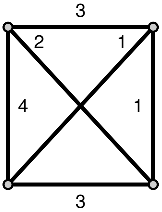

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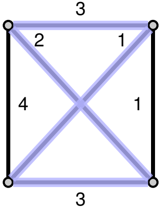

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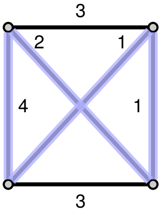

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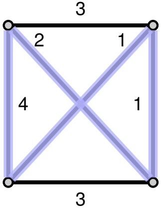

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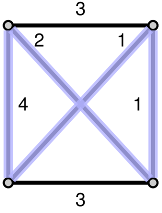

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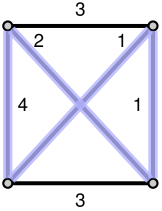

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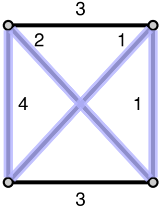

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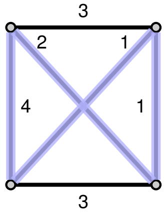

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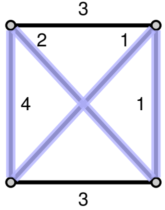

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Even this version is NP hard (Ex. 35.2-2)

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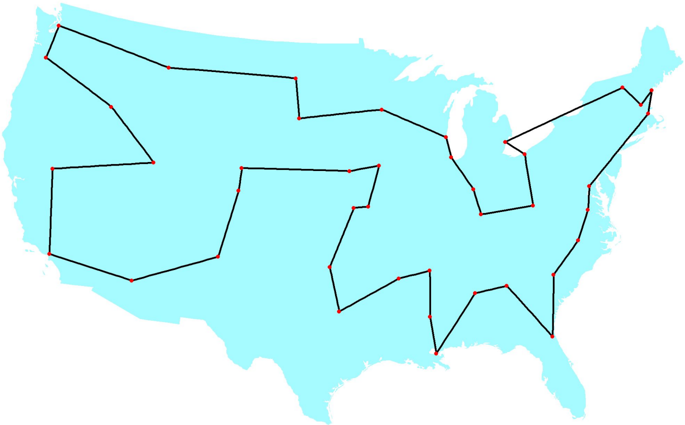
History of the TSP problem (1954)

Dantzig, Fulkerson and Johnson found an optimal tour through 42 cities.



`http://www.math.uwaterloo.ca/tsp/history/img/dantzig_big.html`

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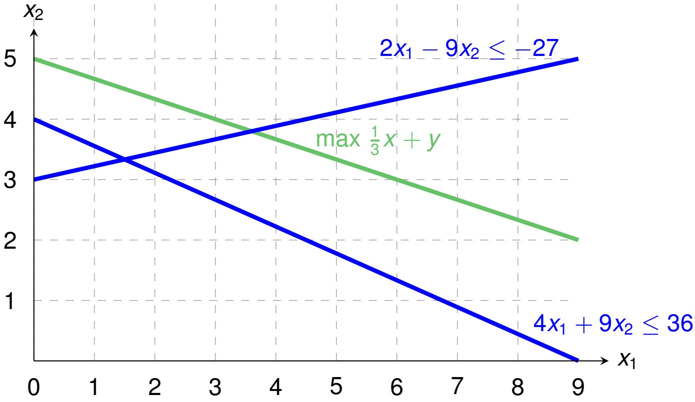

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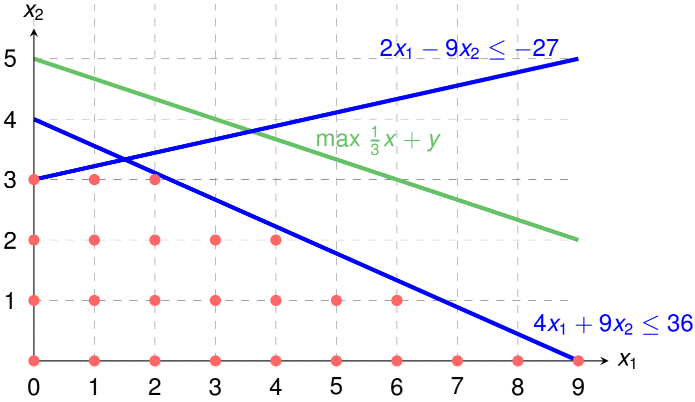

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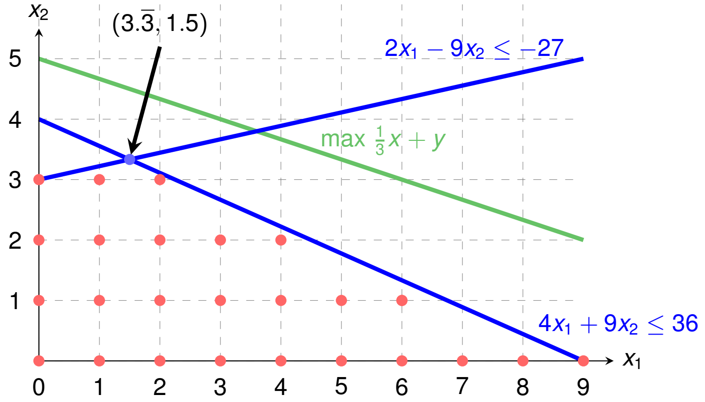

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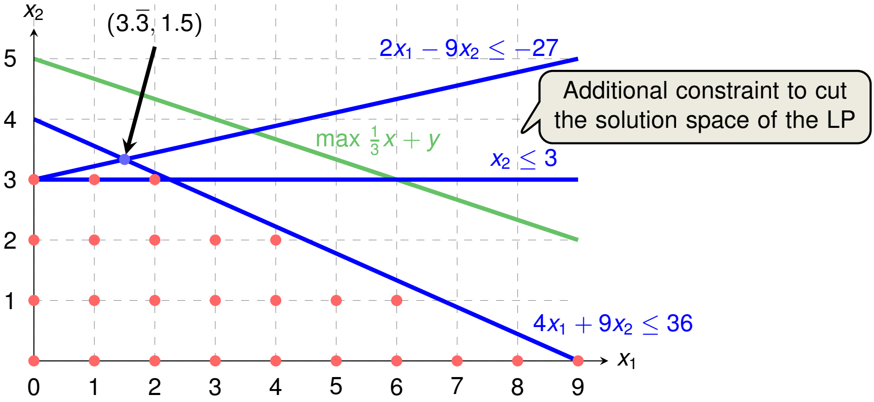

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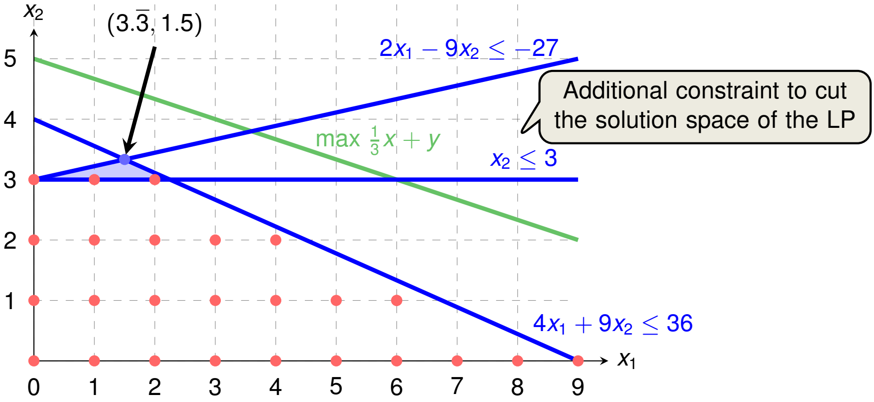

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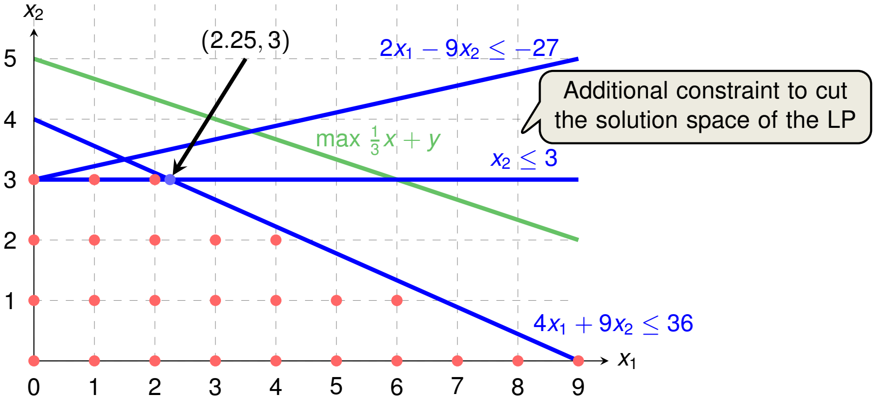

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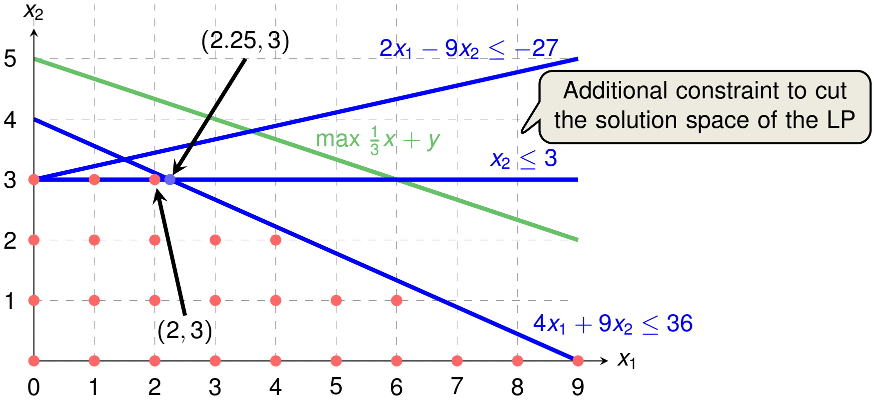

