The modal $\mu$-calculus [§4.2 p48]

\[
A ::= T \mid F \mid A_0 \land A_1 \mid A_0 \lor A_1 \mid \neg A \mid (\lambda)A \mid (\neg)A \mid X \mid \nu X.A
\]

To guarantee monotonicity (and therefore the existence of the fixed point), require the variable $X$ to occur only positively in $A$ in $\nu X.A$. That is, $X$ occurs only under an even number of $\neg$s.

\[
s \models \nu X.A \quad \text{iff} \quad s \in \nu X.A
\]

i.e. \[
s \in \bigcup \{S \subseteq P \mid S \subseteq A[S/X]\}\]

the maximum fixed point of the monotonic function $S \mapsto A[S/X]$
The modal $\mu$-calculus [§4.2 p48]

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As before, we take

$$[\lambda]A \equiv \neg(\lambda)\neg A \quad [\neg]A \equiv \neg(\neg)\neg A$$

Now also take

$$\mu X.A \equiv \neg\nu X.(-A[-X/X])$$
Example

Consider the process

\[ P \overset{\text{def}}{=} a.(a.P + b.c.nil) \]

Which states satisfy

- \( \mu X.(a)X \)
- \( \nu X.(a)X \)
- \( \mu X.[a]X \)
- \( \nu X[a]X \)
Let $\varphi : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ be monotonic. 
$\varphi$ is $\cap$-continuous iff for all decreasing chains $X_0 \supseteq X_1 \supseteq \ldots \supseteq X_n \supseteq \ldots$

$$\bigcap_{n \in \omega} \varphi(X_n) = \varphi\left(\bigcap_{n \in \omega} X_n\right)$$

If the set of states $S$ is finite, continuity certainly holds.

**Theorem**

If $\varphi : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ is $\cap$-continuous:

$$\nu X. \varphi(X) = \bigcap_{n \in \omega} \varphi^n(S)$$
Let $\varphi : \mathcal{P}(S) \to \mathcal{P}(S)$ be monotonic.
$\varphi$ is $\bigcup$-continuous iff for all increasing chains $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots$

$$
\bigcup_{n \in \omega} \varphi(X_n) = \varphi\left( \bigcup_{n \in \omega} X_n \right)
$$

If the set of states $S$ is finite, continuity certainly holds.

**Theorem**

If $\varphi : \mathcal{P}(S) \to \mathcal{P}(S)$ is $\bigcup$-continuous:

$$
\mu X. \varphi(X) = \bigcup_{n \in \omega} \varphi^n(\emptyset)
$$
Proposition

\[ s \models \mu X . \langle a \rangle T \lor \langle \neg \rangle X \] in any transition system iff there exists a sequence of transitions from \( s \) to a state \( t \) where an \( a \)-action can occur.
Proving interpretations

Proposition

$s \models \nu X.\langle a\rangle X$ in a finite-state transition system iff there exists an infinite sequence of $a$-transitions from $s$. 

Proposition

$s \models \nu X.\langle a\rangle X$ in a finite-state transition system iff there exists an infinite sequence of $a$-transitions from $s$.

There are infinite-state transition systems where $\phi(X) = \langle a\rangle X$ is not $\cap$-continuous.
Bisimilarity and modal $\mu$

For finite-state processes, modal-$\mu$ can be encoded in infinitary H-M logic.

If finite-state processes $p$ and $q$ are bisimilar then they satisfy the same modal-$\mu$ assertions.
For finite-state processes, modal-\(\mu\) can be encoded in infinitary H-M logic if finite-state processes \(p\) and \(q\) are bisimilar then they satisfy the same modal-\(\mu\) assertions.

Note that logical equivalence in modal-\(\mu\) does not generally imply bisimilarity (due to the lack of infinitary conjunction).