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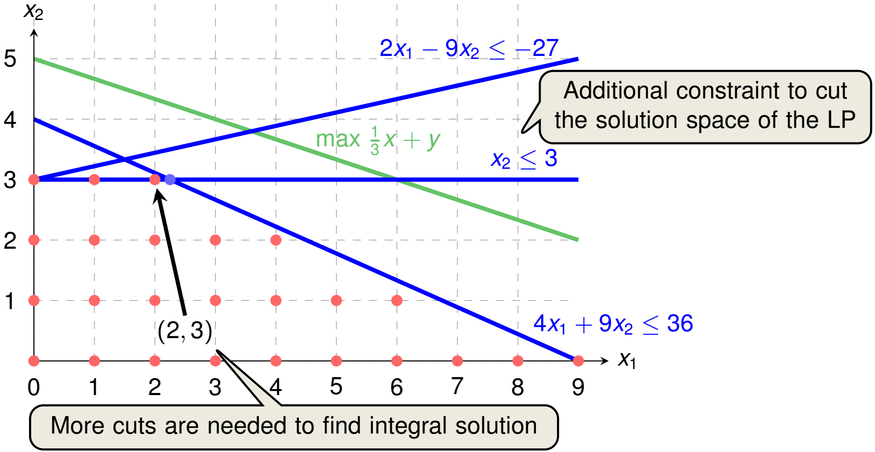

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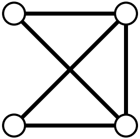

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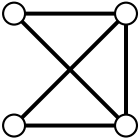

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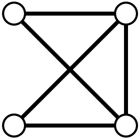

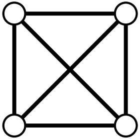

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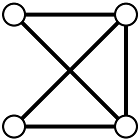

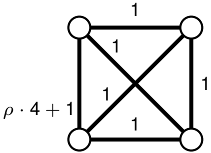

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Large weight will render this edge useless!

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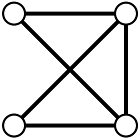

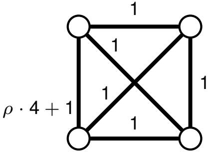

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Can create representations of G' and c in time polynomial in $|V|$ and $|E|$!

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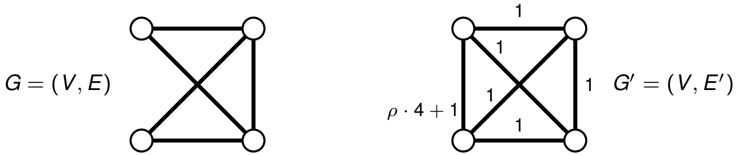

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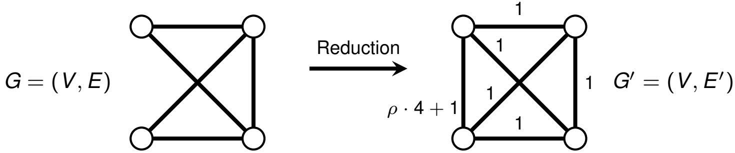

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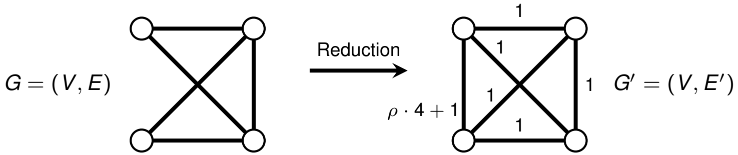

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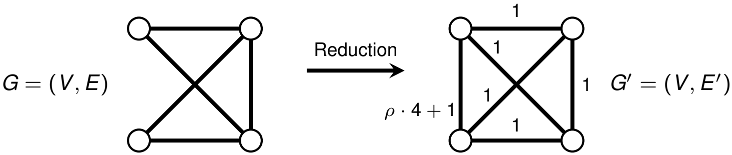

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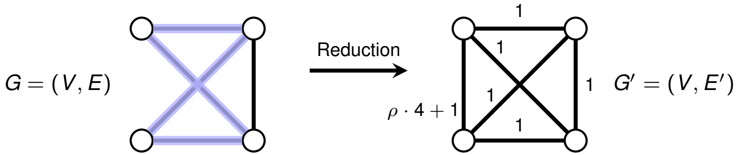

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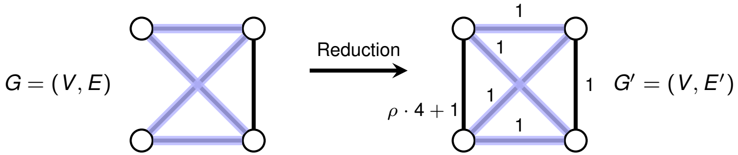

Hardness of Approximation

Theorem 35.3

If P $\neq$ NP, then for any constant $\rho \geq 1$, there is no polynomial-time approximation algorithm with approximation ratio ρ for the general TSP.

Idea: Reduction from the hamiltonian-cycle problem.

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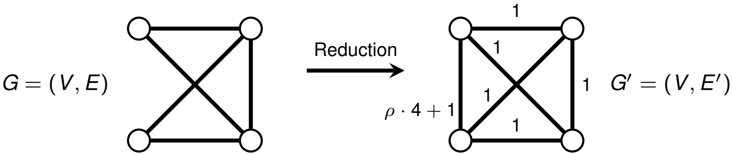

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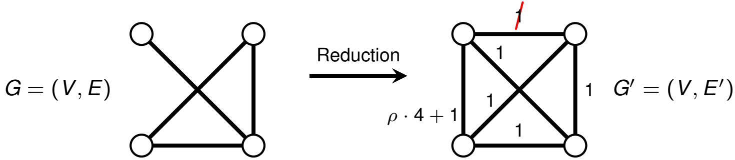

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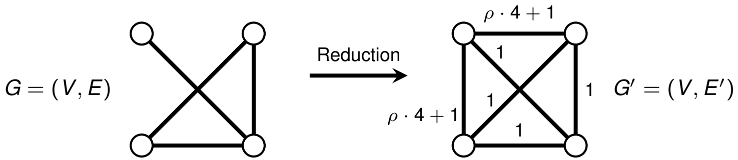

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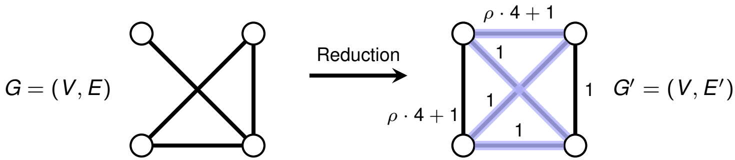

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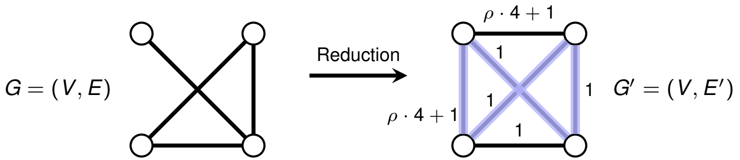

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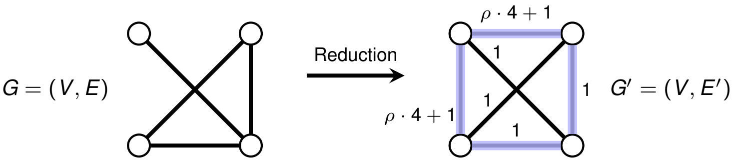

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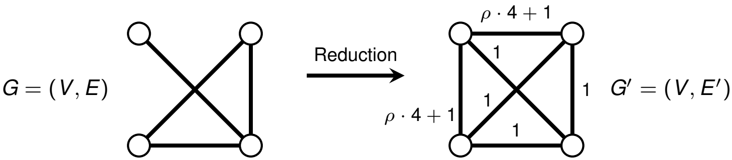

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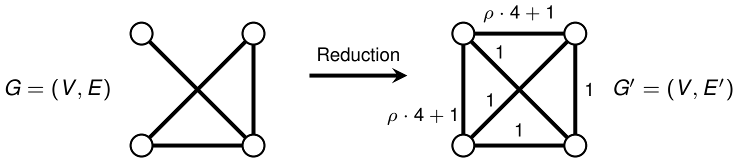

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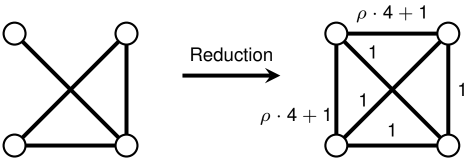

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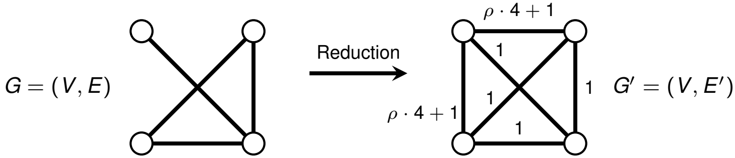

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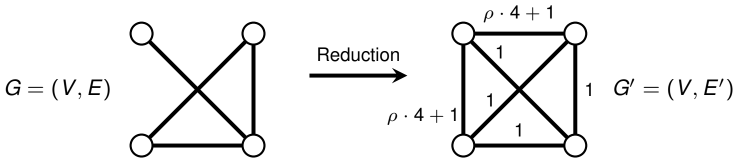
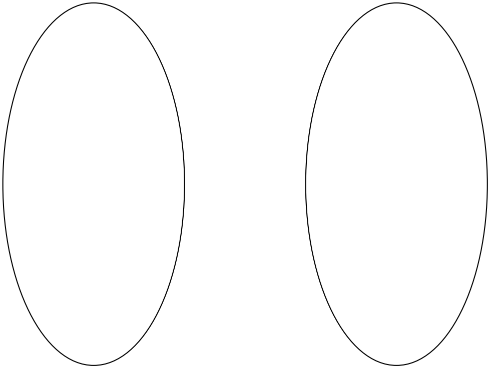


Proof of Theorem 35.3 from a higher perspective



instances of Hamilton instances of TSP

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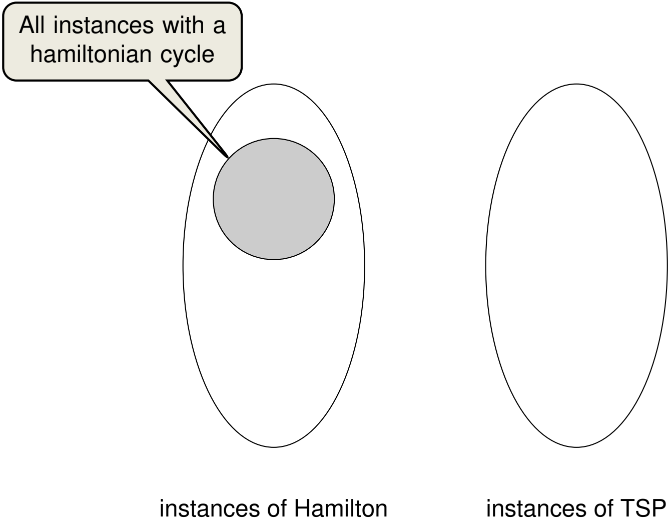

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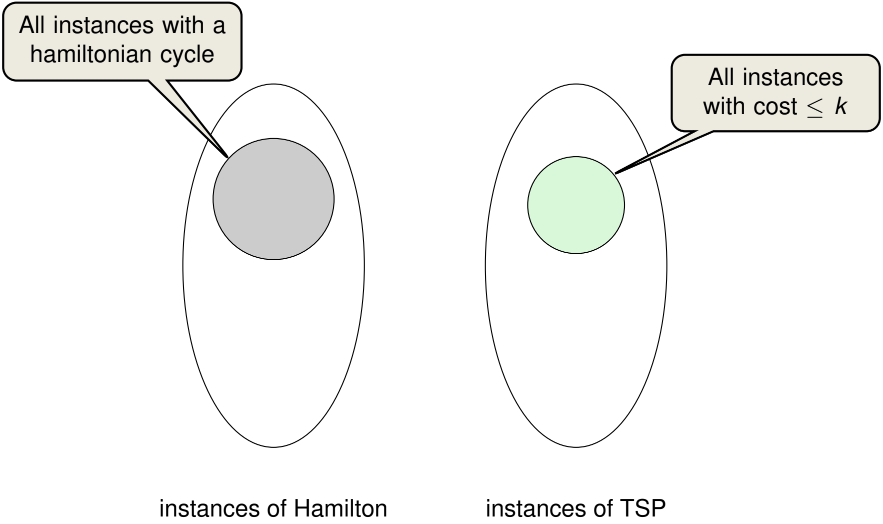

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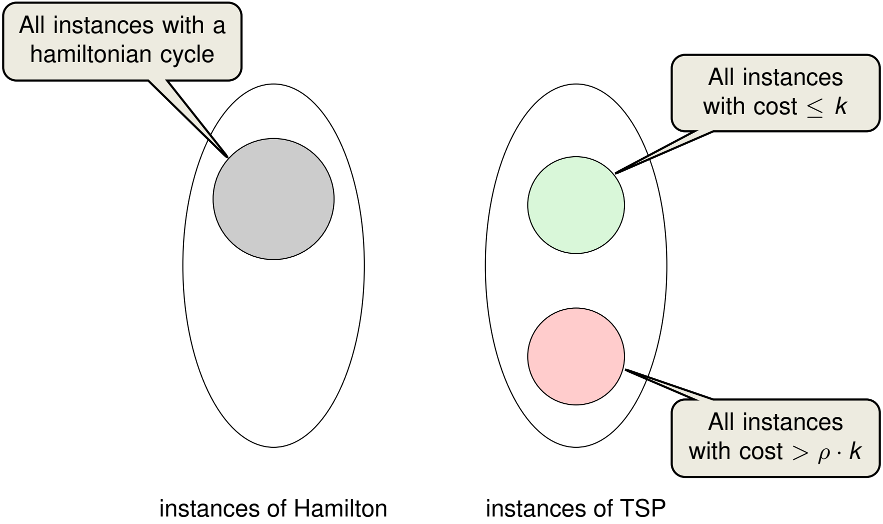

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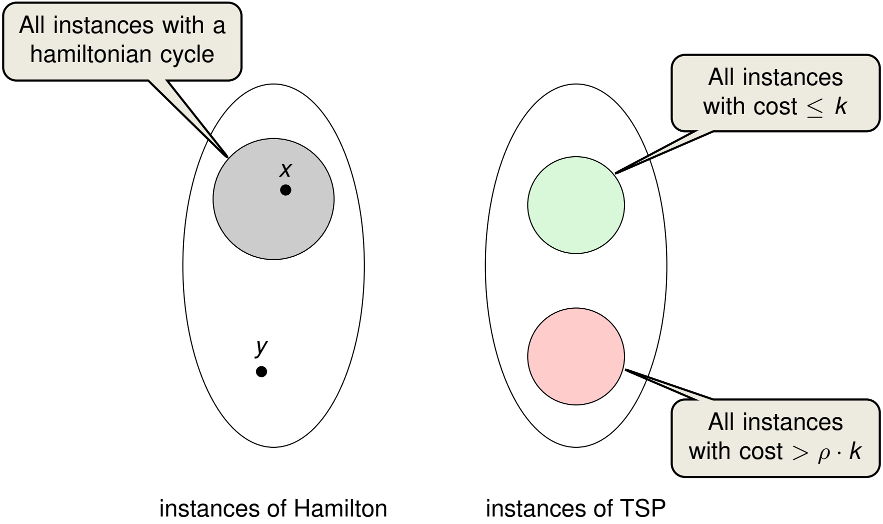

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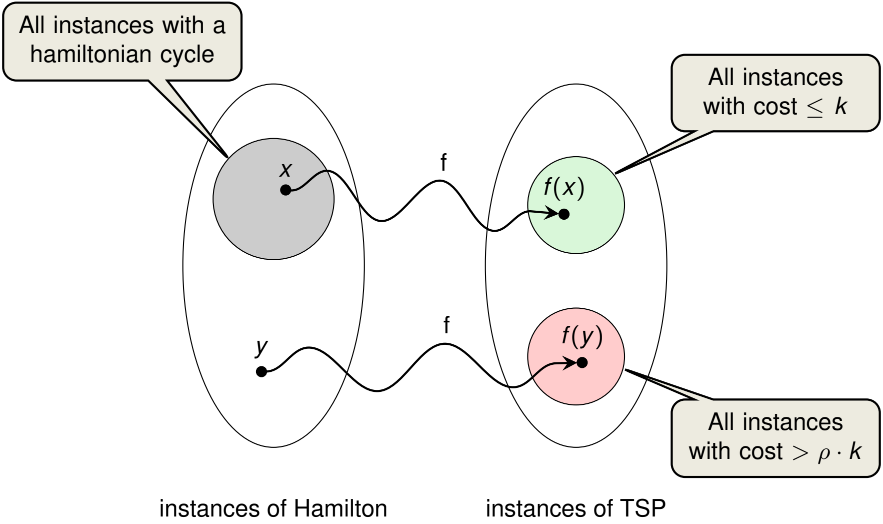

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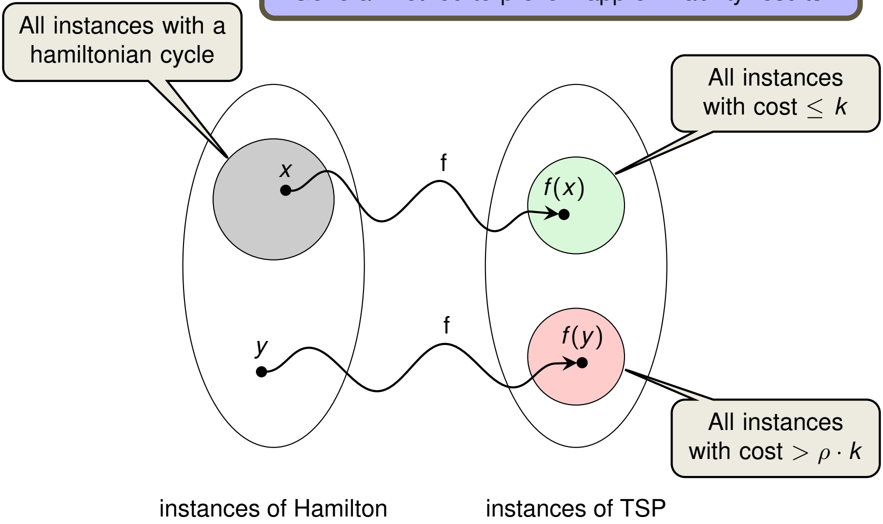
General Method to prove inapproximability results!



Outline

Introduction

General TSP

Metric TSP

Metric TSP (TSP Problem with the Triangle Inequality)

Idea: First compute an MST, and then create a tour based on the tree.

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APPROX-TSP-TOUR(G, c)
select a vertex $r \in G.V$ to be a "root" vertex
compute a minimum spanning tree $T_{\min}$ for G from root r
    using MST-PRIM(G, c, r)
let H be a list of vertices, ordered according to when they are first visited
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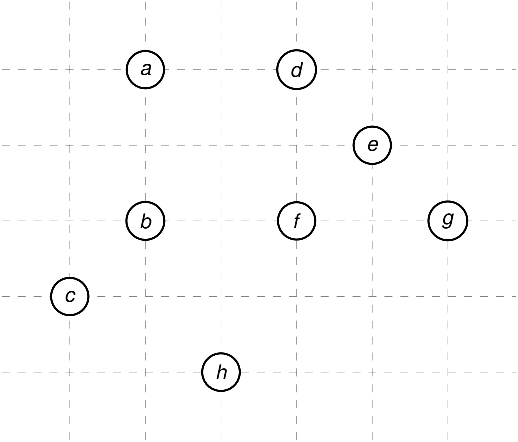
Remember: In the Metric-TSP problem, G is a complete graph.

Run of APPROX-TSP-TOUR



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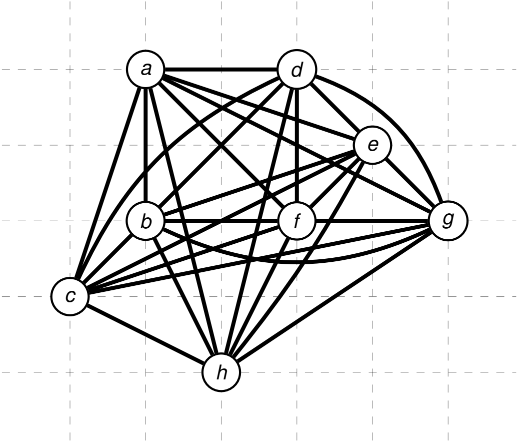

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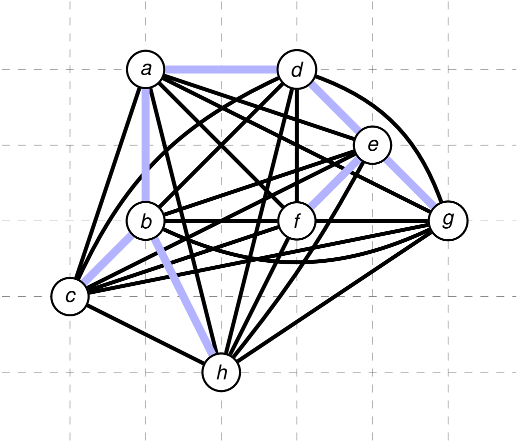
Run of APPROX-TSP-TOUR



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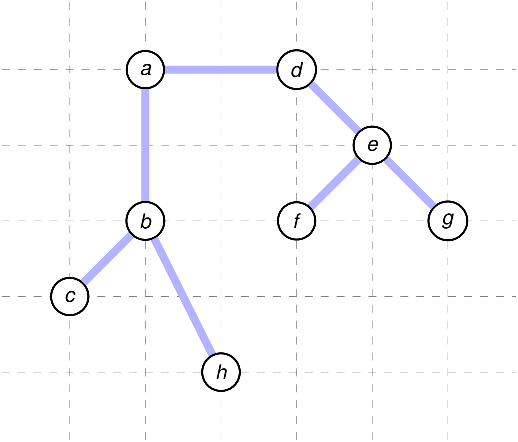

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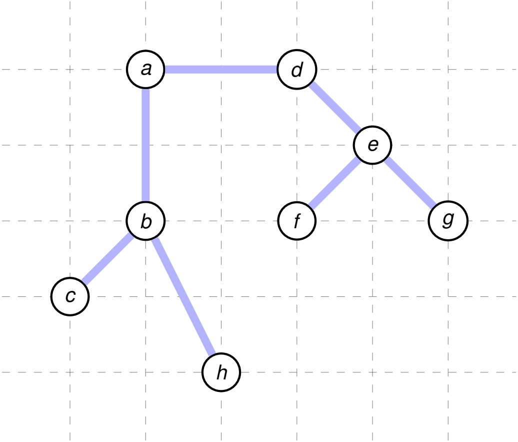

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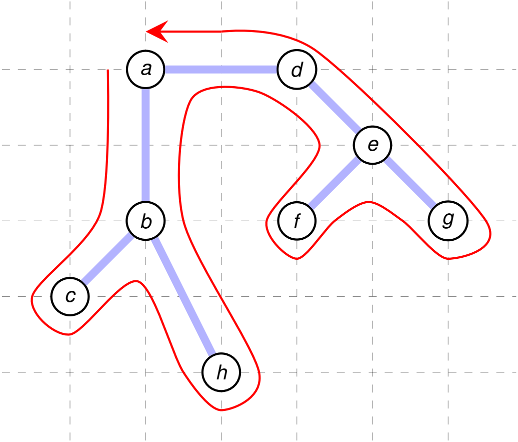

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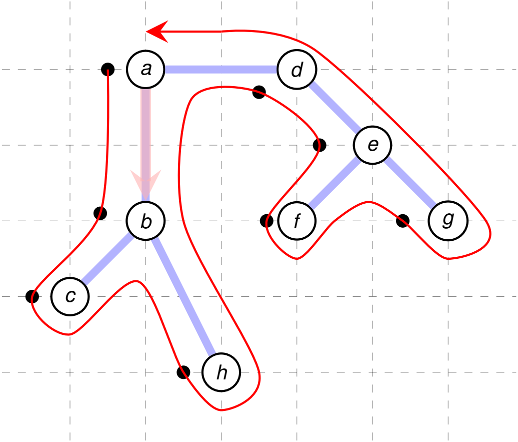

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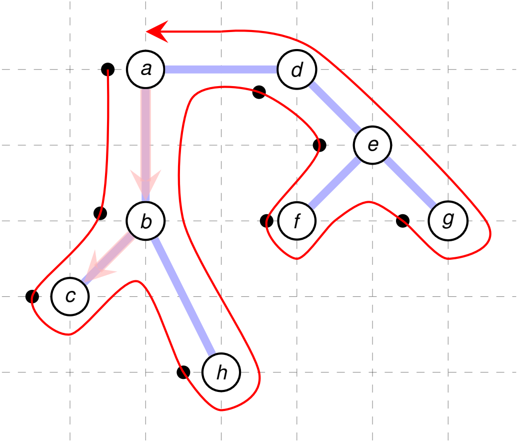

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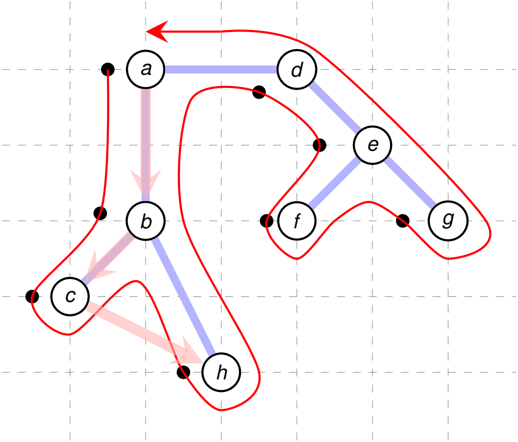

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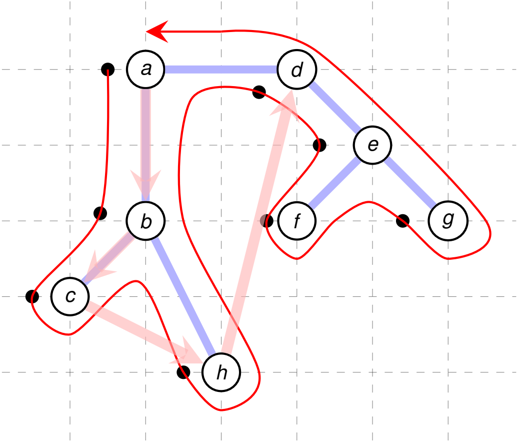

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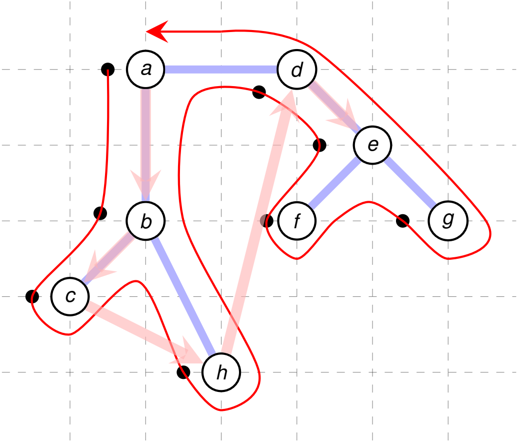

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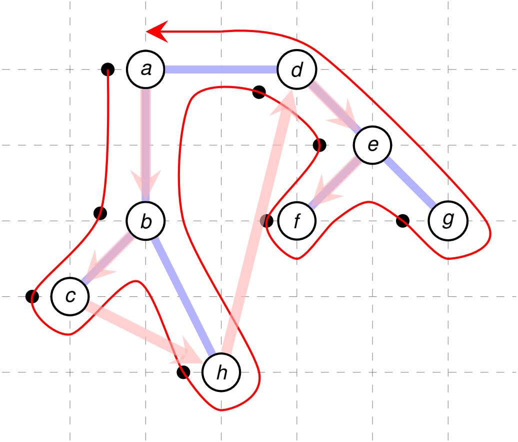

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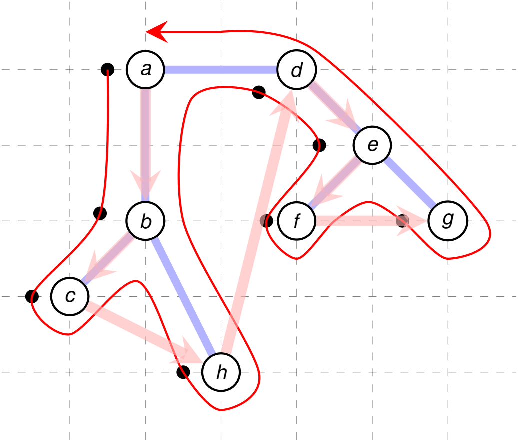

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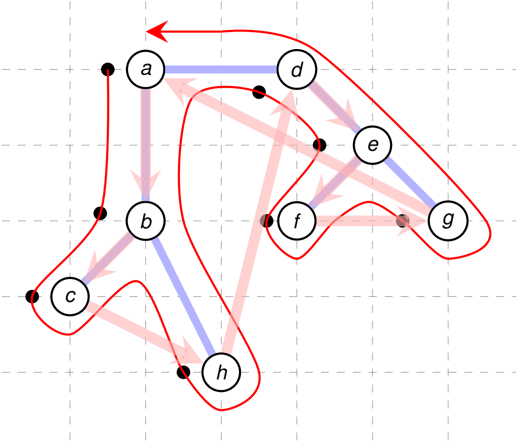

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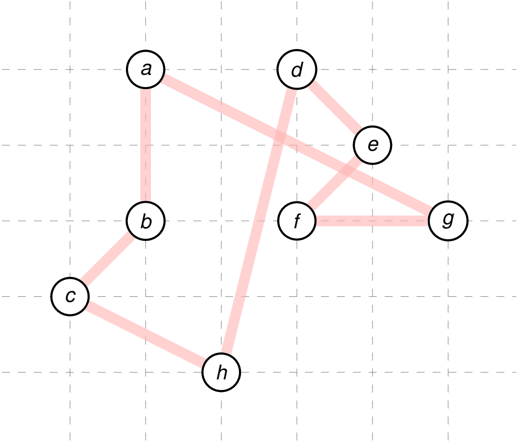

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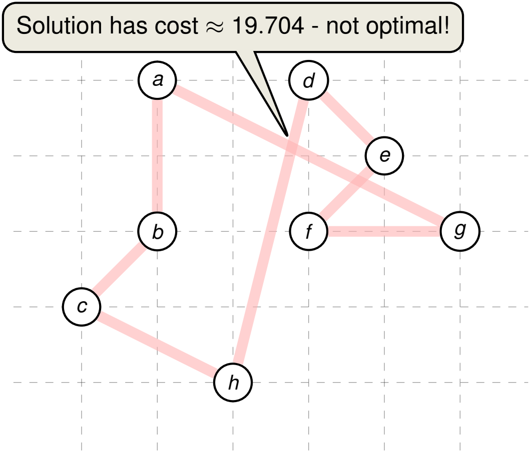

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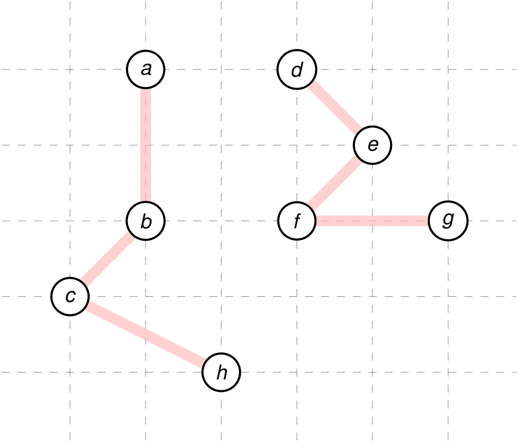

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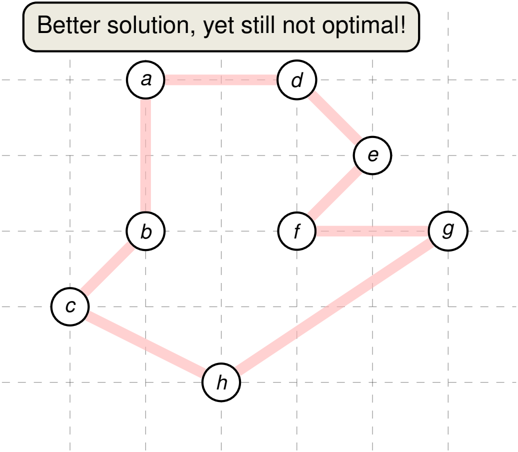
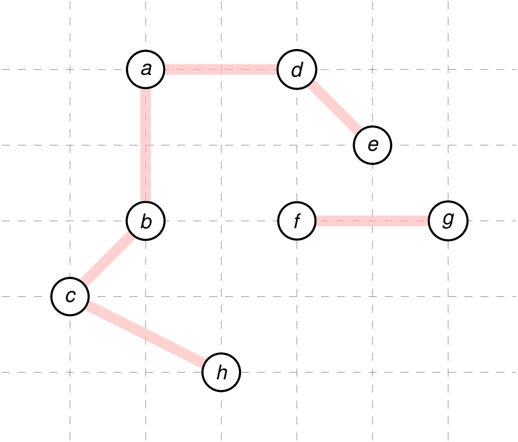


1. Compute MST $T_{\min}$ ✓
2. Perform preorder walk on MST $T_{\min}$ ✓
3. Return list of vertices according to the preorder tree walk ✓

Run of APPROX-TSP-TOUR



1. Compute MST $T_{\min}$ ✓
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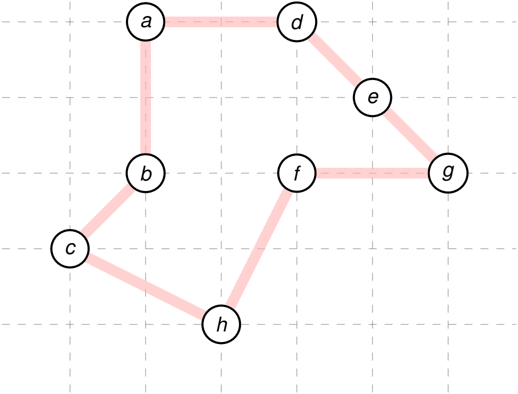
3. Return list of vertices according to the preorder tree walk ✓

Run of APPROX-TSP-TOUR

This is the optimal solution (cost $\approx$ 14.715).



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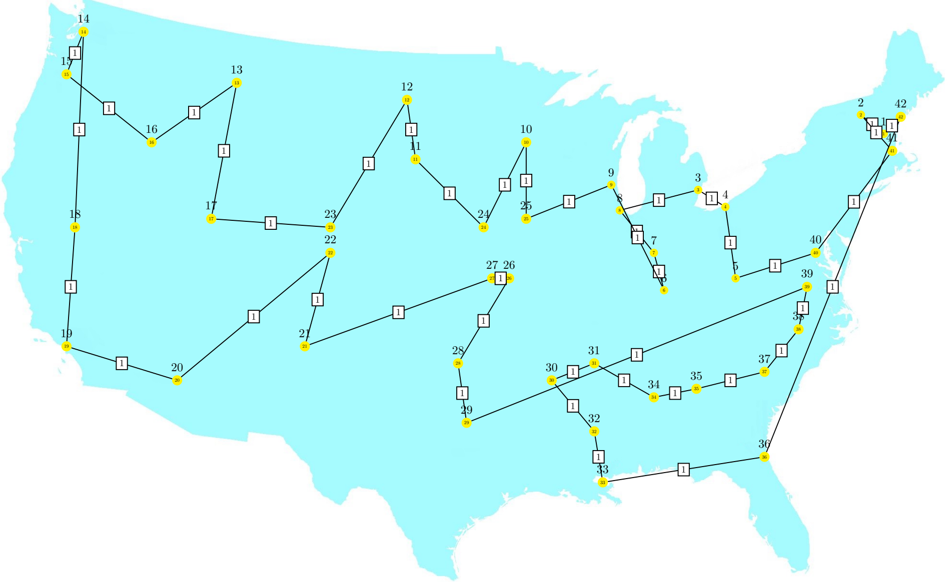
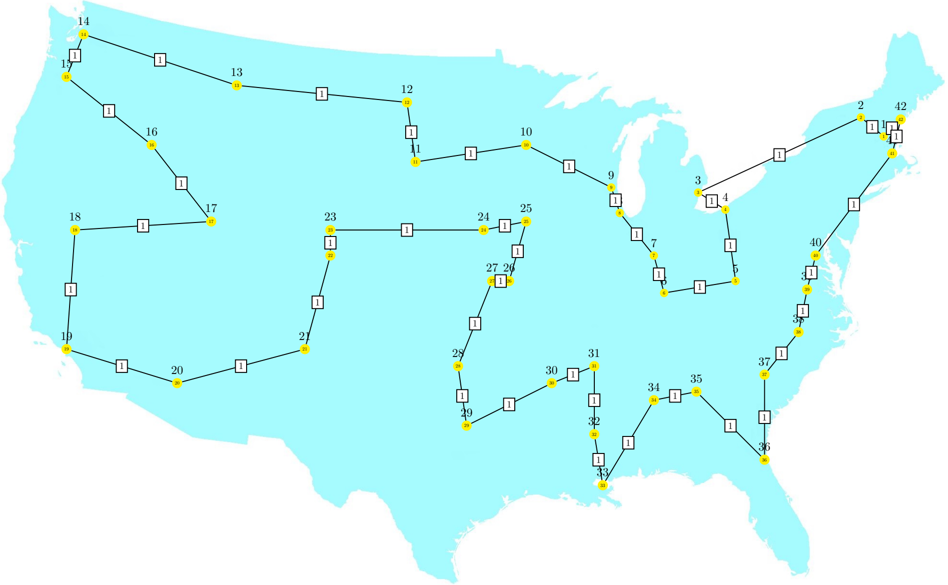
Approximate Solution: Objective 921



Optimal Solution: Objective 699



Proof of the Approximation Ratio

Theorem 35.2

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

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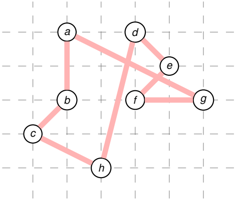

solution H of APPROX-TSP

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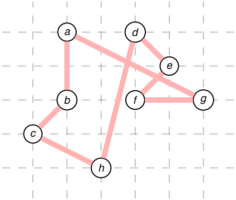

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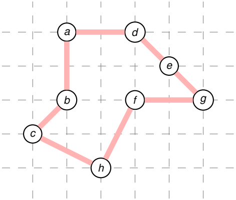

optimal solution H^*

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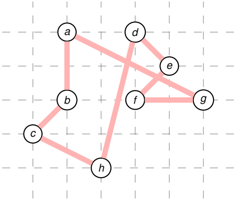

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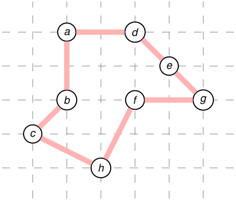

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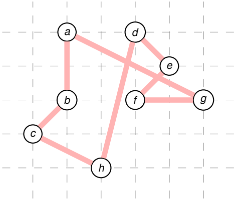

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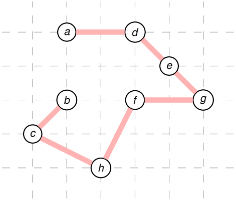

spanning tree T as a subset of H^*

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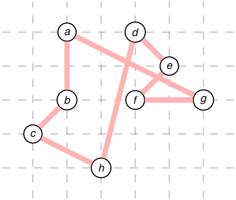

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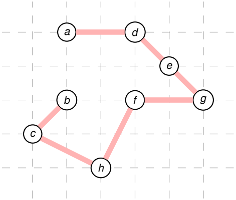

spanning tree T as a subset of H^*

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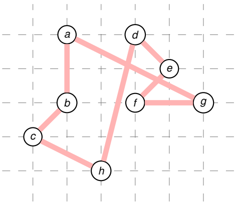

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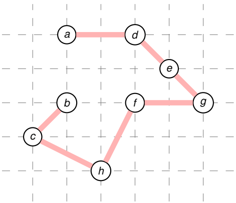

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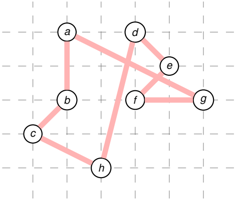
exploiting that all edge costs are non-negative!



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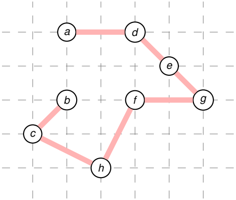

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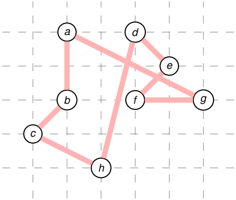

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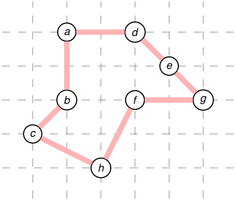

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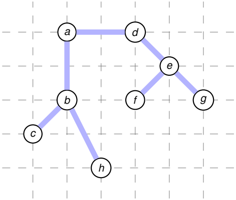
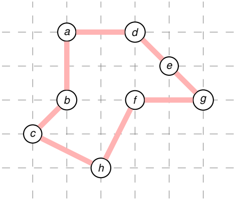


minimum spanning tree $T_{\min}$



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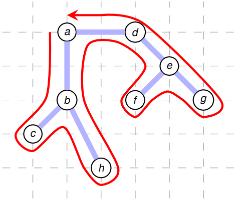
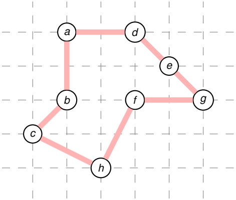


Walk $W = (a, b, c, b, h, b, a, d, e, f, e, g, e, d, a)$



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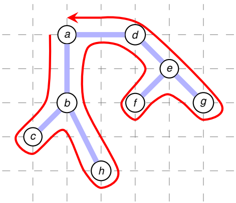

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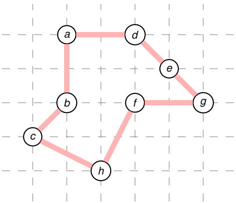

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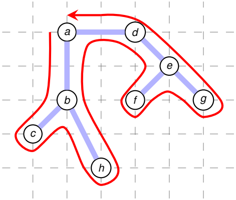

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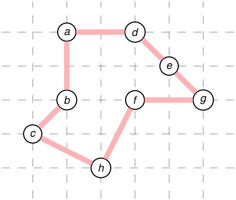

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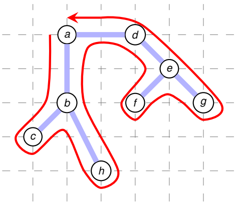

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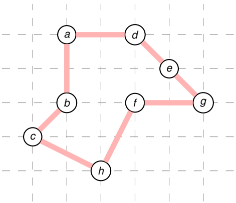

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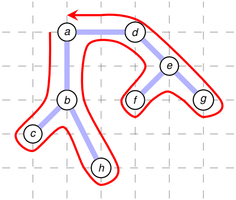
- Deleting duplicate vertices from W yields a tour H



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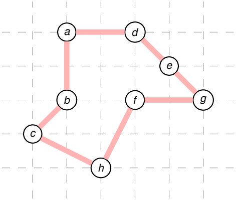

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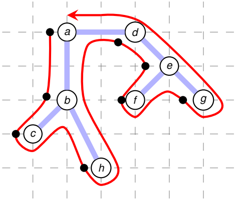

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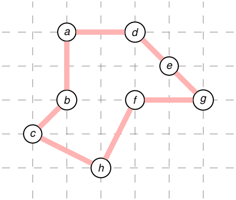

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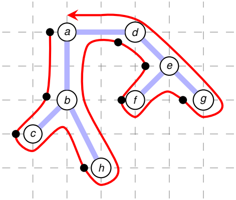

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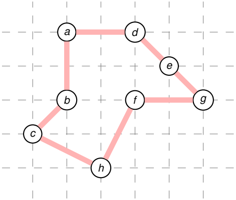

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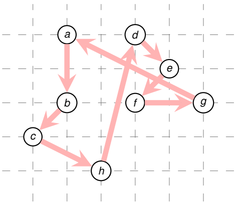
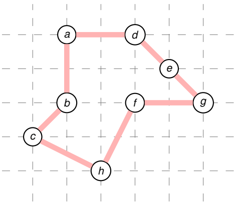


Tour $H = (a, b, c, h, d, e, f, g, a)$



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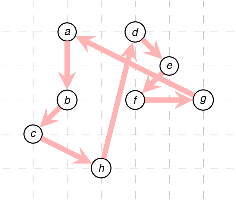
exploiting triangle inequality!

- Deleting duplicate vertices from W yields a tour H with smaller cost:



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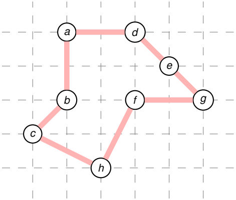

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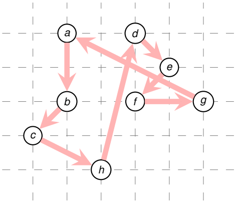

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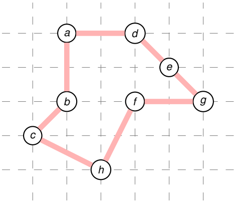

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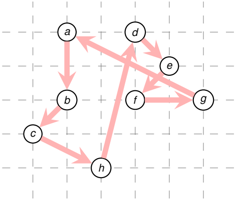

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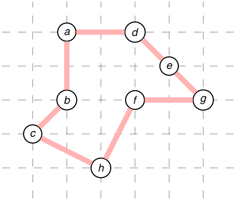

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- Let W be the full walk of the minimum spanning tree $T_{\min}$ (including repeated visits)

$\Rightarrow$ Full walk traverses every edge exactly twice, so

$$c(W) = 2c(T_{\min}) \leq 2c(T) \leq 2c(H^*)$$

exploiting triangle inequality!

- Deleting duplicate vertices from W yields a tour H with smaller cost:

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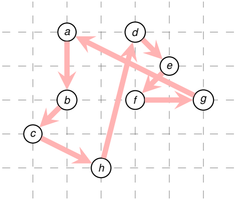
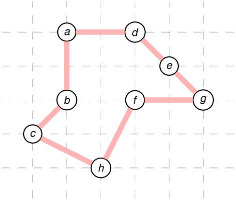


Tour $H = (a, b, c, h, d, e, f, g, a)$



optimal solution H^*

Proof of the Approximation Ratio

Theorem 35.2

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

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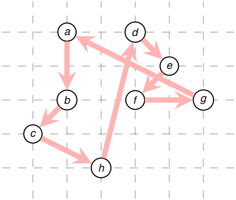

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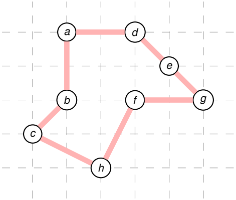

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Christofides Algorithm

Theorem 35.2

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Christofides Algorithm

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Can we get a better approximation ratio?

Christofides Algorithm

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APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

Can we get a better approximation ratio?

CHRISTOFIDES(G, c)
select a vertex $r \in G.V$ to be a “root” vertex
compute a minimum spanning tree $T_{\min}$ for G from root r
    using MST-PRIM(G, c, r)
compute a perfect matching $M_{\min}$ with minimum weight in the complete graph
    over the odd-degree vertices in $T_{\min}$
let H be a list of vertices, ordered according to when they are first visited
    in a Eulearian circuit of $T_{\min} \cup M_{\min}$
return the hamiltonian cycle H

Christofides Algorithm

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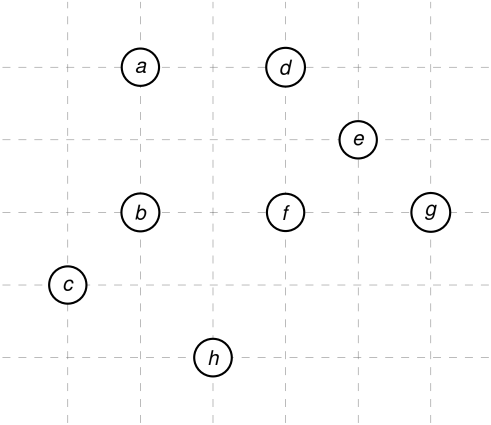
Theorem (Christofides'76)

There is a polynomial-time $\frac{3}{2}$-approximation algorithm for the travelling salesman problem with the triangle inequality.

Run of CHRISTOFIDES



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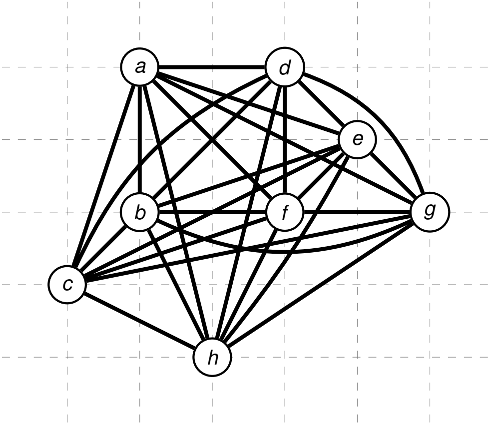
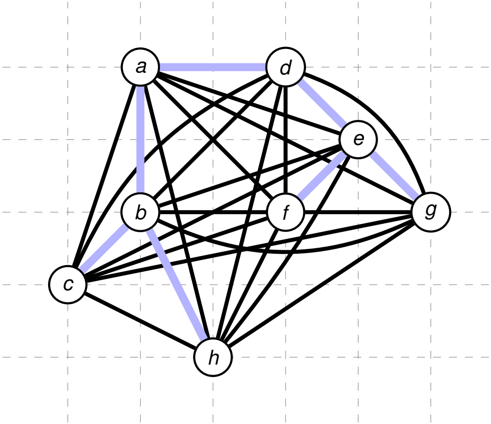


1. Compute MST $T_{\min}$

Run of CHRISTOFIDES



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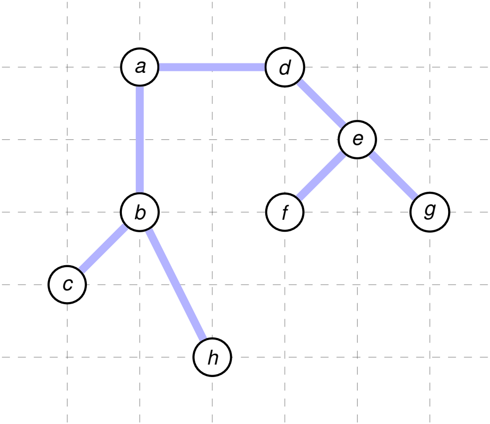

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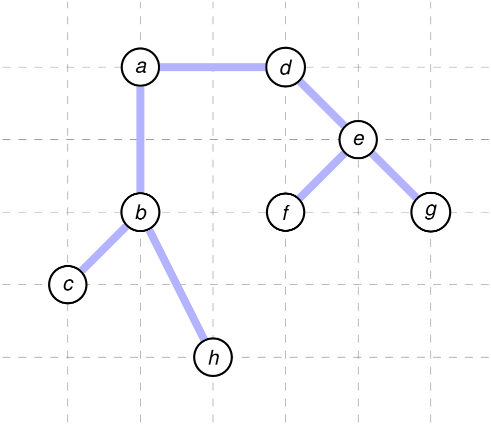
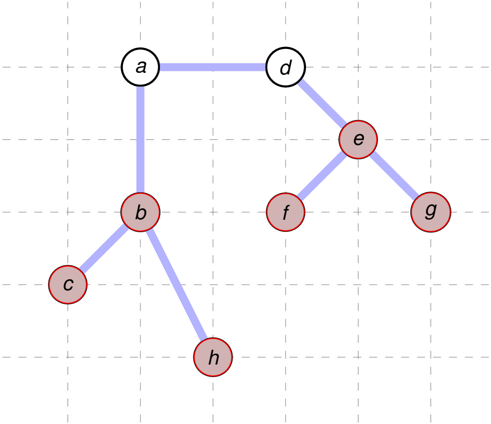


1. Compute MST $T_{\min}$ ✓
2. Add a minimum-weight perfect matching $M_{\min}$ of the odd vertices in $T_{\min}$

Run of CHRISTOFIDES



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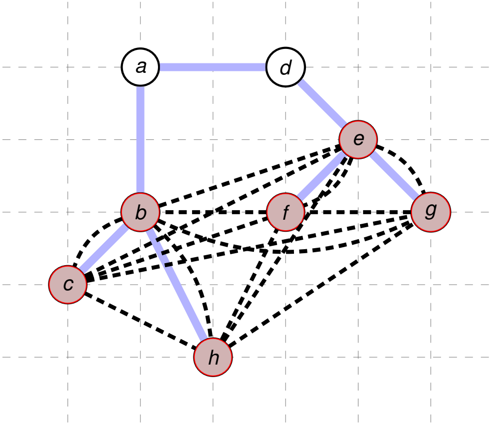

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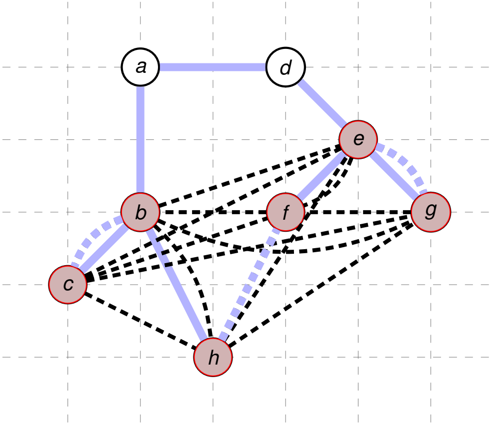

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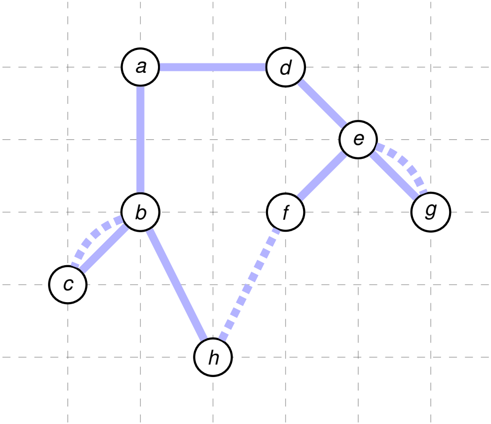

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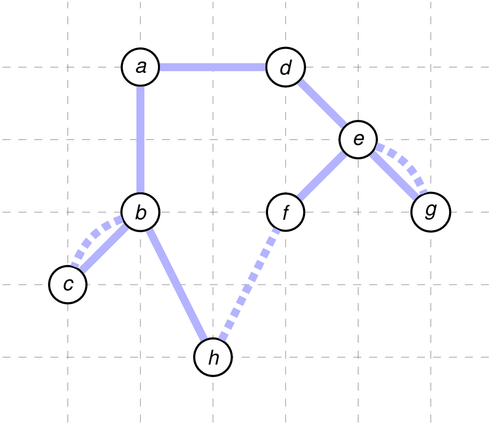

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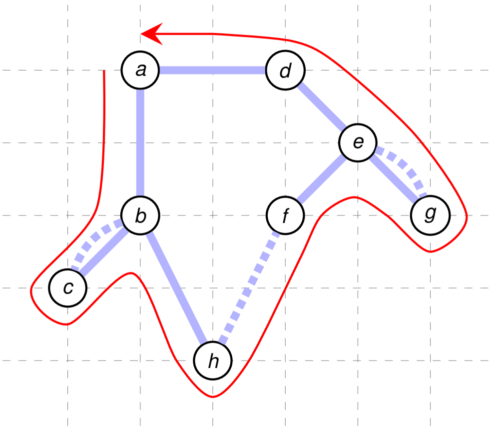
All vertices in $T_{\min} \cup M_{\min}$ have even degree!

Run of CHRISTOFIDES



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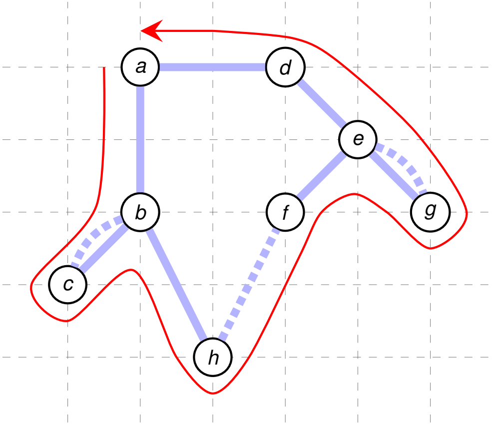

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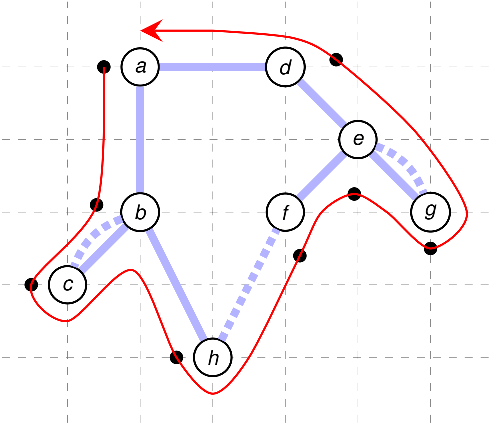
Run of CHRISTOFIDES



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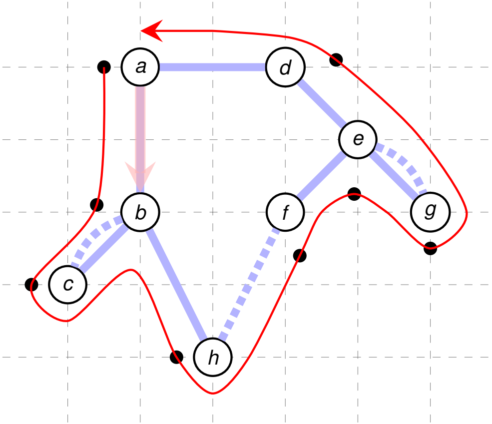

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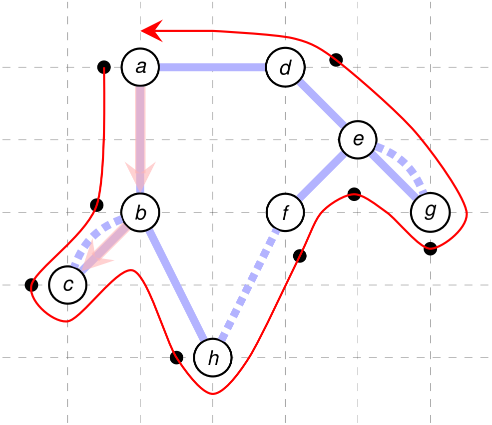

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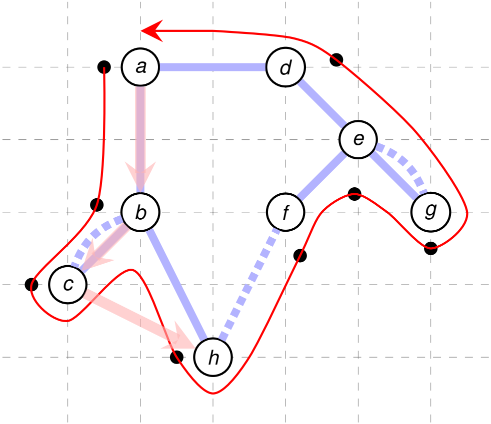

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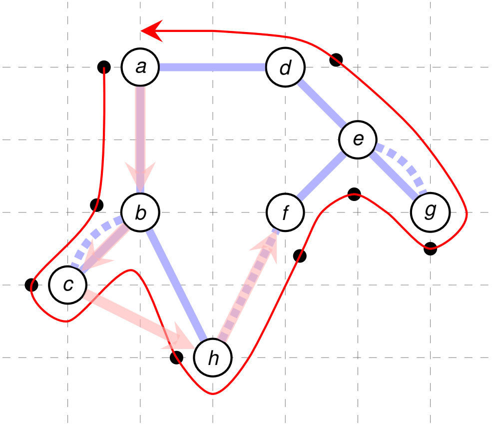

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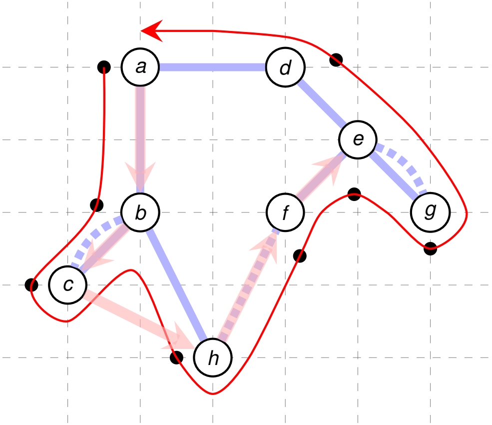

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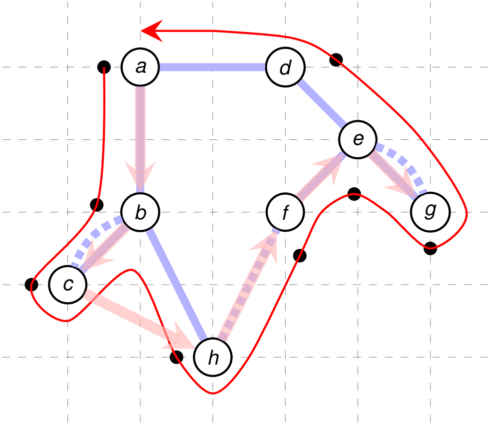

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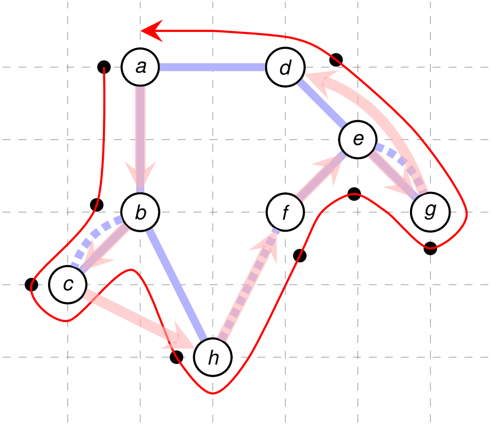

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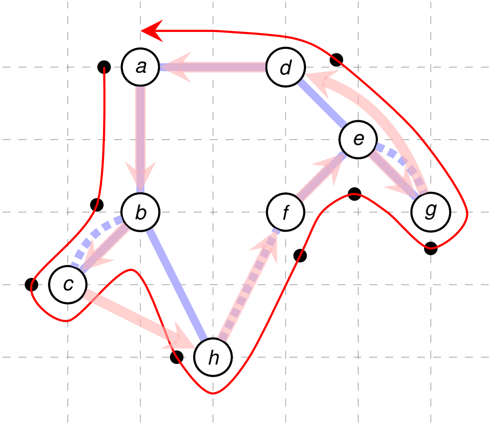

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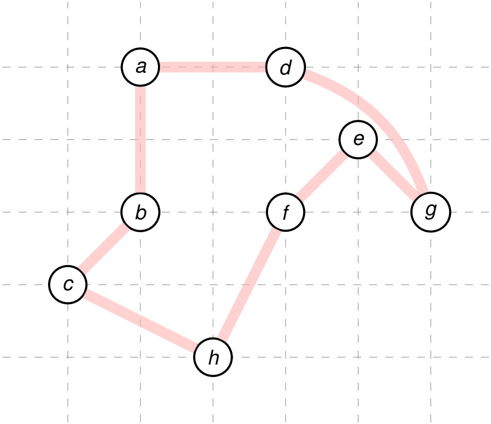

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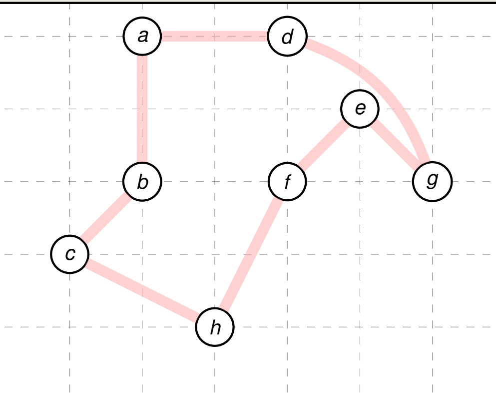
Solution has cost $\approx$ 15.54 - within 10% of the optimum!



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Theorem (Christofides'76)

There is a polynomial-time $\frac{3}{2}$-approximation algorithm for the travelling salesman problem with the triangle inequality.

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